ON PAIRWISE-\(\omega\)-PERFECT FUNCTIONS

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Abstract. In this article, we will present a new form of functions called pairwise-\(\omega\)-- perfect functions and pairwise M-\(\omega\)--perfect functions. We will give some properties of this functions, and we will looking for homeomorphism of different bitopological spaces under the effect these functions. Last but not least, we give the characterizations of product theorems.

Keywords: bitopological spaces; \(\omega\)-open sets; pairwise perfect functions; pairwise-\(\omega\)-- perfect functions.

2010 AMS Subject Classification: 54E55, 54B10, 54D30.

1. INTRODUCTION AND PRELIMINARIES

Firstly, Kelly [10] established the bitopological spaces by generalised any characteristics in single topology into bitopological spaces. For examples for these topics, species of Hausdorff space, continuous functions, lindelöf, compactness, countably compact, normal, and others topics that we can't count it. In this research it will be an abbreviation of pairwise by p-, for example p-perfect functions, it is means pairwise perfect functions.

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Received October 20, 2021
If \((S, \eta_1, \eta_2)\) is a bitopological space and \(N \subseteq S, cl_1(N)\) and \(cl_2(N)\) will denote the closure of \(N\) with respect to \(\eta_1\) and \(\eta_2\) respectively. Let \((S, \eta)\) be a topological space and let \(N\) be a subset of \(S\). A point \(s \in (S, \eta_1, \eta_2)\) is called a condensation point of \(N\), if for each \(K \in \eta\) with \(s \in K\), the set \(K \cap N\) is uncountable. Hdeib presented \(\omega\)-closed sets and \(\omega\)-open sets as: \(N\) is called \(\omega\)-closed if it contains all its condensation points. The complement of an \(\omega\)-closed set is called \(\omega\)-open. Also \(cl^\omega N\) will denote the intersection of all \(\omega\)-closed sets which contains \(N\). The family of all \(\omega\)-open sets in \((S, \eta)\) is denoted by \(W(\eta)\). In [7] Datta defined \(p\)-closed functions, \(p\)-open sets, and in [8] Fletcher presented \(p\)-continuous functions, in addition of these in [9] Fora and Hdeib gived \(p\)-compact and \(p\)-lindelöf. Recently, A.Atoom and H.Z.Hdeib constructed the perfect functions in the bitopological spaces by a function \(\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)\) is called \(p\)-perfect, if \(\Omega\) is \(p\)-continuous, \(p\)-closed, and for each \(t \in T\), \(\Omega^{-1}(t)\) is \(p\)-compact. In this work, we will be presenting pairwise- \(\omega\)-perfect functions, and characterizations of pairwise- \(M\)- \(\omega\)- perfect functions.

2. Definitions and Results

**Definition 2.1.** A subset \(N\) of a bitopological space \((S, \eta_1, \eta_2)\) is pairwise- \(\omega\)-open, (simply \(p\)-\(\omega\)-open) if for each \(s \in N\) there exists a pairwise-\(\omega\) subset \(K_s\) containing \(s\) such that \(K_s \cap N\) is a countable set. The complement of a pairwise \(\omega\)-open is said to be pairwise-\(\omega\)-closed set (simply \(p\)-\(\omega\)-closed). The family of all pairwise \(\omega\)-open (respectively pairwise \(\omega\)-closed) subsets of a space \((S, \eta_1, \eta_2)\) is denoted by \(p\)-\(\omega\)-\(OP(S)\), (respectively \(p\)-\(\omega\)-\(CL(S)\)). Also the family of all pairwise- \(\omega\)-open sets of \((S, \eta_1, \eta_2)\) containing \(s\) is denoted by \(p\)-\(\omega\)-\(OP(S,s)\).

**Definition 2.2.** A function \(\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)\) is a pairwise- \(\omega\)-closed function, if it functions pairwise-\(\omega\) closed sets onto pairwise-\(\omega\)-closed sets.

**Definition 2.3.** A function \(\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)\) is \(p\)-weakly continuous function if for every \(p\)-open set \(K \subseteq T\), \(\Omega^{-1}(K)\) is \(p\)-\(\omega\)-open.

**Definition 2.4.** A function \(\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)\) is \(p\)-strongly \(\omega\)-continuous function if for every \(p\)-\(\omega\)-open set \(K \subseteq T\), \(\Omega^{-1}(K)\) is \(p\)-open.
Definition 2.5. A function \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) is called \( p-\omega \)-continuous at point \( s \in (S, \eta_1, \eta_2) \), if for every \( p \)-open set \( L \) containing \( \Omega(s) \), there is \( p-\omega \)-open set \( K \) containing \( s \) such that \( \Omega(K) \subset L \). If \( \Omega \) is \( p-\omega \)-continuous at each point of \((S, \eta_1, \eta_2)\), then \( \Omega \) is said to be \( p-\omega \)-continuous on \((S, \eta_1, \eta_2)\).

Definition 2.6. A function \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) is called \( p-\omega \)-continuous (resp. \( p-\omega \)-irresolute) if, \( \Omega_1 : (S, \eta_1) \to (T, \gamma_1) \) and \( \Omega_2 : (S, \eta_2) \to (T, \gamma_2) \) are \( \omega \)-continuous (resp. \( \omega \)-irresolute) functions.

Definition 2.7. A family \( \hat{\mathcal{N}} \) of subsets of a bitopological space \((S, \eta_1, \eta_2)\) is called \( \eta_1 \eta_2-\omega \)-open if \( \hat{\mathcal{N}} \subset W(\eta_1) \cup W(\eta_2) \). If, in addition \( \hat{\mathcal{N}} \cap W(\eta_1) \neq \emptyset \) and \( \hat{\mathcal{N}} \cap W(\eta_2) \neq \emptyset \) then \( \hat{\mathcal{N}} \) is called pairwise \( \omega \)-open.

Definition 2.8. A bitopological space \((S, \eta_1, \eta_2)\) is said to be pairwise-\( -\omega \)-compact, (resp. pairwise \( M-\omega \)-compact) if each \( p \)-\( \omega \)-open (resp. \( \eta_1 \eta_2-\omega \)-open) cover of \( S \) has a finite sub-cover. Clearly every \( p.M-\omega.c. \) space is \( p.\omega.c. \), and we can easily show that the converse may not be true.

Definition 2.9. A space \((S, \eta_1, \eta_2)\) is said to be \( p-\omega \)-lindelöf if every \( p-\omega \)-open cover of \((S, \eta_1, \eta_2)\) has a countable subcover.

Definition 2.10. A function \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) is said to be \( p \)-weakly continuous function if for every \( p \)-open set \( K \subset T \), \( \Omega^{-1}(K) \) is \( p-\omega \)-open.

Definition 2.11. A space \((S, \eta_1, \eta_2)\) is said to be \( p-\omega-I_1 \) if for each pair of distinct points \( s \) and \( t \) of \((S, \eta_1, \eta_2)\), there exist \( p-\omega \)-open sets \( K \) and \( L \) containing \( s \) and \( t \), respectively such that \( t \notin K \), and \( s \notin L \).

Definition 2.12. A space \((S, \eta_1, \eta_2)\) is said to be \( p-\omega-I_2 \) if for each pair of distinct points \( s \) and \( t \) of \((S, \eta_1, \eta_2)\), there exist \( p-\omega \)-open sets \( K \) and \( L \) in \((S, \eta_1, \eta_2)\) such that \( s \in K \) and \( t \in L \).

3. MAIN RESULTS IN PAIRWISE-\( \omega \)-PERFECT FUNCTIONS

Definition 3.1. A function \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) is pairwise-\( -\omega \)-perfect, if \( \Omega \) is pairwise-\( -\omega \)-continuous, pairwise-\( -\omega \)-closed, and for each \( t \in T \), \( \Omega^{-1}(t) \) is pairwise-\( -\omega \)-compact.
**Definition 3.2.** A function $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ is called pairwise $-\omega-M$-perfect, if $\Omega$ is pairwise $-\omega-$continuous, pairwise $-\omega-$closed, and for each $t \in T$, $\Omega^{-1}(t)$ is pairwise $M-\omega-$compact.

**Theorem 3.3.** If $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ is a pairwise $-\omega-$perfect function, then for every pairwise $-\omega-$compact subset $Q \subseteq T$, the inverse image $\Omega^{-1}(Q)$ is a pairwise $-\omega-$compact.

**Proof.** Let $K = \{K_\theta : \theta \in \Psi\}$ be a $p$-open cover of $(S, \eta_1, \eta_2)$, because $\Omega$ is a pairwise $-\omega-$perfect function, then $\forall t \in T$, $\Omega^{-1}(t)$ is pairwise $-\omega-$compact, there exists a finite sub-sets $\Psi_t$, $\Psi^*_t$ of $\Psi$, s.t $\Omega^{-1}(t) \subseteq \bigcup \{L_\theta : \theta \in \Psi_t\} \cup \bigcup \{E_\theta : \theta \in \Psi^*_t\}$, where $\{L_\theta : \theta \in \Psi_t\}$ is a pairwise $-\omega-$open $\{E_\theta : \theta \in \Psi^*_t\}$ is pairwise $-\omega-$open. Let $D_t = T - \Omega(S - \bigcup_{\theta \in \Psi_t} L_\theta)$ is a $\gamma_1$-open set containing $t$, and $D^*_t = T - \Omega(S - \bigcup_{\theta \in \Psi^*_t} E_\theta)$ is a $\gamma_2$-open set containing $t$, where $\Omega^{-1}(D_t) \subseteq \bigcup_{\alpha \in \Psi_t} L_\alpha$, $\Omega^{-1}(D^*_t) \subseteq \bigcup_{\alpha \in \Psi^*_t} E_\alpha$. Let $D = \{D_t : t \in T\} \bigcup \{D^*_t : t \in T\}$ is a pairwise $-\omega-$open cover of $T$. $D$ is pairwise $-\omega-$open cover of $Q$. Since $Q$ is pairwise $-\omega-$compact, $Q \subseteq \bigcup_{i=1}^n (D_{t_i}) \bigcup_{j=1}^m (D^*_{t_j})$. Thus, $\Omega^{-1}(Q) \subseteq \bigcup_{i=1}^n \Omega^{-1}(D_{t_i}) \bigcup_{j=1}^m \Omega^{-1}(D^*_{t_j}) \subseteq \text{union of finite of } K$, i.e $\Omega^{-1}(Q)$ is pairwise $-\omega-$compact. \hfill $\square$

**Corollary 3.4.** A pairwise $-\omega-$compact space is inverse invariant under pairwise $-\omega-$perfect function.

**Theorem 3.5.** If $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ is a pairwise $M-\omega-$perfect function, then for every pairwise $-\omega-$compact subset $Q \subseteq T$, the inverse image $\Omega^{-1}(Q)$ is a pairwise $M-\omega-$compact.

**Proof.** We will use the same technique in theorem [2.8]. \hfill $\square$

**Corollary 3.6.** A pairwise $M-\omega-$compact space is constant algebraic expression under pairwise $M-\omega-$perfect function.
Theorem 3.7. If $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ is pair-wise $-\omega-$perfect function and $\Theta : (T, \gamma_1, \gamma_2) \to (Q, \mu_1, \mu_2)$ is pair-wise perfect function, $\Theta \circ \Omega$ is pair-wise $-\omega-$perfect function.

Proof. Suppose $N$ be any $-\omega-$ $\mu_1-$ open set in $Q$, since $\Theta$ is pair-wise $-\omega-$ perfect function, then $\Theta^{-1}(N)$ is $\gamma_1-$ open set in $(T, \gamma_1, \gamma_2)$. □

Because $\Omega$ is pair-wise perfect function, then $\Omega^{-1}(\Theta^{-1}(N))$ is $\eta_1-$ open set in $S$. The same thing, let $G$ be any $-\omega-$ $\mu_2-$ open set in $Q$. Hence $\Theta \circ \Omega$ is pair-wise $-\omega-$ perfect function.

Corollary 3.8. If $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ is pair-wise $-\omega-$semi perfect function and $\Theta : (T, \gamma_1, \gamma_2) \to (Q, \mu_1, \mu_2)$ is pair-wise perfect function $\Theta \circ \Omega$ is pair-wise $M-\omega-$perfect function.

Proposition 3.9. If the composition $\Theta \circ \Omega$ of the pair-wise $-\omega-$continuous function, $\Omega : (S, \eta_1, \eta_2) \xrightarrow{onto} (T, \gamma_1, \gamma_2)$, and pair-wise continuous $\Theta : (T, \gamma_1, \gamma_2) \xrightarrow{onto} (Q, \mu_1, \mu_2)$ is a pair-wise $-\omega-$ closed, then the function $\Theta : (T, \gamma_1, \gamma_2) \xrightarrow{onto} (Q, \mu_1, \mu_2)$ is pair-wise $-\omega-$ closed.

Proof. Let $N$ be a $\gamma_1-$ $-\omega-$ closed in $T$, then $\Omega^{-1}(N)$ is $\eta_1-$ $-\omega-$ closed in $S$. Since $\Theta \circ \Omega$ is pair-wise $-\omega-$ closed, then $\Theta(\Omega^{-1}(N))$ is $\rho_1-$ $-\omega-$ closed in $Q$, i.e $\Theta(N)$ is $\mu_1-$ $-\omega-$ closed in $Q$. Similarly, we can show that if $G$ be a $\gamma_2-$ $-\omega-$ closed in $T$, then $\Theta(G)$ is $\gamma_2-$ $-\omega-$ closed in $Q$. Thus $\Theta$ is a pair-wise $-\omega-$ closed function. □

Theorem 3.10. If the composition $\Theta \circ \Omega$ of the pair-wise $-\omega-$continuous function,

$\Omega : (S, \eta_1, \eta_2) \xrightarrow{onto} (T, \gamma_1, \gamma_2)$, and pair-wise continuous $\Theta : (T, \gamma_1, \gamma_2) \xrightarrow{onto} (Q, \mu_1, \mu_2)$ is pair-wise $-\omega-$ perfect,

then the function $\Theta : (T, \gamma_1, \gamma_2) \xrightarrow{onto} (Q, \mu_1, \mu_2)$ is pair-wise $-\omega-$ perfect.

Proof. For every $q \in Q$, $\Theta^{-1}(q) = \Omega ((\Theta \circ \Omega)^{-1}(q))$ is pair-wise $-\omega-$ compact, because $\Theta \circ \Omega$ is pair-wise $-\omega-$ perfect. Since $\Theta$ is pair-wise $-\omega-$ closed by previous proposition, we get that $\Theta$ is pair-wise $-\omega-$ perfect. □

Theorem 3.11. If $\Omega : (S, \eta_1, \eta_2) \xrightarrow{onto} (T, \gamma_1, \gamma_2)$ is pair-wise $-\omega-$closed function, then for any $G \subset T$ the restriction $\Omega_B : \Omega^{-1}(G) \to G$ is pair-wise $-\omega-$closed.
Proof. Let $G \subset T$. Consider the function $\Omega : (S, \eta_1) \to (T, \gamma_1)$, let $S$ be a $\eta_1 - \omega$-closed. Then $\Omega_G (S \cap \Omega^{-1}(G)) = \Omega(S) \cap G$ is $\gamma_1 - \omega$-closed in $G$. \hfill $\square$

The same thing, we can show that if $S$ a $\gamma_2 - \omega$-closed, $\Omega_G (S \cap \Omega^{-1}(G)) = \Omega(S) \cap G$ is $\sigma_2 - \omega$-closed in $G$. Thus $\Omega_B : \Omega^{-1}(G) \to G$ is pairwise $- \omega$-closed.

**Theorem 3.12.** If $\Omega : (S, \eta_1, \eta_2) \overset{onto}{\to} (T, \gamma_1, \gamma_2)$ is pairwise $- \omega$-perfect function, then for any $G \subset T$ the restriction $f_B : \Omega^{-1}(B) \to B$ is pairwise $- \omega$-perfect.

Proof. We will use the same technique in the above theorem. \hfill $\square$

**Theorem 3.13.** A bitopological space $(S, \eta_1, \eta_2)$ is $p.- \omega- c.$ if and only if each proper $\eta_r - \omega$-closed subset of $(S, \eta_1, \eta_2)$ is $\omega$-compact relative to $(S, \eta_p)$, where $r, p = 1, 2, r \neq p$.

Proof. The proof comes from last theorem. \hfill $\square$

**Theorem 3.14.** If $\Omega : (S, \eta_1, \eta_2) \overset{onto}{\to} (T, \gamma_1, \gamma_2)$ is pairwise $- \omega$-perfect, where $(S, \eta_1, \eta_2)$ is pairwise $- \omega$-compact, and $(T, \gamma_1, \gamma_2)$ is pairwise $- \omega$-Hausdorff, then $\Omega$ is pairwise $- \omega$-closed.

Proof. If $N$ is $\eta_1 - \omega$-closed subset of $(S, \eta_1, \eta_2)$, then it is $\eta_2 - \omega$-compact, because $(S, \eta_1, \eta_2)$ is pairwise $- \omega$-compact. Since $\Omega$ is pairwise $- \omega$-continuous, $\Omega(N)$ is a $\gamma_2 - \omega$-compact subset of $(T, \gamma_1, \gamma_2)$. Since $(T, \gamma_1, \gamma_2)$ is pairwise $- \omega$-Hausdorff, then $\Omega(N)$ is a $\gamma_1 - \omega$-closed. Simillary if $B$ is a $\eta_2 - \omega$-closed subset of $S$, then $\Omega(G)$ is a $\gamma_2 - \omega$-closed subset of $(T, \gamma_1, \gamma_2)$. \hfill $\square$

**Corollary 3.15.** If $\Omega : (S, \eta_1, \eta_2) \overset{onto}{\to} (T, \gamma_1, \gamma_2)$ is pairwise $M- \omega$-perfect, where $(S, \eta_1, \eta_2)$ is pairwise $M- \omega$-compact, and $(T, \gamma_1, \gamma_2)$ is pairwise $- \omega$-Hausdorff, then $\Omega$ is pairwise $- \omega$-closed.

**Definition 3.16.** A function $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ is called pairwise $- \omega$-homeomorphism, if $\Omega$ is pairwise continuous, pairwise $- \omega$-closed (pairwise $- \omega$-open), and $\Omega$ is bijection.
Theorem 3.17. Let $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ be a $p$-continuous bijection function. If $(T, \gamma_1, \gamma_2)$ is pairwise $-\omega-$ Hausdorff space, and $(S, \eta_1, \eta_2)$ is pairwise $-\omega-$ compact, then $\Omega$ is pairwise $-\omega-$ homeomorphism function.

Proof. This is enough to prove that $\Omega$ is pairwise $-\omega-$ closed. Let $H$ be a $\eta_r-$closed proper subset of $S$, and hence $H$ is proper $\eta_p - \omega-$ compact, for $r, p = 1, 2; r \neq p,$, by using theorem[3.13], and so, $\Omega(H)$ is a $\gamma_p - \omega-$ compact, but $(T, \gamma_1, \gamma_2)$ is pairwise $-\omega-$ Hausdorff space, $\Omega(H)$ is $\gamma_r - \omega-$ closed. Hence, $\Omega$ is pairwise $-\omega-$ homeomorphism function. □

Definition 3.18. A function $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ is called pairwise $-\omega-$ strongly function (pairwise $-\omega-$ weakly function), if for every pairwise $-\omega-$ open cover $K = \{K_\theta: \theta \in \Psi\}$, there exists pairwise $-\omega-$ open cover $L = \{L_\theta: \theta \in \Psi_1\}$ of $T$, s.t $\Omega^{-1}(L) \subseteq \bigcup\{K_\theta: \theta \in \Psi_1, \Psi_1 \subset \Psi, \text{finite}\}, \forall L_\theta \in L.$

Theorem 3.19. Let $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ be a pairwise $-\omega-$ strongly onto function, then $(S, \eta_1, \eta_2)$ is pairwise $-\omega-$ compact, if $(T, \gamma_1, \gamma_2)$ is so.

Proof. Suppose $K = \{K_\theta: \theta \in \Psi\}$ be a pairwise $-\omega-$ open cover $(S, \eta_1, \eta_2)$. Because $\Omega$ is pairwise $-\omega-$ strongly function, there exists pairwise open cover $L = \{L_\theta: \theta \in \Psi_1\}$ of $(T, \gamma_1, \gamma_2)$, such that $\Omega^{-1}(L) \subseteq \bigcup\{K_\theta: \theta \in \Psi_1, \Psi_1 \subset \Psi, \text{finite}\}, \forall L_\theta \in L,$ but $(T, \gamma_1, \gamma_2)$ is pairwise $-\omega-$ compact, so there exists $\Psi_1 \subset \Psi$, where $\Psi_1$ is finite, such that, $T = \bigcup_{\theta \in \Psi_1} L_\theta$ and so, $S = \bigcup\Omega^{-1}(L_\theta)$. Each $\Omega^{-1}(L_\theta)$ contains of finite members of $K$, thus $S$ is pairwise $-\omega-$ compact. □

Definition 3.20. If $K$ and $F$ are pairwise $-\omega-$ open covers of the bitopological space $(S, \eta_1, \eta_2)$, then $K$ is called a parallel refinement of $F$, if each $K \in K \cap W(\eta_r)$ is contained in some $F \in F \cap W(\eta_r), r = 1, 2$.

Definition 3.21. If $K$ and $F$ are pairwise $\eta_1 \eta_2 - \omega-$ open covers of the bitopological space $(S, \eta_1, \eta_2)$, then $K$ is called a parallel refinement of $F$, if each $K \in K \cap W(\eta_r)$ is contained in some $F \in F \cap W(\eta_r), r = 1, 2$. 
Definition 3.22. A family $N$ of subsets of a space $(S, \eta_1, \eta_2)$ is locally finite in $(S, W(\eta))$ if for each $s \in S$ there exists a $\omega$-open set $K$ such that $s \in K$ and $K$ intersects at most finitely many elements of $N$.

Definition 3.23. A bitopological space $(S, \eta_1, \eta_2)$ is called pairwise $M-\omega$-paracompact, if each pairwise $-\omega$-open cover of $S$ has a pairwise $-\omega$-locally finite $\eta_1\eta_2 - \omega$-open refinement.

Definition 3.24. A bitopological space $(S, \eta_1, \eta_2)$ is called pairwise $-\omega$-paracompact, if each pairwise $-\omega$-open cover of $S$ has a pairwise $-\omega$-locally finite pairwise $-\omega$-open refinement.

Theorem 3.25. Let $\Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2)$ be a pairwise $-\omega$-perfect function, and $(T, \gamma_1, \gamma_2)$ is a pairwise $M-\omega$-paracompact, then $(S, \eta_1, \eta_2)$ is so.

Proof. Suppose $K = \{K_\theta: \theta \in \Psi\}$ be a pairwise $-\omega$-open cover of $(S, \eta_1, \eta_2)$, because $\Omega$ is a pairwise $-\omega$-perfect function, then $\forall t \in T$, $\Omega^{-1}(t)$ is pairwise $-\omega$-compact. Suppose $K = \{K_\theta: \theta \in \Psi\}$ be a $\rho$-open cover of $(S, \eta_1, \eta_2)$, since $\Omega$ is a pairwise $-\omega$-perfect function, then $\forall t \in T$, $\Omega^{-1}(t)$ is pairwise $-\omega$-compact, $\exists$ a finite subsets $\Psi_t, \Psi^*_t$ of $\Psi$ s.t $\Omega^{-1}(t) \subseteq \bigcup_{\theta \in \Psi_t} \{L_\theta: \theta \in \Psi_t\}$, where $\{L_\theta: \theta \in \Psi_t\}$ is $\eta_1-\omega$-open, $\{E_\theta: \theta \in \Psi^*_t\}$ is $\eta_2-\omega$-open. Let $D_t = T - \Omega(S - \bigcup_{\theta \in \Psi_t} L_\theta)$ is a $\gamma_1 - \omega$-open set containing $t$, and $D^*_t = T - \Omega(S - \bigcup_{\theta \in \Psi^*_t} E_\theta)$ is a $\gamma_2 - \omega$-open set containing $t$, where $\Omega^{-1}(D_t) \subseteq \bigcup_{\alpha \in \Psi^*_t} L_\theta$, $\Omega^{-1}(D^*_t) \subseteq \bigcup_{\alpha \in \Psi^*_t} E_\theta$. Let $D = \{D_t: t \in T\} \bigcup \{D^*_t: t \in T\}$ is a pairwise $-\omega$-open cover of $T$. Since $(T, \gamma_1, \gamma_2)$ is pairwise $M-\omega$-paracompact, $D$ has a pairwise locally finite $\eta_1\eta_2 - \omega$-open, refinement. say: $\square$

$I = \{I_2: Z \in \Xi_1\} \bigcup \{I^*_2: Z \in \Xi_2\}$, where $\{I_2: Z \in \Xi_1\}$ is $\eta_1 - \omega$-locally finite paracompact of $D_t$, and $\{I^*_2: Z \in \Xi_2\}$ is $\eta_2 - \omega$-locally finite paracompact of $D^*_t$, $\Xi = \Xi_1 \bigcup \Xi_2$. Let $J_1 = \{\Omega^{-1}(I_2): r = 1, 2, \ldots, n, Z \in \Xi_1, \theta \in \Psi_t\}$ is $\eta_1 - \omega$-open locally finite refinement of $\{L_\theta: \theta \in \Psi_t\}$, and let $J_2 = \{f^{-1}(I^*_2): r = 1, 2, \ldots, n, Z \in \Xi_2, \theta \in \Psi^*_t\}$ is $\eta_2 - \omega$-open locally finite refinement of $\{E_\theta: \theta \in \Psi^*_t\}$. Since $\Omega$ is a pairwise $-\omega$-perfect function, then $\forall t \in T$, $\Omega^{-1}(t)$ is pairwise $-\omega$-compact, $\exists$ a finite subsets $\Psi_t, \Psi^*_t$ of $\Psi$ s.t $\Omega^{-1}(t) \subseteq \bigcup_{\theta \in \Psi_t} \{L_\theta: \theta \in \Psi_t\}$, where $\{L_\theta: \theta \in \Psi_t\}$ is $\eta_1-\omega$-open, $\{E_\theta: \theta \in \Psi^*_t\}$ is $\eta_2-\omega$-open. Let $D_t = T - \Omega(S - \bigcup_{\theta \in \Psi_t} L_\theta)$ is a $\gamma_1 - \omega$-open set containing $t$, and $D^*_t = T - \Omega(S - \bigcup_{\theta \in \Psi^*_t} E_\theta)$ is a $\gamma_2 - \omega$-open set containing $t$, where $\Omega^{-1}(D_t) \subseteq \bigcup_{\alpha \in \Psi^*_t} L_\theta$, $\Omega^{-1}(D^*_t) \subseteq \bigcup_{\alpha \in \Psi^*_t} E_\theta$. Let $D = \{D_t: t \in T\} \bigcup \{D^*_t: t \in T\}$ is a pairwise $-\omega$-open cover of $T$. Since $(T, \gamma_1, \gamma_2)$ is pairwise $M-\omega$-paracompact, $D$ has a pairwise locally finite $\eta_1\eta_2 - \omega$-open, refinement. say: $\square$
open locally finite refinement of \( \{ E_\theta : \theta \in \Psi_1 \} \). Let \( I = \{ I_1 \cup I_2 \} \), then \( I \) is pairwise \(-\omega-\) locally finite \( \eta_1 \eta_2 \omega-\) open refinement \( U \). Hence \((S, \eta_1, \eta_2)\) is a pairwise \( M-\omega-\) paracompact space.

**Corollary 3.26.** Let \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) be a pairwise \(-\omega-\) perfect function, and \((T, \gamma_1, \gamma_2)\) is a pairwise \(-\omega-\) paracompact, then \((S, \eta_1, \eta_2)\) is so.

**Theorem 3.27.** The pairwise \(-\omega-\) Hausdorff space is constant algebraic expression under pairwise \(-\omega-\) perfect.

**Proof.** Let \((S, \eta_1, \eta_2)\) be a pairwise \(-\omega-\) Hausdorff space, \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) be a pairwise \(-\omega-\) perfect function, and \( t_1 \neq t_2 \) in \((T, \gamma_1, \gamma_2)\), then \( \Omega^{-1}(t_1), \Omega^{-1}(t_2) \) are disjoint and pairwise \(-\omega-\) compact subset of \((S, \eta_1, \eta_2)\). Since \((S, \eta_1, \eta_2)\) be a \( p-\) Hausdorff space , there exists a \( \eta_1 \) neighborhood \( K \) of \( S \), and \( \eta_2 \) neighborhood \( L \), such that \( \Omega^{-1}(t) \subseteq K, \Omega^{-1}(t_2) \subseteq L \). Let the sets \( T - \Omega(S - K) \) be \( \gamma_1 - \omega- \) open set in \((T, \gamma_1, \gamma_2)\) and containing \( t_1 \), \( T - \Omega(S - L) \) be \( \gamma_2 - \omega- \) open set in \((T, \gamma_1, \gamma_2)\) and containing \( t_2 \), s.t \( (T - \Omega(S - K) \cap T - \Omega(S - L) = T - (\Omega(S - L) \cup \Omega(S - L)) = Y - f(X - U \bigcap V) = T - \Omega(S) = \phi \). Hence \((T, \gamma_1, \gamma_2)\) is pairwise \(-\omega-\) Hausdorff space.

**Remark 3.28.** The pairwise \(-\omega-\) Hausdorff space is constant algebraic expression and inverse constant algebraic expression under pairwise \( M-\omega-\) perfect.

**Lemma 3.29.** In a bitopological space \((S, \eta_1, \eta_2)\), \( W(\eta_1) \) is said to be \( \omega-\) regular with respect to \( W(\eta_2) \) if, for each point \( s \) in \( S \) and each \( \eta_1 - \omega- \) closed set \( C \) such that \( s \notin C \), there are a \( \eta_1 - \omega- \) open set \( K \) and a \( \eta_2 - \omega- \) open set \( L \) such that \( s \in K, C \subseteq L \) and \( K \cap L = \phi \). \((S, \eta_1, \eta_2)\) is \( p-\omega-\) regular if \( W(\eta_1) \) is \( \omega-\) regular with respect to \( W(\eta_2) \). Let \( S \) be a pairwise \(-\omega-\) regular space, and \( N \) be \( \eta_r - \omega- \) compact subset of \( S, r = 1, 2 \), then for each \( \tau_r - \omega- \) neighbourhood \( K \) of \( N \), there exists a \( \eta_r - \omega- \) open \( P \), such that \( N \subset P \subset Cl \eta_r(P) \subset U, r, \varepsilon = 1, 2, \) \( r \notin \varepsilon \).
Proof. For each \( n \in N \), there exist a \( \eta_r - \omega - \) neighbourhood \( V(n) \) such that \( Cl \eta_e L(n) \subset K \), so \( N \subset \bigcup_{x=1}^{n} L(n_x) \subset Cl \eta_e \bigcup_{x=1}^{n} L(n_x) \). Let \( P = \bigcup_{x=1}^{n} L(n_x) \), then \( P \) is \( \eta_r - \omega - \) open, but \( Cl \eta_e P = Cl \eta_e \bigcup_{x=1}^{n} L(n_x) = Cl \eta_e \eta \), hence \( N \subset P \subset Cl \eta_e (P) \subset K \), \( r, \epsilon = 1, 2, r \notin \epsilon \).

**Theorem 3.30.** Let \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) be a pairwise \(-\omega-\)perfect function, and \( (S, \eta_1, \eta_2) \) is a pairwise \(-\omega-\)regular, then \( (T, \gamma_1, \gamma_2) \) is so.

Proof. Given \( \gamma_r - \omega - \) open set \( L, t \in L, r, \epsilon = 1, 2, r \notin \epsilon, \Omega^{-1}(t) \in \Omega^{-1}(L) \) in \( T \), since \( S \) is pairwise \(-\omega-\) regular, there exists \( \eta_r - \omega - \) open set \( K \), (by using Lemma 2.52), such that \( \Omega^{-1}(t) \in Cl \eta_e \bigcup_{x=1}^{n} K \subset \Omega^{-1}(L) \). Since \( \Omega \) is \( \eta_r - \omega \), then there exists \( \gamma_r - \omega - \) neighbourhood \( P \) of \( t \), such that \( \Omega^{-1}(t) \in \Omega^{-1}(P) \subset L \), but \( P \subset \Omega(Cl \eta_e K) \subset L \), since \( \Omega(Cl \eta_e K) \) is \( \gamma_r - \omega - \) closed, \( t \in E \subset (Cl \eta_e (P)) \subset (Cl \eta_e K) \subset L \), hence \( T \) is pairwise \(-\omega-\)regular.

**Remark 3.31.** The pairwise \(-\omega-\) regular space is constant algebraic expression and inverse constant algebraic expression under \( M-\omega-\)perfect.

**Definition 3.32.** A bitopological space \( (S, \eta_1, \eta_2) \) is called pairwise \(-\omega-\) normal, if each \( \eta_r - \omega - \) closed set \( N \) and \( \eta_e - \omega - \) closed set \( G \), there exists \( \eta_e - \omega - \) open set \( K \) and \( \eta_r - \omega - \) open set \( L \), such that \( N \subset K, G \subset L, K \cap L = \phi \), \( r, \epsilon = 1, 2, r \notin \epsilon \).

**Theorem 3.33.** Let \( \Omega : (S, \eta_1, \eta_2) \to (T, \gamma_1, \gamma_2) \) be a pairwise \(-\omega-\) perfect function, and \( (S, \eta_1, \eta_2) \) is a pairwise \(-\omega-\) normal, then \( (T, \gamma_1, \gamma_2) \) is so.

Proof. It follows by using Lemma [3.32] and theorem [3.33].

**Theorem 3.34.** Let \( (S, \eta_1, \eta_2), (T, \gamma_1, \gamma_2) \), be any bitopological spaces. If \( (S, \eta_1, \eta_2) \) is pairwise \( M-\omega-\) compact, then the projection function, \( \Phi : (S \times T, \eta_1 \times \gamma_1, \eta_2 \times \gamma_2) \to (T, \gamma_1, \gamma_2) \) is pairwise \(-\omega-\) closed.

Proof. If \( (S, \eta_1, \eta_2) \) is pairwise \( M-\omega- \) compact, then \( (S, \eta_1) \) is \( M-\omega- \) compact, \( (S, \eta_2) \) is \( M-\omega- \) compact, 

thus the projection functions: \( \Phi_1 : (S \times T, \eta_1 \times \gamma_1) \to (T, \gamma_1), \Phi_2 : (S \times T, \eta_2 \times \gamma_2) \to (T, \gamma_2) \), are \( \omega- \) closed, thus \( \Phi \) is pairwise \(-\omega-\) closed.
**Corollary 3.35.** Let \((S, \eta_1, \eta_2), (T, \gamma_1, \gamma_2)\) are pairwise \(M-\omega-\)compact then \((S \times T, \eta_1 \times \gamma_1, \eta_2 \times \gamma_2)\) is pairwise \(M-\omega-\)compact

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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