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GROWTH OF POLYNOMIALS NOT VANISHING IN A DISK

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Abstract. This paper deals with the problem of finding some upper bound estimates for the maximal modulus of a lacunary polynomial on a disk of radius R, $R \geq 1$ under the assumption that the polynomial does not vanish in another disk with radius k, $k \geq 1$. Our results sharpen as well as generalize a result recently proved by Hussain [Indian J. Pure Appl. Math., (https://doi.org/10.1007/s13226-021-00169-7)]. Further, these results generalize as well as sharpen some known results in this direction.

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1. INTRODUCTION

For a polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n and $R > 0$, set $M(p,R) = \max_{|z|=R} |p(z)|$. We denote $M(p,1)$ by $\| p \|$, the uniform norm of a polynomial p on the unit disk $|z| = 1$. The study of inequalities that relate the norm of a polynomial on a larger disk to that of its norm on the unit disk and their various versions is a classical topic in analysis. Over a period, these inequalities have been generalized in different domains and in different norms. It is a simple deduction from the maximum modulus principle (see [13], p. 158) that for $R \geq 1$,

$$M(p,R) \leq R^n \| p \| . \tag{1}$$

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Equality holds in (1) only for $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number. Noting that these extremal polynomials have all zeros at the origin, it is natural to seek improvements under appropriate condition on the zeros of $p(z)$. It was shown by Ankeny and Rivlin [1] that if $p(z)$ is a polynomial having no zero in $|z| < 1$, then inequality (1) can be replaced by

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right) \| p \| . \tag{2}$$

Inequality (2) is sharp and equality holds for $p(z) = \lambda + \mu z^n$ with $|\lambda| = |\mu|$.
As a refinement of (2), it was shown by Govil [8] that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < 1$, then for $R \geq 1$,

$$\begin{aligned} M(p,R) \quad &\leq \quad \left(\frac{R^n+1}{2}\right) \| p \| - \frac{n}{2}\left(\frac{\| p \|^2 - 4|a_n|^2}{\| p \|}\right) \\ &\times \quad \left[\frac{(R-1) \| p \|}{\| p \| + 2|a_n|} - ln\left\{1 + \frac{(R-1) \| p \|}{\| p \| + 2|a_n|}\right\}\right]. \end{aligned} \tag{3}$$

Inequality (3) was sharpened by Dewan and Bhat [3], which was later generalized by Govil and Nyuydinkong [9] that if $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $R \geq 1$,

$$\begin{aligned} M(p,R) \quad &\leq \quad \left(\frac{R^n+k}{1+k}\right)\|p\| - \left(\frac{R^n-1}{1+k}\right)m - \frac{n}{1+k}\left\{\frac{(\|p\|-m)^2 - (1+k)^2|a_n|^2}{\|p\|-m}\right\} \\ &\times \quad \left[\frac{(R-1)(\|p\|-m)}{\|p\|-m+(1+k)|a_n|} - \ln\left\{1 + \frac{(R-1)(\|p\|-m)}{\|p\|-m+(1+k)|a_n|}\right\}\right], \end{aligned} \tag{4}$$

where $m = \min_{|z|=k} |p(z)|$.
Inequality (4) was generalized by Gardner et al. [5] in a different direction by considering polynomials of the form $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$. More precisely, Gardner et al. [5] proved the following result.

Theorem 1. *If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \quad &\leq \quad \left(\frac{R^n+k^\mu}{1+k^\mu}\right) \| p \| - \left(\frac{R^n-1}{1+k^\mu}\right)m - \frac{n}{1+k^\mu}\left\{\frac{(\| p \| - m)^2 - (1+k^\mu)^2|a_n|^2}{(\| p \| - m)}\right\} \\ &\times \quad \left[\frac{(R-1)(\| p \| - m)}{(\| p \| - m) + (1+k^\mu)|a_n|} - ln\left\{1 + \frac{(R-1)(\| p \| - m)}{(\| p \| - m) + (1+k^\mu)|a_n|}\right\}\right], \end{aligned} \tag{5}$$

where $m=\min\limits_{|z|=k}|p(z)|$.

Inequality (4) of Govil and Nyuydinkong [9] is a special case of Theorem 1, when $\mu=1$. For $k=\mu=1$, Theorem 1 reduces to the result of Dewan and Bhat [3], which is a sharpening of inequality (3). Very recently, Dalal and Govil [2] used a recurrence relation and proved the following sharpening of inequality (3).

Theorem 2. *If* $p(z)=\sum\limits_{v=0}^{n}a_v z^v$ *is a polynomial of degree n having no zero in* $|z|<1$, *then for* $R\geq 1$ *and any* $1\leq N\leq n$,

$$M(p,R)\leq\left(\frac{R^n+1}{2}\right)\parallel p\parallel-\frac{n\parallel p\parallel}{2}\left(1-\frac{2|a_n|}{\parallel p\parallel}\right)h(N), \tag{6}$$

where

$$h(N)=(R-1)-\left(1+\frac{2|a_n|}{\|p\|}\right)\ln\left\{1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right\}\ for\ N=1,$$

$$\begin{aligned} h(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left(1+\frac{2|a_n|}{\|p\|}\right)\left(\frac{2|a_n|}{\|p\|}\right)^{v-1}\\ &+ (-1)^N\left(1+\frac{2|a_n|}{\|p\|}\right)\left(\frac{2|a_n|}{\|p\|}\right)^{N-1}\ln\left\{1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right\}\ for\ N\geq 2. \end{aligned} \tag{7}$$

For $k=1$, it is obvious from Lemma 10 that $|a_n|\leq\frac{\|p\|}{2}$, and by Lemma 6, the function $h(N)$ is a non-negative increasing function of N, $1\leq N\leq n$, it easily follows that the bound in (6) is sharper than that obtained from (3). In 2005, Gardner et al. [6] used the coefficients of the polynomial $p(z)$ and proved the following generalization and refinement of inequality (3).

Theorem 3. *If* $p(z)=a_0+\sum\limits_{v=\mu}^{n}a_v z^v$, $1\leq\mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then for* $R\geq 1$,

$$\begin{aligned} M(p,R) &\leq \left(\frac{R^n+s_1}{1+s_1}\right)\parallel p\parallel-\left(\frac{R^n-1}{1+s_1}\right)m-\frac{n}{1+s_1}\left\{\frac{(\parallel p\parallel-m)^2-(1+s_1)^2|a_n|^2}{(\parallel p\parallel-m)}\right\}\\ &\times \left[\frac{(R-1)(\parallel p\parallel-m)}{(\parallel p\parallel-m)+(1+s_1)|a_n|}-ln\left\{1+\frac{(R-1)(\parallel p\parallel-m)}{(\parallel p\parallel-m)+(1+s_1)|a_n|}\right\}\right] \end{aligned}$$

where

$$s_1=k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}+1}\right\},$$

and $m = \min_{|z|=k} |p(z)|$.

For $k=1$, $\mu=1$, we have $s_1=1$, Theorem 3 reduces to the result of Dewan and Bhat [3]. Although the literature on polynomial inequalities involving growth is vast and growing over the last four decades many different authors produced a large number of research papers and monographs on such inequalities. One can see in the literature (for example, refer [2, 5, 6, 11]), the recent research and development in this direction.

2. Lemmas

We need the following lemmas for the proof of the theorem.

Lemma 1. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, *is a polynomial of degree n, then for* $|z| = R \geq 1$,

$$|p(z)| \leq R^n \left\{ 1 - \frac{(\| p \| - |a_n|)(R-1)}{|a_n| + R \| p \|} \right\} \| p \| .$$

Lemma 2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n having no zero in* $|z| < k$, $k > 0$, *then*

$$|p(z)| \geq m, \ for \ |z| \leq k, \tag{8}$$

where $m = \min_{|z|=k} |p(z)|$.

The above Lemmas 1 and 2 are due to Gardner et al. [6].

Lemma 3. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, *is a polynomial of degree n having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\| p' \| \leq \frac{n}{1+s_\mu} \| p \|, \tag{9}$$

where

$$s_\mu = \frac{k^{\mu+1}(\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1)}{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}+1}. \tag{10}$$

Inequality (9) *is sharp and equality holds for the polynomial* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where n is a multiple of* μ.

Lemma 3 is due to Qazi [14, Lemma 1].

Lemma 4. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *and* $r \ge 1$, *then the function*

$$\left\{1 - \frac{(y - n|a_n|)(r-1)}{ry + n|a_n|}\right\} y$$

is an increasing function of y *for* $y > 0$.

Lemma 4 is due to Mir et al. [12, Lemma H]

Lemma 5. *Let*

$$h(N) = \int_1^R \frac{(r-1)r^{N-1}}{r+x} dr, \; x > 0 \, .$$

Then, for $N \ge 2$,

$$\begin{aligned} h(N) &= \left(\frac{R^N - 1}{N}\right) + \sum_{\nu=1}^{N-1} \left(\frac{R^{N-\nu} - 1}{N - \nu}\right) (-1)^\nu (x+1) x^{\nu-1} \\ &+ (-1)^N (x+1) x^{N-1} \ln\left(\frac{R+x}{1+x}\right), \end{aligned} \tag{11}$$

and for $N = 1$,

$$h(1) = (R-1) - (1+x)\ln\left(1 + \frac{R-1}{1+x}\right). \tag{12}$$

Proof of Lemma 5. Although the proof of this lemma was done by Dalal and Govil [2, Lemma 3.6], however, we include it here for the sake of completeness and for reader's convenience. For any $x > 0$, let

$$u(N) = \int_1^R \frac{r^N}{r+x} dr,$$

which is defined for $N \ge 0$. Then it is easy to see that $h(N) = u(N) - u(N-1)$ for $N \ge 1$. Note that for $N \ge 2$,

$$\begin{aligned} u(N) + xu(N-1) &= \int_1^R \frac{r^N + xr^{N-1}}{r+x} dr \\ &= \int_1^R \frac{r^{N-1}(r+x)}{r+x} dr \\ &= \frac{R^N - 1}{N} \\ &= w(N). \end{aligned}$$

Therefore,

$$u(N) = w(N) - xu(N-1), \tag{13}$$

for $N \geq 2$ and on solving the recurrence relation, we have

$$u(N) = \sum_{v=0}^{N-1} (-1)^v w(N-v)x^v + (-1)^N x^N u(0),\ N \geq 1, \tag{14}$$

where

$$u(0) = \int_1^R \frac{1}{r+x} dr = \ln\left(\frac{R+x}{1+x}\right).$$

Substituting the value of $u(0)$ in (14), we have

$$u(N) = \sum_{v=0}^{N-1} (-1)^v w(N-v)x^v + (-1)^N x^N \ln\left(\frac{R+x}{1+x}\right),\ N \geq 1. \tag{15}$$

Using $h(N) = u(N) - u(N-1)$ and value of $w(N) = \frac{R^N-1}{N}$, we have for $N \geq 2$,

$$\begin{aligned} h(N) &= \left(\frac{R^N-1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v(1+x)x^{v-1} \\ &+ (-1)^N(1+x)x^{N-1}\ln\left(\frac{R+x}{1+x}\right). \end{aligned} \tag{16}$$

For $N = 1$, we have

$$\begin{aligned} h(1) &= \int_1^R \frac{(r-1)}{r+x} dr \\ &= (R-1) - (1+x)\ln\left(1 + \frac{R-1}{1+x}\right). \end{aligned}$$

This completes the proof of Lemma 5. □

Lemma 6. *The function $h(N)$ defined in Lemma 5 is a non-negative increasing function of N for $N \geq 1$.*

Proof of Lemma 6. The proof of this lemma was done by Dalal and Govil [2, Lemma 3.7], however, a simple alternative proof is presented here.

Using the method of differentiation under the integral sign, we have

$$\frac{d}{dN}h(N) = \int_1^R \frac{(r-1)(r^{N-1})}{r+x} \ln r dr. \tag{17}$$

Since, for $r \in [1,R]$, $\frac{(r-1)r^{N-1}}{r+x} \ln r \geq 0$, therefore, we have

$$\int_1^R \frac{(r-1)r^{N-1}}{r+x} \ln r dr \geq 0.$$

From equality (17),

$$\frac{d}{dN} h(N) \geq 0, \ for \ N \geq 1.$$

Therefore, $h(N)$ is an increasing function of N for $N \geq 1$.

Further, since $\frac{(r-1)r^{N-1}}{r+x} \geq 0$ for $N \geq 1$, $h(N) \geq 0$ for $N \geq 1$.

This completes the proof of Lemma 6. □

Lemma 7. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, *is a polynomial of degree n having no zero in* $|z| < k$, $k \geq 1$, *then the function*

$$f(u) = \frac{1 + \frac{\mu}{n}\frac{|a_\mu|}{u} k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{u}(k^{\mu+1} + k^{2\mu})} \tag{18}$$

is a non-increasing function of u for $u > 0$*.*

Proof of Lemma 7. Considering the first derivative of f w.r.t u, we have

$$f^{'}(u) \quad = \quad \frac{\frac{\mu}{n}\frac{|a_\mu|}{u^2} k^{2\mu}(1-k^2)}{\left\{1 + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{u}(k^{\mu+1} + k^{2\mu})\right\}^2},$$

which is non-positive, since $(1-k^2) \leq 0$ for $k \geq 1$, and hence $f(u)$ is a non-increasing function of u. □

Lemma 8. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, *is a polynomial of degree n having no zero in* $|z| < k$, $k \geq 1$, *then for any real or complex number* λ *with* $|\lambda| < 1$

$$|a_n| \leq \frac{1}{1+s_2}\left(\|p\| - |\lambda| m\right), \tag{19}$$

where s_2 *is as defined in* (22) *and* $m = \min_{|z|=k} |p(z)|$*.*

Proof of Lemma 8. Since $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, then $p'(z) = \sum_{v=\mu}^{n} v a_v z^{v-1}$.

Hence, applying Cauchy's inequality to $p'(z)$ on the unit circle $|z| = 1$, we have

$$\left|\frac{d^{n-1}}{dz^{n-1}} p'(z)\right|_{z=0} \leq (n-1)! \max_{|z|=1} |p'(z)|.$$

That is,

$$|na_n| \le \|p'\|. \tag{20}$$

Combining inequality (27) in the proof of Theorem 4 and (20), we have inequality (19) and this completes the proof of Lemma 8. □

The next lemma is due to Gardner et al. [4, Lemma 3].

Lemma 9. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then*

$$s_1 \ge k^\mu,$$

where s_1 is as defined in Theorem 3.

Lemma 10. *If $p(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n in $|z| < k$, $k \ge 1$, then*

$$|a_n| \le \frac{1}{1+k^\mu}\left(\| p \| - m\right),$$

where $m = \min\limits_{|z|=k} |p(z)|$.

Lemma 10 is due to Gardner et al. [5].

3. Main Results

In this paper, we prove a result which is a generalization of Theorems 3 due to Gardner et al. [6]. More precisely, we prove

Theorem 4. *If $p(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, then for any real or complex number λ with $|\lambda| < 1$, $R \ge 1$ and any positive integer N, $1 \le N \le n$,*

$$M(p,R) \le \left(\frac{R^n + s_2}{1+s_2}\right)\|p\| - \frac{(R^n-1)|\lambda|m}{1+s_2} - n\left\{\frac{\|p\| - |\lambda|m}{1+s_2} - |a_n|\right\} S(N), \tag{21}$$

where

$$s_2 = \frac{k^{\mu+1}\left(\frac{\mu}{n}\left|\frac{a_\mu}{a_0+\lambda m}\right|k^{\mu-1}+1\right)}{\frac{\mu}{n}\left|\frac{a_\mu}{a_0+\lambda m}\right|k^{\mu+1}+1} \tag{22}$$

and

$$
\begin{aligned}
S(N) &= \left(R-1\right)-\left\{1+\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\} \\
&\times \ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{(\|p\|-|\lambda|m)+(1+s_2)|a_n|}\right\} \ for\ N=1,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
S(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}^{v-1} \\
&+ (-1)^N\left\{1+\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}^{N-1} \\
&\times \ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{(\|p\|-|\lambda|m)+(1+s_2)|a_n|}\right\} \ for\ N\geq 2,
\end{aligned} \tag{24}
$$

and $m=\min_{|z|=k}|p(z)|$.

Proof of Theorem 4. Since the polynomial $p(z)=a_0+\sum_{v=\mu}^{n}a_v z^v$, $1\leq\mu\leq n$, has no zero in $|z|<k$, $k\geq 1$, therefore for every real or complex number λ with $|\lambda|<1$, by Rouche's Theorem the polynomial $p(z)+\lambda m$ has no zero in $|z|<k$, $k\geq 1$, where $m=\min_{|z|=k}|p(z)|$.
Applying Lemma 3 to $p(z)+\lambda m$, we have

$$
\| p^{'} \| \leq \frac{n\|p(z)+\lambda m\|}{1+s_2}, \tag{25}
$$

where s_2 is as defined in (22).
Now, we choose the argument of λ suitably such that

$$
|p(z)+\lambda m| = |p(z)|-|\lambda|m. \tag{26}
$$

Using inequality (26) to (25), we have

$$
\| p^{'} \| \leq \frac{n(\|p\|-|\lambda|m)}{1+s_2}. \tag{27}
$$

Then, for each θ, $0\leq\theta<2\pi$ and $1\leq r\leq R$, we have

$$
p(Re^{i\theta})-p(e^{i\theta})=\int_1^R e^{i\theta}p'(re^{i\theta})dr,
$$

which implies

$$
|p(Re^{i\theta})-p(e^{i\theta})|\leq\int_1^R|p'(re^{i\theta})|dr.
$$

Now, applying Lemma 1 to the polynomial $p'(z)$ which is of degree $n-1$, we get

$$|p(Re^{i\theta})-p(e^{i\theta})| \le \int_1^R r^{n-1}\left\{1-\frac{(\|p'\|-n|a_n|)(r-1)}{n|a_n|+r\|p'\|}\right\}\|p'\|dr. \tag{28}$$

By Lemma 4, the quantity $\left\{1-\frac{(\|p'\|-n|a_n|)(r-1)}{n|a_n|+r\|p'\|}\right\}\|p'\|$ occuring in the integrand of (28) is an increasing function of $\|p'\|$, and hence using inequality (27) for the value of $\| p' \|$, we have for $0 \le \theta < 2\pi$,

$$\begin{aligned}
& |p(Re^{i\theta})-p(e^{i\theta})| \\
\le\ & \int_1^R r^{n-1}\left[1-\frac{\left\{\frac{n}{1+s_2}(\|p\|-|\lambda|m)-n|a_n|\right\}(r-1)}{n|a_n|+r(\frac{n}{1+s_2})(\|p\|-|\lambda|m)}\right]\frac{n}{1+s_2}(\|p\|-|\lambda|m)dr \qquad (29)\\
=\ & \frac{n}{1+s_2}(\|p\|-|\lambda|m)\int_1^R r^{n-1}dr-\frac{n}{1+s_2}(\|p\|-|\lambda|m) \\
\times\ & \int_1^R r^{n-1}\left\{\frac{(\|p\|-|\lambda|m)-(1+s_2)|a_n|}{(1+s_2)|a_n|+r(\|p\|-|\lambda|m)}\right\}(r-1)dr \\
=\ & \frac{R^n-1}{1+s_2}(\|p\|-|\lambda|m)-\frac{n}{1+s_2}(\|p\|-|\lambda|m)(1-f)\int_1^R\frac{(r-1)r^{n-1}}{r+f}dr, \qquad (30)
\end{aligned}$$

where s_2 is as defined in (22) and $f=\frac{|a_n|(1+s_2)}{(\|p\|-|\lambda|m)}$.
Note that from Lemma 6, the integral $\int_1^R\frac{(r-1)r^{N-1}}{r+f}dr$ is a non-negative and increasing function of N for $1 \le N \le n$, therefore, we have

$$\int_1^R\frac{(r-1)r^{N-1}}{r+f}dr \le \int_1^R\frac{(r-1)r^{n-1}}{r+f}dr. \tag{31}$$

Noting from Lemma 8 that $(1-f) \ge 0$ and using inequality (31) to (30), we have for every N, $1 \le N \le n$,

$$|p(Re^{i\theta})-p(e^{i\theta})| \le \frac{R^n-1}{1+s_2}(\|p\|-|\lambda|m)-\frac{n}{1+s_2}(\|p\|-|\lambda|m)(1-f)\int_1^R\frac{(r-1)r^{N-1}}{r+f}dr. \tag{32}$$

Using Lemma 5 (on replacing x by f) for the value of integral in (32), we have for each $0 \le \theta < 2\pi$,

$$|p(Re^{i\theta})-p(e^{i\theta})| \le \frac{R^n-1}{1+s_2}(\|p\|-|\lambda|m)-\frac{n}{1+s_2}(\|p\|-|\lambda|m)(1-f)S(N), \tag{33}$$

where $S(N)$ is as defined in (23) and (24).

Now, substituting the value of f and using the obvious inequality

$$\begin{aligned} |p(Re^{i\theta})| &\leq |p(Re^{i\theta})-p(e^{i\theta})|+|p(e^{i\theta})| \\ &\leq |p(Re^{i\theta})-p(e^{i\theta})|+\|p\| \end{aligned} \tag{34}$$

in (33), we get for $0\leq\theta<2\pi$ and $R\geq 1$,

$$|p(Re^{i\theta})|\leq\left(\frac{R^n+s_2}{1+s_2}\right)\|p\|-\frac{(R^n-1)|\lambda|m}{1+s_2}-\frac{n(\|p\|-|\lambda|m)}{1+s_2}\left\{1-\frac{(1+s_2)|a_n|}{(\|p\|-|\lambda|m)}\right\}S(N),$$

which is inequality (21) and hence the proof of Theorem 4 is completed. □

Remark 1. *By Lemma 6, it is noted that $S(N)$ given by* (23) *and* (24) *of Theorem 4 is an increasing function of N, $1\leq N\leq n$, therefore, the bound in Theorem 4 improves most when $N=n$, the degree of the polynomial.*

Further, when $p(z)$ is a polynomial of degree $n=1$, then we have from inequality (30) *in the proof of Theorem 4 that*

$$|p(Re^{i\theta})|\leq\left(\frac{R+s_2}{1+s_2}\right)\|p\|-\frac{R-1}{1+s_2}|\lambda|m-\frac{1}{1+s_2}\{\|p\|-|\lambda|m-(1+s_2)|a_1|\}\int_1^R\frac{r-1}{r+f}dr. \tag{35}$$

Substituting the value of $f=\frac{(1+s_2)|a_1|}{(\|p\|-|\lambda|m)}$ to the integral of (35) *and using equality* (12) *of Lemma 5, we have*

$$|p(Re^{i\theta})|\leq\left(\frac{R+s_2}{1+s_2}\right)\|p\|-\frac{R-1}{1+s_2}|\lambda|m-\frac{1}{1+s_2}\{(\|p\|-|\lambda|m)-(1+s_2)|a_1|\}S(1), \tag{36}$$

where

$$S(1)=(R-1)-\left\{1+\frac{(1+s_2)|a_1|}{\|p\|-|\lambda|m}\right\}\ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{(\|p\|-|\lambda|m)+(1+s_2)|a_1|}\right\}.$$

On the other hand, we have by a simple calculation

$$M(p,R)=\max_{|z|=R}|p(z)|=\max_{|z|=R}|a_0+za_1|=|a_0|+R|a_1|. \tag{37}$$

Hence, in particular, for a polynomial of degree 1*, instead of using the bound given by* (36) *of $M(p,R)$, we prefer the exact value as given by* (37)*.*

From the above discussion, Theorem 4 in particular, assumes

Corollary 1. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\ge 1$, *then for any real or complex number* λ *with* $|\lambda|<1$ *and* $R\ge 1$,

$$M(p,R)=|a_0|+R|a_1| \; for\; n=1 \tag{38}$$

and

$$M(p,R)\le\left(\frac{R^n+s_2}{1+s_2}\right)\|p\|-\frac{(R^n-1)|\lambda|m}{1+s_2}-n\left\{\frac{\|p\|-|\lambda|m}{1+s_2}-|a_n|\right\}S(n), \tag{39}$$

where

$$\begin{aligned} S(n) &= \left(\frac{R^n-1}{n}\right)+\sum_{\nu=1}^{n-1}\left(\frac{R^{n-\nu}-1}{n-\nu}\right)(-1)^\nu\left\{1+\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}^{\nu-1}\\ &+ (-1)^n\left\{1+\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}^{n-1}\\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{(\|p\|-|\lambda|m)+(1+s_2)|a_n|}\right\} \; for\; n\ge 2, \end{aligned} \tag{40}$$

s_2 *is as defined in* (22) *and* $m=\min_{|z|=k}|p(z)|$.

Remark 2. *In particular, if* $k=1$, *then* $s_2=1$ *and Theorem 4 reduces to the following interesting result which is a generalization of a result proved by Dalal and Govil* [2].

Corollary 2. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, *is a polynomial of degree n having no zero in* $|z|<1$, *then for any real or complex number* λ *with* $|\lambda|<1$, $R\ge 1$ *and any positive integer N,* $1\le N\le n$,

$$M(p,R)\le\left(\frac{R^n+1}{2}\right)\|p\|-\frac{R^n-1}{2}|\lambda|m-n\left\{\frac{\|p\|-|\lambda|m}{2}-|a_n|\right\}g^*(N), \tag{41}$$

where

$$g^*(N) = (R-1)-\left(1+\frac{2|a_n|}{\|p\|-|\lambda|m}\right)\ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{\|p\|-|\lambda|m+2|a_n|}\right\} \; for\; N=1,$$

$$\begin{aligned} g^*(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{\nu=1}^{N-1}\left(\frac{R^{N-\nu}-1}{N-\nu}\right)(-1)^\nu\left(1+\frac{2|a_n|}{\|p\|-|\lambda|m}\right)\left(\frac{2|a_n|}{\|p\|-|\lambda|m}\right)^{\nu-1}\\ &+ (-1)^N\left(1+\frac{2|a_n|}{\|p\|-|\lambda|m}\right)\left(\frac{2|a_n|}{\|p\|-|\lambda|m}\right)^{N-1}\\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{\|p\|-|\lambda|m+2|a_n|}\right\} \; for\; N\ge 2, \end{aligned} \tag{42}$$

and $m = \min_{|z|=k} |p(z)|$.

Remark 3. *For* $\lambda = 0$*, Corollary 2 reduces to Theorem 2 proved by Dalal and Govil* [2].

Remark 4. *By Lemma 2, for* $|z| \leq k$ *and* $m \neq 0$*,*

$$|p(z)| \geq m, \tag{43}$$

where $m = \min_{|z|=k} |p(z)|$*. In particular,*

$$|p(0)| \geq m, \tag{44}$$

which implies

$$|a_0| \geq m. \tag{45}$$

For any real or complex number λ *with* $|\lambda| < 1$*, inequality* (45) *further implies*

$$|a_0| \geq |\lambda| m. \tag{46}$$

Now, using inequality (46)*, we have*

$$\begin{aligned} |a_0 + \lambda m| &\geq \Big| |a_0| - |\lambda| m \Big| \\ &= |a_0| - |\lambda| m. \end{aligned} \tag{47}$$

By Lemma 7, $f(u)$ *is a non-increasing function of u, and hence*

$$f(|a_0 + \lambda m|) \leq f(|a_0| - |\lambda| m),$$

which implies

$$\frac{1}{1+s_2} \leq \frac{1}{1+s_3}, \tag{48}$$

where s_2 *is as defined in* (22) *and*

$$s_3 = \frac{k^{\mu+1}\left(\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-|\lambda|m}k^{\mu-1}+1\right)}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-|\lambda|m}k^{\mu+1}+1}. \tag{49}$$

Since $(\|p\| - |\lambda| m) \geq 0$*, inequality* (48) *gives*

$$\frac{n(\|p\| - |\lambda| m)}{1+s_2} \leq \frac{n(\|p\| - |\lambda| m)}{1+s_3}. \tag{50}$$

Applying Lemma 4 to (50)*, we have for* $r \geq 1$*,*

$$\left[1-\frac{\left\{\frac{n}{1+s_2}(\|p\|-|\lambda|m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+s_2}(\|p\|-|\lambda|m)}\right]\frac{n}{1+s_2}(\|p\|-|\lambda|m)$$
$$\leq \left[1-\frac{\left\{\frac{n}{1+s_3}(\|p\|-|\lambda|m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+s_3}(\|p\|-|\lambda|m)}\right]\frac{n}{1+s_3}(\|p\|-|\lambda|m).$$

For $r \geq 0$*, which is equivalent to*

$$r^{n-1}\left[1-\frac{\left\{\frac{n}{1+s_2}(\|p\|-|\lambda|m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+s_2}(\|p\|-|\lambda|m)}\right]\frac{n}{1+s_2}(\|p\|-|\lambda|m)$$
$$\leq r^{n-1}\left[1-\frac{\left\{\frac{n}{1+s_3}(\|p\|-|\lambda|m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+s_3}(\|p\|-|\lambda|m)}\right]\frac{n}{1+s_3}(\|p\|-|\lambda|m). \tag{51}$$

Integrating both sides of (51) *with respect to r from* 1 *to R and making use of the fact that the right hand side of inequality* (29) *simplifies exactly to that of* (30) *in the proof of Theorem 4, we have*

$$\frac{R^n-1}{1+s_2}(\|p\|-|\lambda|m)-\frac{n}{1+s_2}(\|p\|-|\lambda|m)(1-f)\int_1^R\frac{(r-1)r^{n-1}}{r+f}dr$$
$$\leq \frac{R^n-1}{1+s_3}(\|p\|-|\lambda|m)-\frac{n}{1+s_3}(\|p\|-|\lambda|m)(1-g)\int_1^R\frac{(r-1)r^{n-1}}{r+g}dr, \tag{52}$$

where $f=\frac{|a_n|(1+s_2)}{\|p\|-|\lambda|m}$ *and* $g=\frac{|a_n|(1+s_3)}{\|p\|-|\lambda|m}$.
We notice that the integral $\int_1^R\frac{(r-1)r^{N-1}}{r+g}dr$ *occuring in the right hand side of* (52) *is a non-negative and increasing function of N for* $1 \leq N \leq n$*, therefore, we have*

$$\int_1^R\frac{(r-1)r^{N-1}}{r+g}dr \leq \int_1^R\frac{(r-1)r^{n-1}}{r+g}dr. \tag{53}$$

Since $\|p\|-|\lambda|m \geq 0$*, by Lemma 8, we have*

$$\frac{|a_n|(1+s_2)}{\|p\|-|\lambda|m} \leq 1,$$

and hence

$$1-f = 1-\frac{|a_n|(1+s_2)}{\|p\|-|\lambda|m} \geq 0.$$

Similarly, $1-g \geq 0$.

Using (53) *to the right hand side of* (52) *and noting that* $1-g \geq 0$ *and applying Lemma 5 for the values of integrals involved in the resulting inequality, we have*

$$\begin{aligned} & \left(\frac{R^n-1}{1+s_2}\right)\|p\| - \frac{(R^n-1)|\lambda|m}{1+s_2} - \frac{n(\|p\|-|\lambda|m)}{1+s_2}\left\{1-\frac{(1+s_2)|a_n|}{\|p\|-|\lambda|m}\right\}S(n) \\ \leq & \left(\frac{R^n-1}{1+s_3}\right)\|p\| - \frac{(R^n-1)|\lambda|m}{1+s_3} - \frac{n(\|p\|-|\lambda|m)}{1+s_3}\left\{1-\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}T(N), \end{aligned} \tag{54}$$

where $S(n)$ *is as defined in* (40) *and*

$$\begin{aligned} T(N) &= (R-1) - \left\{1+\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{\|p\|-|\lambda|m+(1+s_3)|a_n|}\right\} \; for\; N=1, \end{aligned}$$

$$\begin{aligned} T(N) &= \left(\frac{R^N-1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}^{N-1} \\ &\times \ln\left(1+\frac{(R-1)(\|p\|-|\lambda|m)}{\|p\|-|\lambda|m+(1+s_3)|a_n|}\right) \; for\; N\geq 2. \end{aligned} \tag{55}$$

Adding $\|p\|$ *on both sides of* (54)*, we have*

$$\begin{aligned} & \left(\frac{R^n+s_2}{1+s_2}\right)\|p\| - \frac{(R^n-1)|\lambda|m}{1+s_2} - \frac{n}{1+s_2}\{(\|p\|-|\lambda|m)-(1+s_2)|a_n|\}S(n) \\ \leq & \left(\frac{R^n+s_3}{1+s_3}\right)\|p\| - \frac{(R^n-1)|\lambda|m}{1+s_3} - \frac{n}{1+s_3}\{(\|p\|-|\lambda|m)-(1+s_3)|a_n|\}T(N), \end{aligned}$$

which clearly shows that Corollary 1 is an improvement of the following result which can be further deduced from Theorem 4.

Corollary 3. *If* $p(z)=a_0+\sum_{v=\mu}^{n} a_v z^v$, $1\leq \mu \leq n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then for any real or complex number* λ *with* $|\lambda|<1$, $R\geq 1$ *and any positive integer N,* $1\leq N\leq n$,

$$M(p,R) \leq \left(\frac{R^n+s_3}{1+s_3}\right)\|p\| - \frac{(R^n-1)|\lambda|m}{1+s_3} - n\left\{\frac{\|p\|-|\lambda|m}{1+s_3}-|a_n|\right\}T(N), \tag{56}$$

where

$$s_3 = \frac{k^{\mu+1}(\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-|\lambda|m}k^{\mu-1}+1)}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-|\lambda|m}k^{\mu+1}+1} \tag{57}$$

and

$$\begin{aligned} T(N) &= \left(R-1\right) - \left\{1+\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{(\|p\|-|\lambda|m)+(1+s_3)|a_n|}\right\} \ for\ N=1, \end{aligned} \tag{58}$$

$$\begin{aligned} T(N) &= \left(\frac{R^N-1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}\left\{\frac{(1+s_3)|a_n|}{\|p\|-|\lambda|m}\right\}^{N-1} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-|\lambda|m)}{(\|p\|-|\lambda|m)+(1+s_3)|a_n|}\right\} \ for\ N\geq 2, \end{aligned} \tag{59}$$

and $m = \min_{|z|=k}|p(z)|$.

Remark 5. *In particular, for* $0 \leq \lambda < 1$*, Corollary 3 becomes the following result of Hussain* [10, Theorem 2].

Corollary 4. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$*,* $1 \leq \mu \leq n$*, is a polynomial of degree n having no zero in* $|z| < k$*,* $k \geq 1$*, then for any real or complex number* λ *with* $0 \leq \lambda < 1$*,* $R \geq 1$ *and any positive integer N,* $1 \leq N \leq n$*,*

$$M(p,R) \leq \left(\frac{R^n+s_4}{1+s_4}\right)\|p\| - \frac{(R^n-1)\lambda m}{1+s_4} - n\left\{\frac{\|p\|-\lambda m}{1+s_4} - |a_n|\right\}T^*(N), \tag{60}$$

where

$$s_4 = \frac{k^{\mu+1}(\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu-1}+1)}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}+1} \tag{61}$$

and

$$\begin{aligned} T^*(N) &= \left(R-1\right) - \left\{1+\frac{(1+s_4)|a_n|}{\|p\|-\lambda m}\right\} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-\lambda m)}{(\|p\|-\lambda m)+(1+s_4)|a_n|}\right\} \ for\ N=1, \end{aligned} \tag{62}$$

$$\begin{aligned} T^*(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+s_4)|a_n|}{\|p\|-\lambda m}\right\}\left\{\frac{(1+s_4)|a_n|}{\|p\|-\lambda m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+s_4)|a_n|}{\|p\|-\lambda m}\right\}\left\{\frac{(1+s_4)|a_n|}{\|p\|-\lambda m}\right\}^{N-1} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-\lambda m)}{(\|p\|-\lambda m)+(1+s_4)|a_n|}\right\} \; for\; N\geq 2, \end{aligned} \tag{63}$$

and $m=\min_{|z|=k}|p(z)|$.

Remark 6. *In Theorem* 2 *of the paper of Hussain* [10], *the value of* $\Psi(N)$ *is*

$$\begin{aligned} \Psi(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+S_\lambda)|a_n|}{\|p\|-\lambda m}\right\}\left\{\frac{(1+S_\lambda)|a_n|}{\|p\|-\lambda m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+S_\lambda)|a_n|}{\|p\|-\lambda m}\right\}\left\{\frac{(1+S_\lambda)|a_n|}{\|p\|-\lambda m}\right\}^{N-1} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-\lambda m)}{(\|p\|-\lambda m)+(1+S_\lambda)|a_n|}\right\} \; for\; N\geq 1. \end{aligned} \tag{64}$$

However, we cannot get the value of $\Psi(1)$ *from inequality* (64). *This drawback has been resolved in Corollary 4 by seperately highlighting the value* $T^*(N)$ *for* $N=1$.

Remark 7. *In the limit* $\lambda\to 1$, *Corollary 4 reduces to the following result which is a generalization of Theorem 3 due to Gardner et al.* [6].

Corollary 5. *If* $p(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leq\mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then for* $R\geq 1$ *and any positive integer N,* $1\leq N\leq n$,

$$M(p,R)\leq\left(\frac{R^n+s_1}{1+s_1}\right)\|p\|-\frac{(R^n-1)m}{1+s_1}-n\left\{\frac{\|p\|-m}{1+s_1}-|a_n|\right\}R(N), \tag{65}$$

where s_1 *is as defined in Theorem* 3 *and*

$$\begin{aligned} R(N) &= \Big(R-1\Big)-\left\{1+\frac{(1+s_1)|a_n|}{\|p\|-m}\right\} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+s_1)|a_n|}\right\} \; for\; N=1, \end{aligned} \tag{66}$$

$$\begin{aligned} R(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+s_1)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+s_1)|a_n|}{\|p\|-m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+s_1)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+s_1)|a_n|}{\|p\|-m}\right\}^{N-1} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+s_1)|a_n|}\right\} \; for\; N\geq 2, \end{aligned} \tag{67}$$

and $m=\min_{|z|=k}|p(z)|$.

Remark 8. *From Lemma 6, we have* $T^*(N)$ *is a non-negative increasing function of* N *for* $N\geq 1$ *and hence* $T^*(1)\leq T^*(N)$, $1\leq N$. *Using this and Lemma 8, Corollary 4 also reduces to the result of Hussain* [10, Corollary 1].

Corollary 6. *If* $p(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leq\mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then for any real or complex number* λ *with* $0\leq\lambda<1$ *and for* $R\geq 1$,

$$\begin{aligned} M(p,R) &\leq \left(\frac{R^n+s_4}{1+s_4}\right)\|p\|-\frac{R^n-1}{1+s_4}\lambda m-\frac{n}{1+s_4}\left\{\frac{(\|p\|-\lambda m)^2-(1+s_4)^2|a_n|^2}{(\|p\|-\lambda m)}\right\} \\ &\times \left[\frac{(R-1)(\|p\|-\lambda m)}{(\|p\|-\lambda m)+(1+s_4)|a_n|}-ln\left\{1+\frac{(R-1)(\|p\|-\lambda m)}{(\|p\|-\lambda m)+(1+s_4)|a_n|}\right\}\right], \end{aligned}$$

where s_4 *is as defined in* (61) *and* $m=\min_{|z|=k}|p(z)|$.

Remark 9. *For* $\lambda=1$, *Corollary 6 assumes a result of Gardner et al.* [6].

Remark 10. *For* $\lambda=0$, s_4 *as defined in Corollary 6 becomes*

$$s_0=\frac{k^{\mu+1}(\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1)}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}+1}, \tag{68}$$

and hence Corollary 6 also becomes

Corollary 7. *If* $p(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leq\mu\leq n$, *is a polynomial of degree n having no zero in* $|z|<k$, $k\geq 1$, *then for* $R\geq 1$,

$$\begin{aligned} M(p,R) &\leq \left(\frac{R^n+s_0}{1+s_0}\right)\|p\|-\frac{n}{1+s_0}\left\{\frac{(\|p\|)^2-(1+s_0)^2|a_n|^2}{\|p\|}\right\} \\ &\times \left[\frac{(R-1)\|p\|}{\|p\|+(1+s_0)|a_n|}-ln\left\{1+\frac{(R-1)\|p\|}{\|p\|+(1+s_0)|a_n|}\right\}\right], \end{aligned}$$

where s_0 *is as defined in* (68).

Remark 11. *For $k=1$ and $\mu=1$, we have $s_0=1$, then Corollary 7 reduces to inequality* (3).

Remark 12. *By Lemma 9, $s_1 \geq k^{\mu}$, therefore*

$$\frac{n}{1+s_1}\Big(\|p\|-m\Big) \leq \frac{n}{1+k^{\mu}}\Big(\|p\|-m\Big), \tag{69}$$

where $m=\min\limits_{|z|=k}|p(z)|$ and s_1 is as defined in Theorem 3.
Applying Lemma 4 to (69)*, we have for $r \geq 1$*

$$\begin{aligned}&\left[1-\frac{\left\{\frac{n}{1+s_1}(\|p\|-m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+s_1}(\|p\|-m)}\right]\frac{n}{1+s_1}(\|p\|-m)\\ \leq\ &\left[1-\frac{\left\{\frac{n}{1+k^{\mu}}(\|p\|-m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+k^{\mu}}(\|p\|-m)}\right]\frac{n}{1+k^{\mu}}(\|p\|-m).\end{aligned}$$

For $r \geq 0$, the above inequality is equivalent to

$$\begin{aligned}&r^{n-1}\left[1-\frac{\left\{\frac{n}{1+s_1}(\|p\|-m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+s_1}(\|p\|-m)}\right]\frac{n}{1+s_1}(\|p\|-m)\\ \leq\ &r^{n-1}\left[1-\frac{\left\{\frac{n}{1+k^{\mu}}(\|p\|-m)-n|a_n|\right\}(r-1)}{n|a_n|+\frac{rn}{1+k^{\mu}}(\|p\|-m)}\right]\frac{n}{1+k^{\mu}}(\|p\|-m).\end{aligned} \tag{70}$$

Integrating both sides of (70) *with respect to r from* 1 *to R and making use of the right hand side of inequality* (4.2) *in the proof of Theorem* 1 *of the paper by Hussain* [10, Theorem 1]*, we have*

$$\begin{aligned}&\frac{R^n-1}{1+s_1}(\|p\|-m)-\frac{n}{1+s_1}(\|p\|-m)(1-c)\int_1^R\frac{(r-1)r^{n-1}}{r+c}dr\\ \leq\ &\frac{R^n-1}{1+k^{\mu}}(\|p\|-m)-\frac{n}{1+k^{\mu}}(\|p\|-m)(1-a)\int_1^R\frac{(r-1)r^{n-1}}{r+a}dr,\end{aligned} \tag{71}$$

where $c=\frac{|a_n|(1+s_1)}{\|p\|-m}$ and $a=\frac{|a_n|(1+k^{\mu})}{\|p\|-m}$.
We notice that the integral $\int_1^R\frac{(r-1)r^{N-1}}{r+a}dr$ occuring in the right hand side of (71) *is a non-negative and increasing function of N for $1 \leq N \leq n$, therefore, we have*

$$\int_1^R\frac{(r-1)r^{N-1}}{r+a}dr \leq \int_1^R\frac{(r-1)r^{n-1}}{r+a}dr. \tag{72}$$

Using (72) *to the right hand side of* (71) *and noting from Lemma 10 that* $1-a\geq 0$ *and applying Lemma 5 for the values of integrals involved in the resulting inequality, we have*

$$\begin{aligned} &\left(\frac{R^n-1}{1+s_1}\right)\|p\|-\left(\frac{R^n-1}{1+s_1}\right)m-n\left(\frac{\|p\|-m}{1+s_1}-|a_n|\right)R(n) \\ \leq\ & \left(\frac{R^n-1}{1+k^\mu}\right)\|p\|-\left(\frac{R^n-1}{1+k^\mu}\right)m-n\left(\frac{\|p\|-m}{1+k^\mu}-|a_n|\right)H(N). \end{aligned} \tag{73}$$

Adding $\|p\|$ *on both sides of* (73)*, we have*

$$\begin{aligned} &\left(\frac{R^n+s_1}{1+s_1}\right)\|p\|-\left(\frac{R^n-1}{1+s_1}\right)m-n\left(\frac{\|p\|-m}{1+s_1}-|a_n|\right)R(n) \\ \leq\ & \left(\frac{R^n+k^\mu}{1+k^\mu}\right)\|p\|-\left(\frac{R^n-1}{1+k^\mu}\right)m-n\left(\frac{\|p\|-m}{1+k^\mu}-|a_n|\right)H(N), \end{aligned} \tag{74}$$

where $R(n)$ *is as defined in* (66) *and* (67) *for* $N=n$ *and*

$$\begin{aligned} H(N) &= \left(R-1\right)-\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^\mu)|a_n|}\right\}\ for\ N=1, \end{aligned}$$

$$\begin{aligned} H(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}^{N-1} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^\mu)|a_n|}\right\}\ for\ N\geq 2. \end{aligned}$$

Since by Lemma 5, the function $R(N)$ *of Corollary 5 is sharpest when* $N=n$*, and by inequality* (74)*, this particular bound is sharper than the bound given by the following result.*

Corollary 8. *If* $p(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$*,* $1\leq\mu\leq n$*, is a polynomial of degree n having no zero in* $|z|<k$*,* $k\geq 1$*, then for* $R\geq 1$ *and any positive integer* N*,* $1\leq N\leq n$*,*

$$M(p,R)\leq\left(\frac{R^n+k^\mu}{1+k^\mu}\right)\|p\|-\frac{(R^n-1)m}{1+k^\mu}-n\left\{\frac{\|p\|-m}{1+k^\mu}-|a_n|\right\}H(N), \tag{75}$$

where

$$\begin{aligned} H(N) &= \left(R-1\right)-\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^\mu)|a_n|}\right\}\ for\ N=1, \end{aligned} \tag{76}$$

$$\begin{aligned} H(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}^{N-1} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^\mu)|a_n|}\right\} \; for\; N\geq 2, \end{aligned} \tag{77}$$

and $m=\min_{|z|=k}|p(z)|$.

Remark 13. *In Theorem* 1 *of the paper of Hussain* [10], *the value of* $H(N)$ *is*

$$\begin{aligned} H(N) &= \left(\frac{R^N-1}{N}\right)+\sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}^{v-1} \\ &+ (-1)^N\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}\left\{\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\}^{N-1} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^\mu)|a_n|}\right\} \; for\; N\geq 1. \end{aligned} \tag{78}$$

However, we cannot get the value of $H(1)$ *from inequality* (78). *This disadvantage has been resolved in Corollary 8 by giving seperately the value of* $H(N)$ *for* $N=1$. *Thus for* $N\geq 2$, *Corollary 8 gives the bound as given by the bound of Theorem* 1 *of Hussain* [10].

Remark 14. *By Lemma 5,* $H(1)\leq H(N)$. *Using this and Lemma 10 in Corollary 8, we have*

$$M(p,R)\leq\left(\frac{R^n+k^\mu}{1+k^\mu}\right)\|p\|-\frac{(R^n-1)m}{1+k^\mu}-n\left\{\frac{\|p\|-m}{1+k^\mu}-|a_n|\right\}H(1), \tag{79}$$

where $m=\min_{|z|=k}|p(z)|$ *and*

$$\begin{aligned} H(1) &= \left(R-1\right)-\left\{1+\frac{(1+k^\mu)|a_n|}{\|p\|-m}\right\} \\ &\times \ln\left\{1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+k^\mu)|a_n|}\right\}. \end{aligned} \tag{80}$$

Hence, when $N=1$, *Corollary 8 reduces to Theorem 1 which further deduces to inequality* (4) *for* $\mu=1$.

Remark 15. *It is easy to observe that for* $\mu=1$, *Corollary 8 gives a generalization of Theorem 2 due to Dalal and Govil* [2].

Remark 16. *For $\lambda = 0$ and using the fact of inequality* (68)*, Corollary 4 is a generalization of Theorem 2.*

Corollary 9. *If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $R \geq 1$ and any positive integer N, $1 \leq N \leq n$,*

$$M(p,R) \leq \left(\frac{R^n + s_0}{1+s_0}\right)\|p\| - n\left\{\frac{\|p\|}{1+s_0} - |a_n|\right\}\phi(N), \tag{81}$$

where s_0 is as defined in (68) *and*

$$\begin{aligned} \phi(N) &= \Big(R-1\Big) - \left\{1 + \frac{(1+s_0)|a_n|}{\|p\|}\right\} \\ &\times \ \ln\left\{1 + \frac{(R-1)\|p\|}{\|p\| + (1+s_0)|a_n|}\right\} \ for\ N = 1, \end{aligned} \tag{82}$$

$$\begin{aligned} \phi(N) &= \left(\frac{R^N - 1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{R^{N-v}-1}{N-v}\right)(-1)^v\left\{1 + \frac{(1+s_0)|a_n|}{\|p\|}\right\}\left\{\frac{(1+s_0)|a_n|}{\|p\|}\right\}^{v-1} \\ &+ \ (-1)^N\left\{1 + \frac{(1+s_0)|a_n|}{\|p\|}\right\}\left\{\frac{(1+s_0)|a_n|}{\|p\|}\right\}^{N-1} \\ &\times \ \ln\left\{1 + \frac{(R-1)\|p\|}{\|p\| + (1+s_0)|a_n|}\right\} \ for\ N \geq 2. \end{aligned} \tag{83}$$

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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