WOVEN K-G-FRAMES IN HILBERT C*-MODULES

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Abstract. The aim of this paper is to introduce woven K-g-frames in Hilbert C*-modules, to characterize them in term of atomic system for K, and to discuss the erasures and perturbations of weaving of K-g-frames in Hilbert C*-modules.

Keywords: K-g-frames; woven K-g-frames; C*-algebra; Hilbert C*-modules.

2010 AMS Subject Classification: 42C15, 46L06.

1. INTRODUCTION

As a generalization of bases in Hilbert spaces, frames were first introduced in 1952 by Duf-fin and Schaefer [2] in the study of nonharmonic fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields.

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Received October 18, 2021
The notion of weaving was recently proposed by Bemrose et al. [1] to simulate a question in distributed signal processing and wireless sensor networks.

$K-g-$frames, which are more general than ordinary $g-$frames, naturally have become one of the most active fields in frame theory in recent years. $K-g-$frames share many properties with $g-$frames, but they have their own properties, like the inversibility of frame operator of $K-g-$frames for more see [4, 6, 8, 9, 10, 11, 12].

Hilbert $C^*$-modules are generalization of Hilbert spaces in that they allow the inner product to take values in a $C^*$-algebra rather than the field of complex numbers. There are many differences between Hilbert $C^*$-modules and Hilbert spaces. For example, we know that any closed subspace in a Hilbert space has an orthogonal complement, but it is not true for Hilbert $C^*$-modules. And the Riesz representation theorem of continuous functionals in Hilbert $C^*$-modules is invalid in general.

In this paper, we introduce the weaving of $K-g$-frames in Hilbert $C^*$-modules, we will characterize them in term of atomic system for $K$, and we will discuss the erasures and perturbations of weaving of $K-g$-frames in Hilbert $C^*$-modules.

A frame in a separable Hilbert space $H$ is a sequence $\{x_i\}_{i \in I}$ for which there exist positive constants $A, B > 0$ such that:

$$A \|x\|^2 \leq \sum_{i \in I} |\langle x, x_i \rangle|^2 \leq B \|x\|^2,$$

for all $x \in H$. The constants $A, B$ are respectively called lower and upper bounds. If $A = B$, it is called a tight frame and it is said to be a normalized tight or Parseval frame if $A = B = 1$. The collection $\{x_i\}_{i \in I} \subset H$ is called Bessel if the above second inequality holds. In this case, $B$ is called the Bessel bound.

2. Background Material

Let $I$ and $J$ be finite or countable index sets and let $\mathbb{N}$ be the set of natural numbers. Throughout this paper, we assume that $\mathcal{U}$ and $\mathcal{V}$ are finitely or countably generated Hilbert $A$-modules, where $A$ is a complex $C^*$-algebra with the norm $\| \cdot \|_A$, and $\{ \mathcal{V}_i : i \in I \}$ is a sequence of closed Hilbert submodules of $\mathcal{V}$. $\text{End}_{A}^*(\mathcal{U}, \mathcal{V}_i)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{U}$ to $\mathcal{V}_i$ and $\text{End}_{A}^*(\mathcal{U}, \mathcal{U})$ is abbreviated for $\text{End}_{A}^*(\mathcal{U}, \mathcal{U})$. 


In this section, we recall the definitions of $g$-frames, $K-g$-frames in Hilbert $C^*$-modules and some lemmas which are needed later.

**Definition 2.1.** [5] A sequence $\{\Lambda_i \in \text{End}_A^*(U, V_i), i \in I\}$ is called a $g$-frame or a generalized frame in $U$ with respect to $\{V_i : i \in I\}$ if there exist constants $C; D > 0$ such that for every $f \in U$,

$$C\langle f, f \rangle \leq \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \leq D\langle f, f \rangle$$

**Definition 2.2.** [13] Let $K \in \text{End}_A^*(U)$, a sequence $\{\Lambda_i \in \text{End}_A^*(U, V_i), i \in I\}$ is called a $K-g$-frame if there exist constants $C; D > 0$ such that for every $f \in U$,

$$C\langle K^* f, K^* f \rangle \leq \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle \leq D\langle f, f \rangle$$

**Lemma 2.3.** [3] Let $U, V$ and $W$ be Hilbert $A$-modules, let $S \in \text{End}_A^*(W, V)$ and $T \in \text{End}_A^*(U, V)$ with $\mathcal{R}(T^*)$ orthogonally complemented. The following statements are equivalent.

(i) $SS^* \leq \lambda TT^*$ for some $\lambda > 0$

(ii) There exists $\mu > 0$ such that $\|S^* z\| \leq \mu \|T^* z\|, \quad \forall z \in V$

(iii) There exists a $D \in \text{End}_A^*(W, V)$ such that $S = TD$, i.e. $T X = S$ has a solution.

(iv) $\mathcal{R}(S) \subset \mathcal{R}(T)$

**Lemma 2.4.** [7] Let $U$ and $V$ be Hilbert $A$-modules over a $C^*$-algebra $A$, and let $T : U \longrightarrow V$ be a linear map. Then the following conditions are equivalent:

1. The operator $T$ is bounded and $A$-linear.

2. There exists $k \geq 0$ such that $\langle Tx, Tx \rangle \leq k \langle x, x \rangle$ for all $x \in U$.

One of the advantages of this equivalent definition of K-g-frames is that it is much easier to compare the norms of two elements than to compare two elements in $C^*$-algebras.

**Theorem 2.5.** Let $K \in \text{End}_A^*(U)$, a sequence $\{\Lambda_i \in \text{End}_A^*(U, V_i), i \in I\}$ is a K-g-frame if and only if there exists $0 < C; D < \infty$ such that:

$$C \|\langle K^* f, K^* f \rangle\| \leq \|\sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle\| \leq D \|\langle f, f \rangle\|$$

for every $f \in U$. 
Proof. ($\implies$) immediate.

($\impliedby$) Assume that there exist constants $0 < C; D < \infty$ such that for all $f \in \mathcal{U}$

$$C\|\langle K^* f, K^* f \rangle \| \leq \sum_{i \in I} \|\langle \Lambda_i f, \Lambda_i f \rangle \| \leq D \| f, f \|$$

Let $S$ the frame operator of the bessel $g$-sequence $\{\Lambda_i\}_{i \in I}$.

$S$ is a bounded positive self-adjoint operator, hence $S$ has a unique positive square root, denoted by $S_{\frac{1}{2}}$. then

$$\sqrt{C}\|K^* f\| \leq \|S_{\frac{1}{2}} f\| \leq \sqrt{D}\|f\|.$$ 

By lemma 2.4, we obtain

$$\langle S_{\frac{1}{2}} f, S_{\frac{1}{2}} f \rangle = \langle S f, f \rangle \leq B\langle f, f \rangle.$$

From $(i) \iff (ii)$ in lemma (2.3) there exist some $\lambda > 0$ such that:

$$KK^* \leq \lambda S_{\frac{1}{2}}(S_{\frac{1}{2}})^*.$$ 

Then

$$\frac{1}{\lambda} \langle K^* f, K^* f \rangle \leq \langle S f, f \rangle = \sum_{i \in I} \langle \Lambda_i f, \Lambda_i f \rangle, \forall f \in \mathcal{U}.$$ 

□

Lemma 2.6. Let $H$ be an Hilbert $A$-module, let $T, P, Q \in \text{End}_A^*(H)$ with $\overline{\mathcal{R}}(P^*)$ and $\overline{\mathcal{R}}(Q^*)$ are orthogonally complemented. The following statements are equivalent:

(i) $\mathcal{R}(T) \subset \mathcal{R}(P) + \mathcal{R}(Q)$

(ii) $TT^* \leq \lambda (PP^* + QQ^*)$ for some $\lambda > 0$

(iii) There exists $X, Y \in \text{End}_A^*(H)$ such that $T = PX + QY$.

3. WOVEN $K$-G-FRAMES

Definition 3.1. Two $K$-$g$-frames $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ for $\mathcal{U}$ are said to be woven $K$-$g$-frames if there exist universal positive constants $A$ and $B$ such that for any partition $\sigma$ of $I$, the family $\{\Lambda_i\}_{i \in \sigma} \cup \{\Gamma_i\}_{i \in \sigma^c}$ is a $K$-$g$-frame for $\mathcal{U}$ with lower and upper $K$-$g$-frame bounds $A$ and $B$, respectively, that is

$$A\langle K^* f, K^* f \rangle \leq \sum_{i \in \sigma} \langle \Lambda_i f, \Lambda_i f \rangle + \sum_{i \in \sigma^c} \langle \Gamma_i f, \Gamma_i f \rangle \leq B\langle f, f \rangle, \forall f \in \mathcal{U}.$$
Definition 3.2. A family of $K$-g-frames $\{\Lambda_j = \{\Lambda_{ij}\}_{i \in I}, \ j \in [m]\}$ for $\mathcal{U}$ are said to be woven $K$-g-frames if there exist universal positive constants $A$ and $B$ such that for any partition $\{\sigma_j\}_{j \in [m]}$ of $I$, the family $\{\Lambda_{ij}\}_{i \in \sigma_j, j \in [m]}$ is a $K$-g-frame for $\mathcal{U}$ with lower and upper $K$-g-frame bounds $A$ and $B$, respectively, that is

$$A\langle K^* f, (K^*)^* f \rangle \leq \sum_{j \in [m]} \sum_{i \in \sigma_j} \langle \Lambda_{ij} f, \Lambda_{ij} f \rangle \leq B\langle f, f \rangle, \ \forall f \in \mathcal{U}. $$

Suppose that $\{\Lambda_i\}_{i \in I}$ is a $K$-g-Bessel sequence for $\mathcal{U}$, then the synthesis operator of $\{\Lambda_i\}_{i \in I}$ is defined by $T_\Lambda : \bigoplus_{i \in I} \mathcal{V}_i \rightarrow \mathcal{U}$,

$$T_\Lambda(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* f_i, \ \forall \{f_i\}_{i \in I} \in \bigoplus_{i \in I} \mathcal{V}_i.$$

Its adjoint operator, which is called the analysis operator $T^*_\Lambda : \mathcal{U} \rightarrow \bigoplus_{i \in I} \mathcal{V}_i$,

$$T^*_\Lambda(f) = \{\Lambda_i f\}_{i \in I}, \ \forall f \in \mathcal{U}.$$

And the $K$-g-frame operator $S_\Lambda : \mathcal{U} \rightarrow \mathcal{U}$,

$$S_\Lambda f = T_\Lambda T^*_\Lambda f = \sum_{i \in I} \Lambda_i^* \Lambda_i f, \ \forall f \in \mathcal{U}. $$

For any partition $\{\sigma_j\}_{j \in [m]}$ of $I$, we define these operators,

$$T^*_\Lambda(\{f_i\}_{i \in I}) = \sum_{i \in \sigma_j} \Lambda_i^* f_i, \ \forall \{f_i\}_{i \in I} \in \bigoplus_{i \in I} \mathcal{V}_i, \ j \in [m], $$

$$(T^*_\Lambda)^*(f) = \{\Lambda_i f\}_{i \in \sigma_j}, \ \forall f \in \mathcal{U}, \ j \in [m], $$

$$S^*_\sigma_j f = T_\Lambda T^*_\Lambda f = \sum_{i \in \sigma_j} \Lambda_i^* \Lambda_i f, \ \forall f \in \mathcal{U}.$$ 

Theorem 3.3. Let $K \in \text{End}_A^\sigma(\mathcal{U})$, $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ be two $K$-g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$. Then for every partition $\sigma$ of $I$, $\Lambda$ and $\Gamma$ are woven $K$-g-frames for $\mathcal{U}$ with universal lower and upper $K$-g-frame bounds $A$ and $B$, respectively, if and only if

$$A\|\langle K^* f, (K^*)^* f \rangle\| \leq \|\sum_{i \in \sigma} \langle \Lambda_i f, \Lambda_i f \rangle + \sum_{i \in \sigma'} \langle \Gamma_i f, \Gamma_i f \rangle\| \leq B\|\langle f, f \rangle\|, \ \forall f \in \mathcal{U}. $$

Proof. It follows from Theorem (2.4) \qed
Proposition 3.4. Let $K \in L(U)$ and $\{\Lambda_{ij}\}_{i \in \sigma_j, j \in [m]}$ be a family of woven $K$-g-frames for $U$. Then the frame operator $S$ is self-adjoint, positive, bounded on $U$, and $KK^* \leq \lambda S$ for some $\lambda > 0$.

Proof. For every $f \in U$

$$Sf = \sum_{j \in [m]} \sum_{i \in \sigma_j} \Lambda_{i j}^* \Lambda_{i j} f$$

then

$$\langle Sf, f \rangle = \langle \sum_{j \in [m]} \sum_{i \in \sigma_j} \Lambda_{i j}^* \Lambda_{i j} f, f \rangle = \sum_{j \in [m]} \sum_{i \in \sigma_j} \langle \Lambda_{i j} f, \Lambda_{i j} f \rangle$$

then

$$A \langle K^* f, K^* f \rangle \leq \langle Sf, f \rangle \leq B \langle f, f \rangle$$

hence

$$AKK^* \leq S \leq BI.$$  

So, the frame operator $S$ is bounded and positive.

Therefore, $S^* = (TT^*)^* = TT^* = S$ then $S$ is self-adjoint. □

Proposition 3.5. Suppose for every $j \in [m]$: $\Lambda_j = \{\Lambda_{ij}\}_{i \in I}$ is a $g$-Bessel sequence for $U$ with bound $B_j$. Then every weaving $\{\Lambda_{ij}\}_{i \in \sigma_j, j \in [m]}$ is a $g$-Bessel sequence with bound $\sum_{j \in [m]} B_j$.

Proof.

$$\sum_{j \in [m]} \sum_{i \in \sigma_j} \langle \Lambda_{ij} f, \Lambda_{ij} f \rangle \leq \sum_{j=1}^{m} \sum_{i \in \sigma} \langle \Lambda_{ij} f, \Lambda_{ij} f \rangle \leq \sum_{j=1}^{m} B_j \langle f, f \rangle. \quad \square$$

The following theorem gives a characterization for weaving $K$-g-frames in term of atomic system for $K$

Definition 3.6. Let $K \in \text{End}_A^*(U)$, then the family $\{\Lambda_i \in \text{End}_A^*(U, \mathcal{V}_i), i \in I\}$ is called an atomic system for $K$, if the following conditions are satisfied

(i) The family $\{\Lambda_i \in \text{End}_A^*(U, \mathcal{V}_i), i \in I\}$ is a $g$-Bessel sequence,
(ii) For every \( f \in \mathcal{U} \), there exists \( f_i \in \bigoplus_{i \in I} \mathcal{V}_i \) such that \( \| \{ f_i \}_{i \in I} \| \leq C \| f \| \) for some \( C > 0 \) and \( K f = \sum_{i \in I} \Lambda_i^* f_i \).

**Theorem 3.7.** Let \( K \in \text{End}^*_A(\mathcal{U}) \), the families \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) be two \( K \)-g-frames for \( \mathcal{U} \).

The following statements are equivalent

(i) The families \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) are woven \( K \)-g-frames.

(ii) The family \( \{ \Lambda_i \}_{i \in \sigma} \cup \{ \Gamma_i \}_{i \in \sigma^c} \) is an atomic system for \( K \), where \( \sigma \) is any subset of \( I \).

**Proof.** \( i \implies ii \). Suppose that the families \( \{ \Lambda_i \}_{i \in I} \) and \( \{ \Gamma_i \}_{i \in I} \) are woven \( K \)-g-frames with bounds \( A \) and \( B \).

For every partition \([ \sigma, \sigma^c] \) of \( I \), we have

\[
A(\langle K^* f, \langle K^* f \rangle \rangle \leq \sum_{i \in \sigma} \langle \Lambda_i f, \Lambda_i f \rangle + \sum_{i \in \sigma^c} \langle \Gamma_i f, \Gamma_i f \rangle \leq B \langle f, f \rangle, \quad \forall f \in \mathcal{U}.
\]

Then the family \( \{ \Lambda_i \}_{i \in \sigma} \cup \{ \Gamma_i \}_{i \in \sigma^c} \) is g-Bessel sequence with bound \( B \).

On the other hand

\[
A(\langle K^* f, \langle K^* f \rangle \rangle \leq \langle S^\sigma_{\Lambda} f, f \rangle + \langle S^\sigma_{\Gamma} f, f \rangle
\]

This imply that

\[
AKK^* \leq T^\sigma_{\Lambda} (T^\sigma_{\Lambda})^* + T^\sigma_{\Gamma} (T^\sigma_{\Gamma})^*
\]

by lemma (2.6), there exist two bounded operators \( L_1, L_2 : \mathcal{U} \rightarrow \bigoplus_{i \in I} \mathcal{V}_i \) such that

\[
K f = T^\sigma_{\Lambda} L_1 f + T^\sigma_{\Gamma} L_2 f, \quad \forall f \in \mathcal{U}.
\]

Let \( L_1 f = \{ f_i \}_{i \in I} \in \bigoplus_{i \in I} \mathcal{V}_i \) and \( L_2 f = \{ g_i \}_{i \in I} \in \bigoplus_{i \in I} \mathcal{V}_i \), then

\[
K f = T^\sigma_{\Lambda} L_1 f + T^\sigma_{\Gamma} L_2 f
\]

\[
= T^\sigma_{\Lambda} \{ f_i \}_{i \in I} + T^\sigma_{\Gamma} \{ g_i \}_{i \in I}.
\]

\[
= \sum_{i \in \sigma} \Lambda_i^* f_i + \sum_{i \in \sigma^c} \Gamma_i^* g_i.
\]

and

\[
\| \{ f_i \}_{i \in I} \| = \| L_1 f \| \leq \| L_1 \| \| f \|
\]

\[
\| \{ g_i \}_{i \in I} \| = \| L_2 f \| \leq \| L_2 \| \| f \|.
\]

So \( \{ \Lambda_i \}_{i \in \sigma} \cup \{ \Gamma_i \}_{i \in \sigma^c} \) is an atomic system for \( K \).
\( ii) \implies i \). Suppose that the family \( \{\Lambda_i\}_{i \in \sigma} \cup \{\Gamma_i\}_{i \in \sigma^c} \) is an atomic system for \( K \), for any partition \([\sigma, \sigma^c]\) of \( I \), then the family \( \{\Lambda_i\}_{i \in \sigma} \cup \{\Gamma_i\}_{i \in \sigma^c} \) is a g-Bessel sequence for \( \mathcal{Y} \), then for any \( g \in \mathcal{Y} \), there exist \( \{f_i\}_{i \in I} \in \bigoplus_{i \in I} \mathcal{Y}_i \) such that

\[
K g = \sum_{i \in \sigma} \Lambda_i^* f_i + \sum_{i \in \sigma^c} \Gamma_i^* f_i
\]

where

\[
\|\{f_i\}_{i \in I}\| \leq C \|g\|.
\]

Then

\[
\|K^* f\|^2 = \sup_{g \in \mathcal{Y}, \|g\|=1} \|\langle K^* f, g \rangle\|
\]

\[
= \sup_{g \in \mathcal{Y}, \|g\|=1} \|\langle f, K g \rangle\|
\]

\[
= \sup_{g \in \mathcal{Y}, \|g\|=1} \|\langle f, \sum_{i \in \sigma} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f, f_i \rangle\|
\]

\[
= \sup_{g \in \mathcal{Y}, \|g\|=1} \|\sum_{i \in I} |\langle f_i, f_i \rangle| \sum_{i \in \sigma} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f, f_i \|.
\]

\[
\leq \sup_{g \in \mathcal{Y}, \|g\|=1} \sum_{i \in I} |\langle f_i, f_i \rangle| \|\sum_{i \in \sigma} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f, f_i \|.
\]

\[
\leq C \| \sum_{i \in \sigma^c} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f, f_i \|.
\]

\[
\leq \| \sum_{i \in \sigma^c} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f, f_i \| + \sum_{i \in \sigma^c} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f \|.
\]

Hence

\[
\frac{1}{C} |K^* f|^2 \leq \| \sum_{i \in \sigma^c} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f, f_i \| + \sum_{i \in \sigma^c} \Lambda_i f + \sum_{i \in \sigma^c} \Gamma_i f \|.
\]

Therefore, the family \( \{\Lambda_i\}_{i \in \sigma} \cup \{\Gamma_i\}_{i \in \sigma^c} \) is a g-Bessel sequence, then the families \( \{\Lambda_i\}_{i \in I} \) and \( \{\Gamma_i\}_{i \in I} \) are woven \( K \)-g-frames. \( \square \)

**Proposition 3.8.** Let \( \Lambda = \{\Lambda_i\}_{i \in I} \) and \( \Gamma = \{\Gamma_i\}_{i \in I} \) be two g-Bessel sequences in \( \mathcal{Y} \) with respect to \( \{\mathcal{Y}_i : i \in I\} \) with g-Bessel bounds \( B_1, B_2 \), respectively. If for \( J \subset I \); \( \Lambda_J = \{\Lambda_i\}_{i \in J} \) and \( \Gamma_J = \{\Gamma_i\}_{i \in J} \) are woven \( K \)-g-frames, then \( \Lambda \) and \( \Gamma \) are woven \( K \)-g-frames for \( \mathcal{Y} \).

**Proof.** Let \( A \) be universal lower bound for the woven \( K \)-g-frame \( \Lambda_J \) and \( \Gamma_J \), and let \( \sigma \subset I \) be a subset of \( I \). Then,
It follows that, 

\[ A(K^* f, K^* f) \leq \sum_{j \in \sigma \cup J} \langle \Lambda_j f, \Lambda_j f \rangle + \sum_{j \in \sigma \cap J} \langle \Gamma_j f, \Gamma_j f \rangle \]

\[ \leq \sum_{j \in \sigma} \langle \Lambda_j f, \Lambda_j f \rangle + \sum_{j \in \sigma^c} \langle \Gamma_j f, \Gamma_j f \rangle \]

\[ \leq (B_1 + B_2) \langle f, f \rangle. \]

Hence, \( \Lambda \) and \( \Gamma \) are woven K-g-frames for \( \mathcal{U} \).

Theorem 3.9. Let \( \Lambda = \{ \Lambda_i \}_{i \in I} \) and \( \Gamma = \{ \Gamma_i \}_{i \in I} \) be woven K-g-frames for \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_i : i \in I \} \) with universal K-g-frame bounds \( A \) and \( B \). If for all \( f \in \mathcal{U} \) \( \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq D \langle K^* f, K^* f \rangle \) for some \( 0 < D < A \) and some \( J \subset I \) Then \( \Lambda_0 = \{ \Lambda_i \}_{i \in I \setminus J} \) and \( \Gamma_0 = \{ \Gamma_i \}_{i \in I \setminus J} \) are woven K-g-frames for \( \mathcal{U} \) with universal K-g-frame bounds \( A - D \) and \( B \).

Proof. Let \( \sigma \) be a subset of \( I \setminus J \), then

\[ \sum_{j \in \sigma} \langle \Lambda_j f, \Lambda_j f \rangle + \sum_{j \in \sigma \setminus J} \langle \Gamma_j f, \Gamma_j f \rangle = (\sum_{j \in \sigma \cup J} \langle \Lambda_j f, \Lambda_j f \rangle - \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle) + \sum_{j \in \sigma \setminus J} \langle \Gamma_j f, \Gamma_j f \rangle. \]

\[ = (\sum_{j \in \sigma \cup J} \langle \Lambda_j f, \Lambda_j f \rangle + \sum_{j \in \sigma \setminus J} \langle \Gamma_j f, \Gamma_j f \rangle - \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle). \]

\[ \geq A \langle K^* f, K^* f \rangle - D \langle K^* f, K^* f \rangle \]

\[ = (A - D) \langle K^* f, K^* f \rangle, \quad \forall f \in \mathcal{U}. \]

And for the upper bound

\[ \sum_{j \in \sigma} \langle \Lambda_j f, \Lambda_j f \rangle + \sum_{j \in \sigma \setminus J} \langle \Gamma_j f, \Gamma_j f \rangle \leq \sum_{j \in \sigma \cup J} \langle \Lambda_j f, \Lambda_j f \rangle + \sum_{j \in \sigma \setminus J} \langle \Gamma_j f, \Gamma_j f \rangle \]

\[ \leq B \langle f, f \rangle. \]

It follows that, \( \Lambda_0 \) and \( \Gamma_0 \) are woven K-g-frames for \( \mathcal{U} \) with the universal lower and upper K-g-frame bounds \( A - D \) and \( B \), respectively.

Theorem 3.10. Let \( \Lambda = \{ \Lambda_i \}_{i \in I} \) and \( \Gamma = \{ \Gamma_i \}_{i \in I} \) be a pair of K-g-frames for \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_i : i \in I \} \) with universal K-g-frame bounds \( A_1, B_1 \) and \( A_2, B_2 \), respectively. Assume that there are constants \( 0 < \alpha, \beta, \mu < 1 \) such that

\[ \alpha \sqrt{B_1} + \beta \sqrt{B_2} + \mu < \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_1})} \]
and

\[
\left\| \sum_{i \in I} \langle (\Lambda_i^* - \Gamma_i^*) f_i, (\Lambda_i^* - \Gamma_i^*) f_i \rangle \right\| \leq \alpha \left\| \sum_{i \in I} \langle \Lambda_i^* f, \Lambda_i^* f \rangle \right\|^{1/2} + \beta \left\| \sum_{i \in I} \langle \Gamma_i^* f, \Gamma_i^* f \rangle \right\|^{1/2} + \mu \left\| \{ f_i \}, \{ f_i \} \right\|^{1/2}
\]

for all \( \{ f_i \} \in (\oplus \mathcal{Y}_i)_{i \in I} \). Then, \( \Lambda \) and \( \Gamma \) are woven \( K \)-g-frames with universal lower and upper frame bounds \( A_1 - \frac{A_1}{2} \| K^\dagger \| \) and \( B_1 + B_2 \), respectively.

**Proof.**

\[
\left\| \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f - \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\| = \left\| T_\sigma^0 (\{ \Lambda_i f \}_{i \in \sigma}) - T_\sigma^0 (\{ \Gamma_i f \}_{i \in \sigma}) \right\|
\]

\[
= \left\| T_\sigma^0 (T_\Lambda^0) f - T_\sigma^0 (T_\Gamma^0) f \right\|
\]

\[
\leq \left\| T_\sigma^0 (T_\Lambda^0) f - T_\Lambda^0 (T_\Gamma^0) f \right\| + \left\| T_\Lambda^0 (T_\Gamma^0) f - T_\Gamma^0 (T_\Gamma^0) f \right\|
\]

\[
\leq \left\| T_\Lambda^0 (T_\Gamma^0) f - T_\Lambda^0 (T_\Gamma^0) f \right\| + \left\| T_\Lambda^0 (T_\Gamma^0) f - T_\Gamma^0 (T_\Gamma^0) f \right\|
\]

\[
\leq \left\| T_\Lambda^0 \right\| \left\| T_\Lambda^0 - T_\Gamma^0 \right\| \| K^\dagger \| \| K^* f \| + \left\| T_\Lambda^0 - T_\Gamma^0 \right\| \| T_\Gamma^0 \| \| K^\dagger \| \| K^* f \|
\]

\[
\leq (\alpha \left\| T_\Lambda^0 \right\| + \beta \left\| T_\Gamma^0 \right\| + \mu \left( \left\| T_\Lambda^0 \right\| + \left\| T_\Gamma^0 \right\| \right)) \| K^\dagger \| \| K^* f \|
\]

\[
< \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_1})} (\sqrt{B_1} + \sqrt{B_1}) \| K^\dagger \| \| K^* f \|
\]

\[
= \frac{A_1}{2} \| K^\dagger \| \| K^* f \|
\]

On the other hand

\[
\left\| \sum_{i \in \sigma'} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\| = \left\| \sum_{i \in \sigma'} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f - \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\|
\]

\[
= \left\| \sum_{i \in I} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f - \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f \right\|
\]

\[
\geq \left\| \sum_{i \in I} \Lambda_i^* \Lambda_i f \right\| - \left\| \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f - \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\|
\]

\[
\geq A_1 \| K^* f \| - \frac{A_1}{2} \| K^\dagger \| \| K^* f \|
\]
\[ = (A_1 - \frac{A_1}{2} \|K^\dagger\|) \|K^*f\|. \]

So \((A_1 - \frac{A_1}{2} \|K^\dagger\|)\) is an universal lower bound, and one can see that \(B_1 + B_2\) is an universal upper bound. \(\square\)

**Theorem 3.11.** Let \(\Lambda = \{\Lambda_i\}_{i \in I}\) and \(\Gamma = \{\Gamma_i\}_{i \in I}\) be woven K-g-frames for \(\mathcal{U}\) with respect to \(\{\forall_i : i \in I\}\) with universal K-g-frame bounds \(A_1, B_1\) and \(A_2, B_2\), respectively. Assume that there are constants \(0 < \alpha, \beta, \mu < 1\) such that

\[
\alpha B_1 \|K^\dagger\| + \beta B_2 \|K^\dagger\| + \mu \|K^\dagger\| < A_1
\]

and

\[
\left\| \sum_{i \in \sigma} (\Lambda_i^* \Lambda_i - \Gamma_i^* \Gamma_i) f_i, (\Lambda_i^* \Lambda_i - \Gamma_i^* \Gamma_i) f_i \right\| \leq \alpha \left\| \sum_{i \in \sigma} (\Lambda_i^* \Lambda_i f_i, \Lambda_i^* \Lambda_i f_i) \right\|^\frac{1}{2} + \beta \left\| \sum_{i \in \sigma} (\Gamma_i^* \Gamma_i f_i, \Gamma_i^* \Gamma_i f_i) \right\|^\frac{1}{2} + \mu \left( \sum_{i \in \sigma} \|\Lambda_i f_i\| \right)^\frac{1}{2}
\]

for all \(f \in \mathcal{U}\) and \(\sigma \subset I\). Then, \(\Lambda\) and \(\Gamma\) are woven K-g-frames with universal lower and upper frame bounds \((A_1 - \alpha B_1 \|K^\dagger\| - \beta B_2 \|K^\dagger\| - \mu\) and \((B_1 + \alpha B_1 + \beta B_2 + \mu \sqrt{B_1})\), respectively.

**Proof.** For any \(\sigma \in I\), we have by hypothesis

\[
\left\| \sum_{i \in \sigma} (\Lambda_i^* \Lambda_i) \right\| \leq \alpha \left\| \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f_i \right\| + \beta \left\| \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f_i \right\| + \mu \left( \sum_{i \in \sigma} \|\Lambda_i f_i\| \right)^\frac{1}{2}
\]

then

\[
\left\| \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\| = \left\| \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f - \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\|
\]

\[
= \left\| \sum_{i \in I} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f - \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f \right\|.
\]

\[
\geq \left\| \sum_{i \in I} \Lambda_i^* \Lambda_i f \right\| - \left\| \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f \right\| - \left\| \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\|.
\]

\[
\geq A_1 \|K^*f\| - \alpha \left\| \sum_{i \in \sigma} \Lambda_i^* \Lambda_i f \right\| - \beta \left\| \sum_{i \in \sigma} \Gamma_i^* \Gamma_i f \right\| - \mu \left( \sum_{i \in \sigma} \|\Lambda_i f_i\| \right)^\frac{1}{2}.
\]

\[
\geq (A_1 - \alpha B_1 \|K^\dagger\| - \beta B_2 \|K^\dagger\| - \mu \|K^\dagger\|) \|K^*f\|
\]
On the other hand
\[
\| \sum_{i \in \sigma} \Lambda_i^\ast \Lambda_i f + \sum_{i \in I} \Gamma_i^\ast \Gamma_i f \| = \| \sum_{i \in \sigma} \Lambda_i^\ast \Lambda_i f + \sum_{i \in \sigma} \Gamma_i^\ast \Gamma_i f - \sum_{i \in \sigma} \Lambda_i^\ast \Lambda_i f \|.
\]
\[
\leq \| \sum_{i \in \sigma} \Lambda_i^\ast \Lambda_i f \| + \| \sum_{i \in \sigma} \Lambda_i^\ast \Lambda_i f - \sum_{i \in \sigma} \Gamma_i^\ast \Gamma_i f \|. 
\]
\[
\leq B_1 \|f\| + \alpha \| \sum_{i \in \sigma} \Lambda_i^\ast \Lambda_i f \| + \beta \| \sum_{i \in \sigma} \Gamma_i^\ast \Gamma_i f \| + \mu \left( \sum_{i \in \sigma} \| \Lambda_i f \| \right)^{\frac{1}{2}}. 
\]
\[
\leq (B_1 + \alpha B_1 + \beta B_2 + \mu \sqrt{B_1}) \|f\|.
\]

Then, \( \Lambda \) and \( \Gamma \) are woven \( k \)-g-frames with the universal lower and upper bounds \((A_1 - \alpha B_1 \|K^\dagger\| - \beta B_2 \|K^\dagger\| - \mu \|K^\dagger\|)\) and \((B_1 + \alpha B_1 + \beta B_2 + \mu \sqrt{B_1})\), respectively. \( \square \)

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.

**References**


