Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:36
https://doi.org/10.28919/jmcs/6921
ISSN: 1927-5307

ON THE NUMERICAL INVESTIGATIONS OF THE TIME-FRACTIONAL MODIFIED BURGERS' EQUATION WITH CONFORMABLE DERIVATIVE, AND ITS STABILITY ANALYSIS

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Abstract: In this paper, we aim to introduce the cubic non-polynomial spline functions to develop a computational method for solving the fractional modified Burgers' equation. Using the Von Neumann method, the proposed approach is shown to be conditionally stable. The proposed approach has been implemented on two test problems. The obtained results indicate that the proposed approach is a good option for solving the fractional modified Burgers' equation. The error norms l_2 and l_∞ have been determined to validate the accuracy and efficiency of the proposed method. The numerical solution of such kinds of models has been the key interest of researchers due to their wide range of applications in real life, optical fibers, solid-state physics, biology, plasma physics, fluid dynamics, number theory, chemical kinetics, turbulence theory, heat conduction, gas dynamics.

Keywords: Conformable fractional derivative; Fractional modified Burgers' equation; Cubic non-polynomial spline; Von Neumann stability.

2010 AMS Subject Classification: 35R11, 65M22.

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Received October 22, 2021

1. INTRODUCTION

Fractional calculus, as a generalization of integer order calculus, has aided scientists in understanding and modeling a wide range of phenomena in physics and engineering branches [1–3]. In the works of literature, there are some common methods used to obtain analytical or approximate solutions of nonlinear-fractional ordinary and partial differential equations. For instance, variational iteration method (VIM) [4] for space- and time-fractional Burgers equations, differential transformation method (DTM) [5] for fractional Fornberg-Whitham equation, combination of DTM and generalized Taylor's formula [6] for nonlinear fractional partial differential equations, Adomian decomposition method (ADM) [7] for the fractional nonlinear Schrödinger equation, radial basis functions (RBFs) [8] for fractional partial differential equations, homotopy analysis method (HAM) [9] for nonlinear fractional differential equations.

The fractional derivative of order $\alpha > 0$ has been defined in a variety of ways. The Riemann-Liouville, Caputo, Riesz, and Grunwald–Letnikov definitions are the most widely utilized [1–3]. The fractional integral is used to define these definitions, which are commonly used in non-integer calculus literature. As a result, fractional derivative operators behave as non-local operators and do not satisfy classical properties of normal (integer) derivatives such as product, chain, and quotient rules, which allow us to obtain analytical solutions in standard calculus. Algebraic operations in non-integer calculus involve many challenges and inconvenience in mathematical handling because these basic rules cannot be used. R. Khalil et al. [10] have presented a new definition, namely "conformable fractional derivative", which obeys basic classical properties and allows us to solve fractional differential equations analytically. This definition is more simple than other fractional definitions because it has received a lot of attention, many phenomena and applications can be modeled based on the conformable sense [10–15], and it has a lot of interesting advantages such as generalizes all concepts of standard calculus and can be solved numerous fractional differential equations in all cases.

In recent years, several studies have been made further studies and explanations on the physical applications and physical meaning of the Burgers equation. The modified Burgers equations, which is a different form of the Burgers' equation, are discussed in this paper and can be applied to a wide range of scientific fields, including optical fibers, solid-state physics, biology, plasma physics, fluid dynamics, number theory, chemical kinetics, turbulence theory, heat conduction, gas dynamics, etc. [16–19]. In this article, the collocation method with cubic non-polynomial spline functions is used to obtain approximate solutions of the time-fractional modified Burgers' equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \nu \frac{\partial^2 u}{\partial x^2} + u^{\mathrm{p}} \frac{\partial u}{\partial x} = 0,\ t > 0,\ 0 < \alpha \le 1, \tag{1}$$

subject to the conditions

$$u(a,\ t) = \Phi_1(t),\ u(b,\ t)\ = \Phi_2(t), t > 0, \tag{2}$$

and

$$u(x,\ t_0) = r(x),\ \mathrm{a} \le x \le \mathrm{b}, \tag{3}$$

where ν, p are parameters and, α is the parameter describing the order of the fractional time derivatives. The fractional derivatives are considered in the conformable sense.

Definition 1.1. The Riemann-Liouville fractional derivative of order α is defined as [1–3]:

$$D_a^{\alpha}(F)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{F(x)}{(t-x)^{\alpha-n+1}} dx,$$

for $n \in \mathbb{N}$ and $\alpha \in [n-1, n)$.

Definition 1.2. The Caputo fractional derivative of order α is defined as [1–3]:

$$D_a^{\alpha}(F)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{F^{(\mathrm{n})}(x)}{(t-x)^{\alpha-n+1}} dx,$$

for $n \in \mathbb{N}$ and $\alpha \in [n-1, n)$.

Definition 1.3. Let $F: [0, \infty) \to \mathbb{R}$ and $t > 0$. Then the "conformable fractional derivative" of F of order α is defined by [10]:

$$T_{\alpha}(F)(t) = \lim_{\varepsilon \to 0} \frac{F(t + \varepsilon\, t^{1-\alpha}) - F(t)}{\varepsilon} \text{ for all } \alpha \in (0,1)$$

The properties of "conformable fractional derivative" are given in the following theorem [10]

Theorem 2.1. Let F, G are α - differentiable functions at a point $t > 0$, and $\alpha \in (0,1]$. Then

(1) $T_{\alpha}(mF + nG) = mT_{\alpha}(F) + nT_{\alpha}(G)$ for all m , $n \in \mathbb{R}$

(2) $T_{\alpha}(\, t^q) = q\, t^{q-\alpha}$ for all $q \in \mathbb{R}$

(3) $T_\alpha(c) = 0$, for all constant function $F(t) = c$.

(4) $T_\alpha(FG) = FT_\alpha(G) + GT_\alpha(F)$

(5) $T_\alpha\left(\frac{F}{G}\right) = \frac{GT_\alpha(F) - FT_\alpha(G)}{G^2}$.

(6) In addition, if F is differentiable, then $T_\alpha(F)(t) = t^{1-\alpha}\frac{dF(t)}{dt}$.

This paper is organized as follows. In Section 2, a presented method that depends on the use of cubic non-polynomial spline functions is derived. In Section 3, the local truncation error is studied. In Section 4, stability analysis of the method is discussed by using the Von Neumann method. In Section 5, we illustrate two numerical examples that are introduced to present the efficiency and accuracy of the presented method. Finally, Section 6 contains the conclusion.

2. DERIVATION OF THE METHOD

Let the region $R = [a,\ b] \times [0,\ \infty]$ be discretized by a set of points $(x_i,\ t_j)$ where $x_i = a + ih$, with $h = \frac{b-a}{N}$ for $i = 0,\ 1,\ ...,\ N$ and $t_j = jk$, $k = \Delta t$ for each $j = 0,\ 1,\ ....$

Let $W_i^{j+1/2}$ be an approximation to $u(x_i,\ t_{j+1/2})$ obtained by the segment $P_i(x_i,\ t_{j+1/2})$ of the spline function passing through the points $(x_i,\ W_i^{j+1/2})$ and $(x_{i+1},\ W_i^{j+1/2})$. Each segment has the form [20,21]:

$$P_i\left(x, t_{j+\frac{1}{2}}\right) = a_i\left(t_{j+\frac{1}{2}}\right) + b_i\left(t_{j+\frac{1}{2}}\right)(x - x_i) + c_i\left(t_{j+\frac{1}{2}}\right)\sin\mu(x - x_i) + d_i\left(t_{j+\frac{1}{2}}\right)\cos\mu(x - x_i), \quad (4)$$

for each $i = 0,\ 1,\ ...,\ N-1$. The four coefficients in (4) need to be obtained in terms of $S_i^{j+1/2}$, $S_{i+1}^{j+1/2}$, $W_i^{j+1/2}$and $W_{i+1}^{j+1/2}$, we first define

$$\begin{aligned} P_i(x_i,\ t_{j+1/2}) &= W_i^{j+1/2}, \\ P_i(x_{i+1},\ t_{j+1/2}) &= W_{i+1}^{j+1/2}, \\ P_i^{(2)}(x_i,\ t_{j+1/2}) = \frac{\partial^2}{\partial x^2}P_i(x_i,\ t_{j+1/2}) &= S_i^{j+1/2}, \\ P_i^{(2)}(x_{i+1},\ t_{j+1/2}) = \frac{\partial^2}{\partial x^2}P_i(x_{i+1},\ t_{j+1/2}) &= S_{i+1}^{j+1/2}, \end{aligned} \quad (5)$$

By using (4), and (5), we get

$$\begin{aligned} a_i + d_i &= W_i^{j+1/2}, \\ a_i + b_i h + c_i \sin\omega + d_i \cos\omega &= W_{i+1}^{j+1/2}, \\ -\mu^2 d_i &= S_i^{j+1/2}, \\ -c_i\,\mu^2 \sin\omega - d_i\,\mu^2\cos\omega &= S_{i+1}^{j+1/2}, \end{aligned} \tag{6}$$

where $\omega = \mu h$, $d_i \equiv d_i(t_{j+1/2})$, $c_i \equiv c_i(t_{j+1/2})$, $b_i \equiv b_i(t_{j+1/2})$ and $a_i \equiv a_i(t_{j+1/2})$. We solve the last four equations in (6), to get the following expressions

$$\begin{aligned} a_i &= \frac{h^2}{\omega^2} S_i^{j+1/2} + W_i^{j+1/2}, \\ b_i &= \frac{W_{i+1}^{j+1/2} - W_i^{j+1/2}}{h} + \frac{h\left(S_{i+1}^{j+1/2} - S_i^{j+1/2}\right)}{\omega^2}, \\ c_i &= \frac{h^2\left(\cos\omega\, S_i^{j+1/2} - S_{i+1}^{j+1/2}\right)}{\omega^2\,\sin\omega}, \\ d_i &= \frac{-h^2}{\omega^2} S_i^{j+1/2}. \end{aligned} \tag{7}$$

Now from the continuity condition of the first derivative at $x = x_i$, that is $P_i^{(1)}(x_i,\ t_{j+1/2}) = P_{i-1}^{(1)}(x_i,\ t_{j+1/2})$, we obtain

$$b_i + c_i\mu = b_{i-1} + \mu c_{i-1}\cos\omega - \mu d_i \sin\omega. \tag{8}$$

Using expressions (7) in (8) becomes

$$W_{i+1}^{j+\frac{1}{2}} - 2W_i^{j+\frac{1}{2}} + W_{i-1}^{j+\frac{1}{2}} = \left(\frac{-h^2}{\omega^2} + \frac{h^2}{\omega\sin\omega}\right) S_{i+1}^{j+\frac{1}{2}} + \left(\frac{-2h^2\cos\omega}{\omega\sin\omega} + \frac{2h^2}{\omega^2}\right) S_i^{j+\frac{1}{2}} + \left(\frac{h^2\sin\omega}{\omega} + \frac{h^2\cos^2\omega}{\omega\sin\omega} - \frac{h^2}{\omega^2}\right) S_{i-1}^{j+1/2}. \tag{9}$$

After slight rearrangements, the last equation becomes

$$W_{i+1}^{j+1/2} - 2W_i^{j+1/2} + W_{i-1}^{j+1/2} = \rho S_{i-1}^{j+1/2} + \sigma S_i^{j+1/2} + \rho S_{i+1}^{j+1/2},\ i = 0,1,\dots,N-1, \tag{10}$$

where $\rho = \frac{-h^2}{\omega^2} + \frac{h^2}{\omega\sin\omega}$ and $\sigma = -\frac{2h^2\cos\omega}{\omega\sin\omega} + \frac{2h^2}{\omega^2}$.

Using the collocation method, W_i^j and its derivatives satisfy (1) at collocating knots

$$\frac{\partial^\alpha W_i^{j+1/2}}{\partial t^\alpha} - \nu\frac{\partial^2 W_i^{j+1/2}}{\partial x^2} + \left(W_i^{j+1/2}\right)^{\mathrm{p}} \frac{\partial W_i^{j+1/2}}{\partial x} = 0. \tag{11}$$

Eq. (11) can be rewritten in the form

$$S_i^{j+1/2} = \frac{\partial^2 W_i^{j+1/2}}{\partial x^2} = \frac{1}{\nu}\left(\frac{\partial^\alpha W_i^{j+1/2}}{\partial t^\alpha} + \delta_i^{j+1/2}\frac{\partial W_i^{j+1/2}}{\partial x}\right), \tag{12}$$

where $\delta_i^{j+1/2} = \left(W_i^{j+1/2}\right)^{\mathrm{p}}$. Using the conformable derivative properties, Eq. (12) becomes

$$S_i^{j+1/2} = \frac{1}{\nu}\left(t_{j+1/2}^{1\text{-}\alpha}\frac{\partial W_i^{j+1/2}}{\partial t} + \delta_i^{j+1/2}\frac{\partial W_i^{j+1/2}}{\partial x}\right). \tag{13}$$

Eq. (13) can be discretized

$$\begin{aligned}
S_{i-1}^{j+1/2} &= \frac{t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\left(W_{i-1}^{j+1} - W_{i-1}^{j}\right) + \frac{\delta_{i-1}^{j+1/2}}{\nu}\frac{\partial W_{i-1}^{j+1/2}}{\partial x},\\
S_{i}^{j+1/2} &= \frac{t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\left(W_{i}^{j+1} - W_{i}^{j}\right) + \frac{\delta_{i}^{j+1/2}}{\nu}\frac{\partial W_{i}^{j+1/2}}{\partial x},\\
S_{i+1}^{j+1/2} &= \frac{t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\left(W_{i+1}^{j+1} - W_{i+1}^{j}\right) + \frac{\delta_{i+1}^{j+1/2}}{\nu}\frac{\partial W_{i+1}^{j+1/2}}{\partial x}.
\end{aligned} \tag{14}$$

Using the finite difference method in (14), we obtain

$$\begin{aligned}
S_{i-1}^{j+1/2} &= \frac{t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\left(W_{i-1}^{j+1} - W_{i-1}^{j}\right) + \frac{\delta_{i-1}^{j+1/2}}{\nu h}\left(W_{i}^{j+1/2} - W_{i-1}^{j+1/2}\right),\\
S_{i}^{j+1/2} &= \frac{t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\left(W_{i}^{j+1} - W_{i}^{j}\right) + \frac{\delta_{i}^{j+1/2}}{2\nu h}\left(W_{i+1}^{j+1/2} - W_{i-1}^{j+1/2}\right),\\
S_{i+1}^{j+1/2} &= \frac{t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\left(W_{i+1}^{j+1} - W_{i+1}^{j}\right) + \frac{\delta_{i+1}^{j+1/2}}{\nu h}\left(W_{i+1}^{j+1/2} - W_{i}^{j+1/2}\right).
\end{aligned} \tag{15}$$

Substituting (15) into (10) gives us the following

$$\begin{aligned}
&\left(1 + \frac{\delta_{i-1}^{j+1/2}\rho}{\nu h} + \frac{\delta_{i}^{j+1/2}\sigma}{2\nu h}\right)W_{i-1}^{j+1/2} + \left(-2 - \frac{\delta_{i-1}^{j+1/2}\rho}{\nu h} + \frac{\delta_{i+1}^{j+1/2}\rho}{\nu h}\right)W_{i}^{j+1/2} +\\
&\left(1 - \frac{\delta_{i+1}^{j+1/2}\rho}{\nu h} - \frac{\delta_{i}^{j+1/2}\sigma}{2\nu h}\right)W_{i+1}^{j+1/2} = \frac{t_{j+1/2}^{1\text{-}\alpha}\rho}{\nu k}\left(W_{i-1}^{j+1} + W_{i+1}^{j+1} - W_{i-1}^{j} - W_{i+1}^{j}\right) +\\
&\frac{t_{j+1/2}^{1\text{-}\alpha}\sigma}{\nu k}\left(W_{i}^{j+1} - W_{i}^{j}\right).
\end{aligned} \tag{16}$$

Suppose that $W_i^{j+1/2}$ is linearly interpolated between time levels $j+1$ and j as

$$W_i^{j+1/2} \approx \frac{W_i^{j+1} + W_i^{j}}{2}. \tag{17}$$

Substituting (17) into (16) gives the following system:

$$A_i^* W_{i-1}^{j+1} + B_i^* W_i^{j+1} + C_i^* W_{i+1}^{j+1} = A_i W_{i-1}^{j} + B_i W_i^{j} + C_i W_{i+1}^{j}\ i = 0,1,\dots,N-1,\ \ j = 0,1,\dots, \tag{18}$$

where

$$A_i^* = \left(1 + \frac{\delta_{i-1}^{j+1/2}\rho}{\nu h} + \frac{\delta_{i}^{j+1/2}\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\rho}{\nu k}\right), \qquad A_i = \left(-1 - \frac{\delta_{i-1}^{j+1/2}\rho}{\nu h} - \frac{\delta_{i}^{j+1/2}\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\rho}{\nu k}\right),$$

$$B_i^* = \left(-2 - \frac{\delta_{i-1}^{j+1/2}\,\rho}{\nu h} + \frac{\delta_{i+1}^{j+1/2}\,\rho}{\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\,\sigma}{\nu k}\right), \qquad B_i = \left(2 + \frac{\delta_{i-1}^{j+1/2}\,\rho}{\nu h} - \frac{\delta_{i+1}^{j+1/2}\,\rho}{\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\,\sigma}{\nu k}\right),$$

$$C_i^* = \left(1 - \frac{\delta_{i+1}^{j+1/2}\,\rho}{\nu h} - \frac{\delta_{i}^{j+1/2}\,\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\,\rho}{\nu k}\right), \qquad C_i = \left(-1 + \frac{\delta_{i+1}^{j+1/2}\,\rho}{\nu h} + \frac{\delta_{i}^{j+1/2}\,\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\,\rho}{\nu k}\right).$$

System (18) consists of $N-1$ equations in the unknowns $W_i,\ i = 0,\ 1,\ \ldots,\ N$. To find a solution of this system, we need two additional equations. These equations are obtained from boundary conditions in (2) which can be written as

$$W_0^{j+1} = \Phi_1(t_{j+1}),\ W_N^{j+1} = \Phi_2(t_{j+1}),\ j = 0,\ 1,\ \ldots. \tag{19}$$

Writing (18) and (19) in matrix form gives

$$\emptyset^* W^{j+1} = \emptyset W^{j} + r^{j+1} \tag{20}$$

where

$$W^{j+1} = \left(W_0^{j+1},\ W_1^{j+1},\ W_2^{j+1}, \cdots,\ W_{N-1}^{j+1},\ W_N^{j+1}\right)^T,$$

$$\emptyset^* = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_1^* & B_1^* & C_1^* & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_2^* & B_2^* & C_2^* & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{N-1}^* & B_{N-1}^* & C_{N-1}^* \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix},$$

$$\emptyset = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_1 & B_1 & C_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_2 & B_2 & C_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

and $\quad r^{j+1} = \left(\Phi_1(t_{j+1}), 0, 0, \cdots, 0,\ \Phi_2(t_{j+1})\right)^T,$

where $\emptyset^*$ and $\emptyset$ are $(N+1)\times(N+1)$ matrices, and W^{j+1} and r^{j+1} are $(N+1)$ dimensional vectors.

The initial condition $u(x,\ t_0) = r(x)$, for $a \le x \le b$ implies that $W_i^0 = r(x)$, for each $i = 0,\ 1,\ \ldots,\ N$. These values can be used in (20) to calculate the value of W_i^1, for each $i = 0$, $1,\ \ldots,\ N$. If the procedure is reapplied once all the approximations W_i^1 are known, the values of $W_i^2,\ W_i^3$, ... can be calculated in a similar manner.

Remark 1. The following steps are used to linearize the nonlinear term in the system (20)

I. At $j = 0$, we approximate $\delta_i^{1/2}$ by $\delta_i^{1/2*}$ computed from W_i^0 and get a first approximation to W_i^1, then we compute $\delta_i^{1/2}$ from $\frac{W_i^1 + W_i^0}{2}$ to refine the approximation to W_i^1.

II. At $j = n$, we approximate $\delta_i^{n+1/2}$ by $\delta_i^{n+1/2*}$ computed from W_i^n and get a first approximation to W_i^{n+1}, then we compute $\delta_i^{n+1/2}$ from $\frac{W_i^{n+1} + W_i^n}{2}$ to refine the approximation to W_i^{n+1}.

3. TRUNCATION ERROR

We can obtain the local truncation error $T_i^{*j+1/2}$ associated with Eq. (10) by rewriting (10) in the form

$$\left(u_{i-1}^{j+1/2} + u_{i+1}^{j+1/2}\right) - 2u_i^{j+1/2} = \rho\left(D_x^2\, u_{i-1}^{j+1/2} + D_x^2 u_{i+1}^{j+1/2}\right) + \sigma D_x^2 u_i^{j+1/2} + T_i^{*j+1/2}, \qquad (21)$$

where $D_x^{\mathrm{n}} \equiv \frac{\partial^{\mathrm{n}}}{\partial x^{\mathrm{n}}}$ and $u_i^j \equiv (x_i,\ t_j)$.

After expanding $u_i^{j+1/2}$, $u_{i-1}^{j+1/2}$, $u_{i+1}^{j+1/2}$, $D_x^2 u_i^{j+1/2}$, $D_x^2\, u_{i-1}^{j+1/2}$ and $D_x^2 u_{i+1}^{j+1/2}$ around the point $(x_i,\ t_{j+1/2})$ using Taylor's series gives us the following important expression for $T_i^{*j+1/2}$

$$T_i^{*j+1/2} = \left(h^2 - (\sigma + 2\rho)\right) D_x^2 u_i^{j+1/2} + h^2\left(\frac{h^2}{12} - \rho\right) D_x^4 u_i^{j+1/2} + h^4\left(\frac{h^2}{360} - \frac{\rho}{12}\right) D_x^6 u_i^{j+1/2} + \cdots\ . \qquad (22)$$

Remark 2. From the last expression of the local truncation error:

I. For $\sigma + 2\rho = h^2$ our scheme is of $O(h^2)$, but for $\sigma + 2\rho = h^2$ and $\rho = h^2/12$ our scheme is of $O(h^4)$.

II. As $\mu \to 0$, that is $\omega(\mu) \to 0$, then $(\rho,\ \sigma) \to (\frac{h^2}{6}, \frac{4h^2}{6})$ and $\sigma + 2\rho = h^2$ and system (10) reduces to ordinary cubic spline

$$W_{i+1}^{j+1/2} - 2W_i^{j+1/2} + W_{i-1}^{j+1/2} = \frac{h^2}{6}\left(S_{i-1}^{j+1/2} + 4S_i^{j+1/2} + S_{i+1}^{j+1/2}\right).$$

4. Stability Analysis

The Von Neumann technique will be used to investigate the stability of our system (18). To do this, we must linearize the nonlinear term u^{p} of Burgers' equation, Eq. (1), by assuming that the corresponding quantities $\delta_{i-1}^{j+1/2}$, $\delta_{i}^{j+1/2}$ and $\delta_{i+1}^{j+1/2}$ are equal to a local constant γ^* in (18). Assuming a solution of the form [22, 23]:

$$W_i^j = \xi^j \exp\,(\phi I i h) \tag{23}$$

where $I^2 = -1$, h is the element size, ϕ is the mode number, and ξ^j is the amplification factor at time level j. As j increases, more time steps are computed. Inserting the latter expression for W_i^j in the system (18) gives

$$\begin{aligned}\xi^{j+1}\big(\, A_i^*\exp(\phi I(i-1)h) + B_i^*\exp(\phi I i h) + C_i^*\exp(\phi I(i+1)h)\big) \\ = \xi^j\big(A_i\exp(\phi I(i-1)h) + B_i\exp(\phi I i h) + C_i\exp(\phi I(i+1)h)\big)\end{aligned} \tag{24}$$

where

$$A_i^* = \left(1 + \frac{\gamma^*\rho}{\nu h} + \frac{\gamma^*\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\rho}{\nu k}\right), \qquad A_i = \left(-1 - \frac{\gamma^*\rho}{\nu h} - \frac{\gamma^*\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\rho}{\nu k}\right),$$

$$B_i^* = \left(-2 - \frac{2t_{j+1/2}^{1\text{-}\alpha}\sigma}{\nu k}\right), \qquad B_i = \left(2 - \frac{2t_{j+1/2}^{1\text{-}\alpha}\sigma}{\nu k}\right),$$

$$C_i^* = \left(1 - \frac{\gamma^*\rho}{\nu h} - \frac{\gamma^*\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\rho}{\nu k}\right), \qquad C_i = \left(-1 + \frac{\gamma^*\rho}{\nu h} + \frac{\gamma^*\sigma}{2\nu h} - \frac{2t_{j+1/2}^{1\text{-}\alpha}\rho}{\nu k}\right).$$

After utilizing some manipulations, Eq. (24) becomes

$$\xi = \frac{A_i\exp(-I\varphi) + B_i + C_i\exp(I\varphi)}{A_i^*\exp(-I\varphi) + B_i^* + C_i^*\exp(I\varphi)} \tag{25}$$

where $\varphi = \phi h$. After using Euler's formula $\exp(I\varphi) = \cos\varphi + I\sin\varphi$, the last equation (25) can be written as

$$\xi = \frac{(A_i + C_i)\cos\varphi + B_i + I(C_i - A_i)\sin\varphi}{(A_i^* + C_i^*)\cos\varphi + B_i^* + I(C_i^* - A_i^*)\sin\varphi}$$

or

$$\xi = \frac{\left(-2 - \dfrac{4\rho\, t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\right)\cos\varphi + \left(2 - \dfrac{2\sigma\, t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\right) + \dfrac{I\gamma^*}{\nu h}(2\,\rho + \,\sigma)\sin\varphi}{\left(2 - \dfrac{4\rho\, t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\right)\cos\varphi + \left(-2 - \dfrac{2\sigma\, t_{j+1/2}^{1\text{-}\alpha}}{\nu k}\right) - \dfrac{I\gamma^*}{\nu h}(2\,\rho + \,\sigma)\sin\varphi}$$

After slight rearrangement

$$\xi = \frac{a - b + Ic}{-a - b - Ic}$$

where $a = 2\nu h(1 - \cos\varphi)$, $b = \frac{2h\, t_{j+1/2}^{1-\alpha}}{k}(2\rho\cos\varphi + \sigma)$, and $c = \gamma^*(2\rho + \sigma)\sin\varphi$. Then

$$|\xi| = \sqrt{\frac{(a-b)^2 + c^2}{(a+b)^2 + c^2}}$$

The quantity $(1 - \cos\varphi)$ is equal to zero or positive, $2\rho\cos\varphi + \sigma$ is surely positive if we choose $\rho > 0$ and $\sigma > 0$ such that $\sigma > 2\rho$. Then $a \geq 0$, $b > 0$ and $(a+b)^2 \geq (a-b)^2$.

For stability, we must have $|\xi| \leq 1$, otherwise ξ_j in (23) would expand in an unbounded manner. This condition is valid for $\rho > 0$ and $\sigma > 0$ such that $\sigma > 2\rho$. Finally, we can say that the proposed method is conditionally stable for $\rho > 0$ and $\sigma > 0$ such that $\sigma > 2\rho$.

5. Numerical Results

In this section, we obtain numerical solutions of fractional modified Burgers' equation for two test problems. The accuracy of our presented numerical method is measured by computing the difference between the numerical and analytical solutions at each mesh point and use these to compute l_2 and l_∞ error norms. Error norms are defined as

$$l_2 = \left\|u_{exact} - W_{approx}\right\|_2 \simeq \sqrt{h\sum_{i=0}^{N}\left|(u_i)_{exact} - (W_i)_{approx}\right|^2},$$

and

$$l_\infty = \left\|u_{exact} - W_{approx}\right\|_\infty \simeq \max_{0\leq i\leq N}\left|(u_i)_{exact} - (W_i)_{approx}\right|.$$

Example 1

Consider the following fractional Burgers' equation [17, 19]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \nu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0,\ 0 < \alpha < 1, \tag{26}$$

BCs:

$$u(-1,\ t) = c - \sqrt{2b\nu + c^2}\,\mathrm{Tanh}\left(\frac{\sqrt{2b\nu + c^2}\left(-1 - \frac{ct^\alpha}{\alpha} - 2\gamma\nu\right)}{2\nu}\right),$$

$$u(1,\ t)\ = c - \sqrt{2b\nu + c^2}\text{Tanh}\left(\frac{\sqrt{2b\nu+c^2}\left(1-\frac{ct^{\alpha}}{\alpha}-2\gamma\nu\right)}{2\nu}\right),\ t > 0,$$

IC:

$$u(x,\ 0) = c - \sqrt{2b\nu + c^2}\text{Tanh}\left(\frac{\sqrt{2b\nu+c^2}(x-2\gamma\nu)}{2\nu}\right),\ -1 \leq x \leq 1.$$

The exact solution to this problem is

$$u(x,\ t) = c - \sqrt{2b\nu + c^2}\text{Tanh}\left(\frac{\sqrt{2b\nu+c^2}}{2\nu}\left(x - \frac{ct^{\alpha}}{\alpha} - 2\nu\gamma\right)\right).$$

The approximate $W(x,\ t)$, exact solutions, and absolute errors are presented in Tables 1-3, for different values of α with $k = 0.01$, $h = 0.01$, $c = 0.1$, $b = 0.9$, $\nu = 1.5$ and $\gamma = 1.5$. Tables 1-3 show a very close agreement between the approximate and exact solutions. In Table 4, the error norms have been shown for $\alpha = 0.7$. The surface plots of the approximate solutions are shown in Fig. 1, for $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$. The exact and approximate solutions of (26) are graphically depicted in Fig. 2, for $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$.

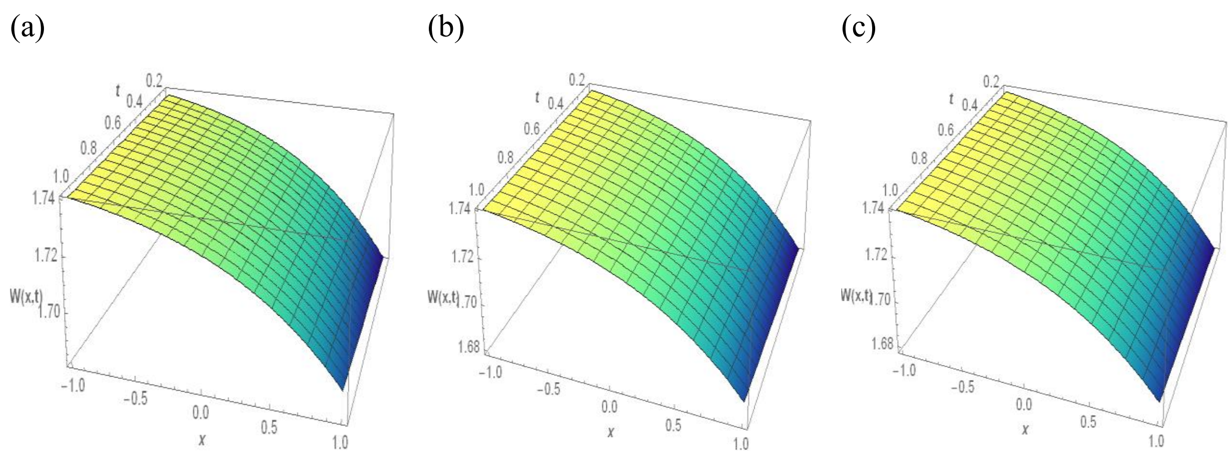

Figure 1: Three-dimensional view of numerical results of Example 1 for $k = 0.01$, $h = 0.01$, $c = 0.1$, $b = 0.9$, $\nu = 1.5, \gamma = 1.5$, and for (a) $\alpha = 0.5$, (b) $\alpha = 0.7$ and (c) $\alpha = 0.9$.

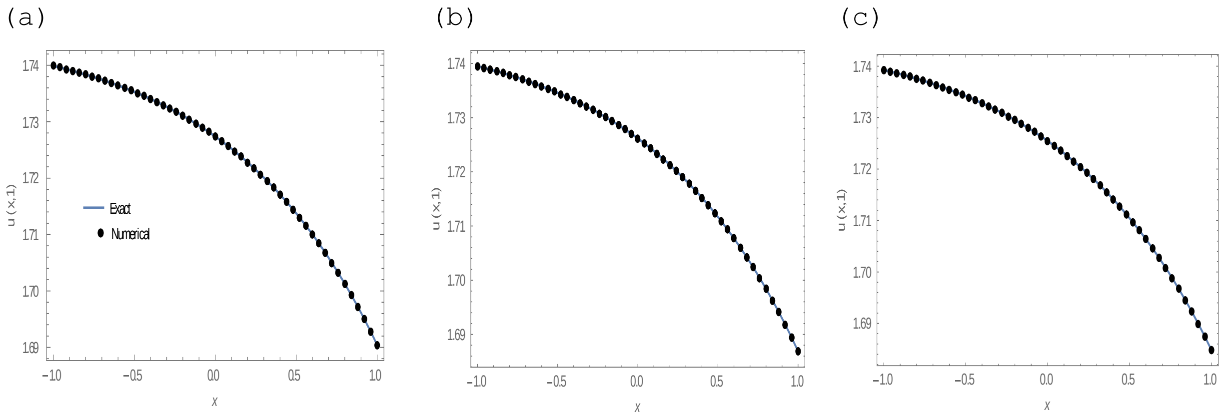

Figure 1: The profile of approximate and exact solutions for $k = 0.01$, $h = 0.01$, $c = 0.1$, $b = 0.9$, $\nu = 1.5$, $\gamma = 1.5$, and for (a) $\alpha = 0.5$, (b) $\alpha = 0.7$ and (c) $\alpha = 0.9$.

x	**Proposed method**	**Exact**	**Abs. error**
0.0	1.727376	1.727376	7.9187×10^{-8}
0.1	1.725205	1.725205	8.56727×10^{-8}
0.2	1.722787	1.722787	9.05121×10^{-8}
0.3	1.720091	1.720092	9.3251×10^{-8}
0.4	1.717089	1.717089	9.33889×10^{-8}
0.5	1.713745	1.713745	9.0364×10^{-8}
0.6	1.710021	1.710021	8.35308×10^{-8}
0.7	1.705874	1.705875	7.2126×10^{-8}
0.8	1.701260	1.701260	5.51523×10^{-8}
0.9	1.696126	1.696126	3.08054×10^{-8}
1.0	1.6904152	1.6904152	0.000000

Table 1: The comparison between the approximate $W(x, t)$ and exact solutions of Example 1 with absolute errors for $t = 1.0$ and $\alpha = 0.5$.

x	**Proposed method**	**Exact**	**Abs. error**
0.0	1.726164	1.726165	7.3342×10^{-8}
0.1	1.723855	1.723855	7.99689×10^{-8}
0.2	1.721282	1.721282	8.51791×10^{-8}
0.3	1.718416	1.718416	8.85191×10^{-8}
0.4	1.715222	1.715222	8.94707×10^{-8}
0.5	1.711666	1.711666	8.74291×10^{-8}
0.6	1.707706	1.707706	8.16716×10^{-8}
0.7	1.703298	1.703298	7.1313×10^{-8}
0.8	1.698393	1.698393	5.52391×10^{-8}
0.9	1.692937	1.692937	3.1989×10^{-8}
1.0	1.686869	1.686869	0.000000

Table 2: The comparison between the approximate $W(x,\ t)$ and exact solutions of Example 1 with absolute errors for $t = 1.0$ and $\alpha = 0.7$.

x	**Proposed method**	**Exact**	**Abs. error**
0.0	1.725458	1.725458	7.84028×10^{-9}
0.1	1.723069	1.723069	1.16324×10^{-8}
0.2	1.720406	1.720406	1.57152×10^{-8}
0.3	1.717439	1.717439	1.9872×10^{-8}
0.4	1.714134	1.714134	2.37854×10^{-8}
0.5	1.710454	1.710454	2.70013×10^{-8}
0.6	1.706357	1.706357	2.88823×10^{-8}
0.7	1.701797	1.701797	2.85472×10^{-8}
0.8	1.696723	1.696723	2.47937×10^{-8}
0.9	1.691080	1.691080	1.60132×10^{-8}
1.0	1.684804	1.684804	0.000000

Table 3: The comparison between the approximate $W(x,\ t)$ and exact solutions of Example 1 with absolute errors for $t = 1.0$ and $\alpha = 0.9$.

N	$t = 1.0$		$t = 2.0$	
	l_2 **Error Norm**	**Max. Abs. Error**	l_2 **Error Norm**	**Max. Abs. Error**
20	1.23961×10^{-5}	1.26126×10^{-5}	4.28434×10^{-5}	2.19661×10^{-5}
40	2.93732×10^{-6}	2.98975×10^{-6}	2.79757×10^{-6}	2.84818×10^{-6}
80	5.76341×10^{-7}	5.84464×10^{-7}	6.38573×10^{-7}	6.48584×10^{-7}
160	1.54773×10^{-8}	1.83971×10^{-8}	9.94056×10^{-8}	9.98317×10^{-8}

Table 4: Error norms for $k = 0.01,\ c = 0.1,\ b = 0.9,\ \nu = 1.5$ and $\alpha = 0.7$ for Example 1.

Example 2

Consider the following fractional modified Burgers' equation [17, 19]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \nu \frac{\partial^2 u}{\partial x^2} + u^2 \frac{\partial u}{\partial x} = 0,\ 0 < \alpha < 1, \tag{27}$$

BCs:

$$u(-11,\ t) = -\sqrt{3c}/\sqrt{1 - \mathrm{Cosh}\left(\frac{2c}{\nu}\left(-11 - \frac{ct^\alpha}{\alpha} + 3\nu r\right)\right) - \mathrm{Sinh}\left(\frac{2c}{\nu}\left(-11 - \frac{ct^\alpha}{\alpha} + 3\nu r\right)\right)},$$

$$u(-1,\ t) = -\sqrt{3c}/\sqrt{1 - \mathrm{Cosh}\left(\frac{2c}{\nu}\left(-1 - \frac{ct^\alpha}{\alpha} + 3\nu r\right)\right) - \mathrm{Sinh}\left(\frac{2c}{\nu}\left(-1 - \frac{ct^\alpha}{\alpha} + 3\nu r\right)\right)},\ t > 0,$$

IC:

$$u(x,\ 0) = -\sqrt{3c}/\sqrt{1 - \mathrm{Cosh}\left(\frac{2c}{\nu}(x + 3\nu r)\right) - \mathrm{Sinh}\left(\frac{2c}{\nu}(x + 3\nu r)\right)},\ -11 \le x \le -1.$$

The exact solution to this problem is

$$u(x,\ t) = -\sqrt{3c}/\sqrt{1 - \mathrm{Cosh}\left(\frac{2c}{\nu}\left(x - \frac{ct^\alpha}{\alpha} + 3\nu r\right)\right) - \mathrm{Sinh}\left(\frac{2c}{\nu}\left(x - \frac{ct^\alpha}{\alpha} + 3\nu r\right)\right)}.$$

The approximate $W(x,\ t)$, exact solutions, and absolute errors are presented in Tables 5-7, for different values of α with $t = 0.01,\ h = 1/45,\ c = 0.1,\ \nu = 1$ and $r = 1.5$. Tables 5-7 show a very close agreement between the approximate and exact solutions. In Table 8, the error norms have been shown for $\alpha = 0.7$. The 3D-Graphs of the approximate solutions are shown in Fig. 3, for $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$. The exact and approximate solutions of (27) are graphically depicted in Fig. 4, for $\alpha = 0.5$, $\alpha = 0.7$ and $\alpha = 0.9$.

(a) (b) (c)

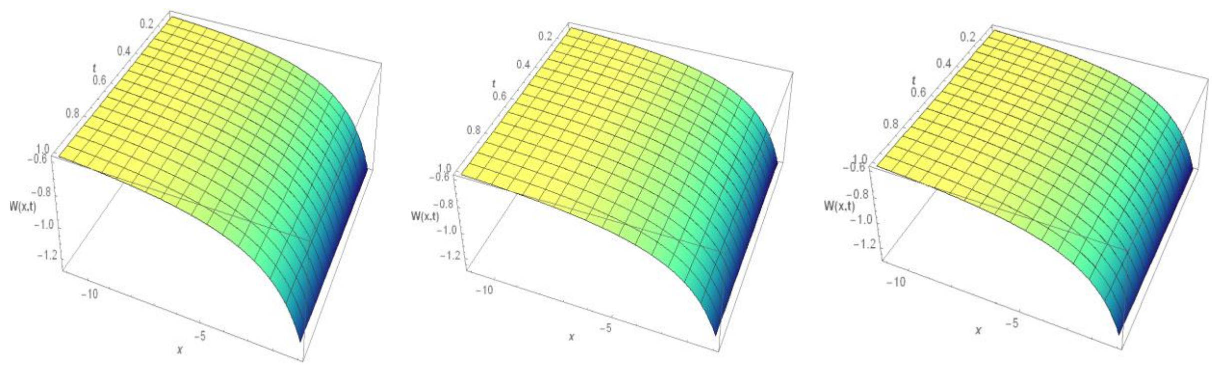

Figure 3: Three-dimensional view of numerical results of Example 2 for $k = 0.01$, $h = 1/45$, $c = 0.1$, $\nu = 1$ and $r = 1.5$, and for (a) $\alpha = 0.5$, (b) $\alpha = 0.7$ and (c) $\alpha = 0.9$.

(a) (b) (c)

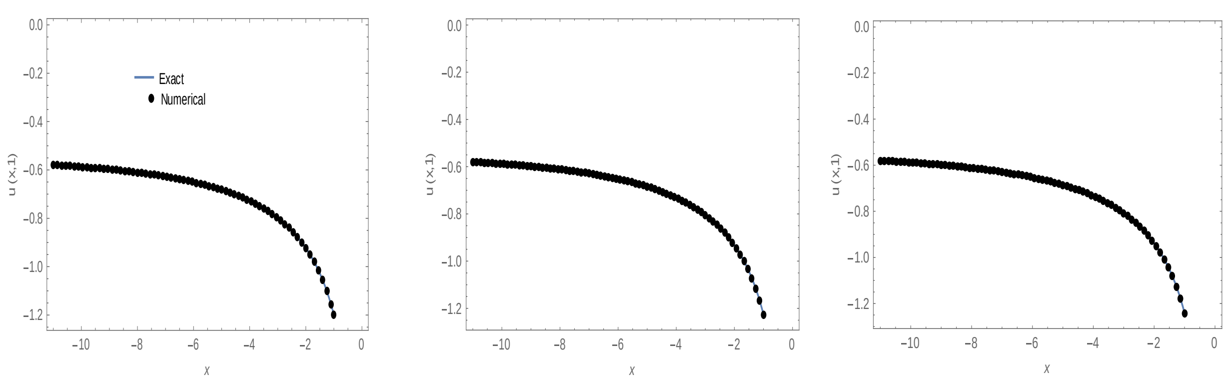

Figure 4: The profile of approximate and exact solutions for $k = 0.01$, $h = 1/45$, $c = 0.1$, $\nu = 1, r = 1.5$, and for (a) $\alpha = 0.5$, (b) $\alpha = 0.7$ and (c) $\alpha = 0.9$.

x	Proposed method	Exact	Abs. error
-11.0	-0.579640	-0.579640	0.000000
-10.0	-0.587517	-0.587494	2.33164×10^{-5}
-9.0	-0.597584	-0.597533	5.05194×10^{-5}
-8.0	-0.610603	-0.610523	7.98113×10^{-5}
-7.0	-0.627713	-0.627601	1.12327×10^{-4}
-6.0	-0.650693	-0.650543	1.50851×10^{-4}
-5.0	-0.682497	-0.682302	1.94985×10^{-4}
-4.0	-0.728467	-0.728233	2.33284×10^{-4}
-3.0	-0.799568	-0.799331	2.37026×10^{-4}
-2.0	-0.923238	-0.923071	1.67036×10^{-4}
-1.0	-1.199122	-1.199122	0.000000

Table 5: The comparison between the approximate $W(x,\ t)$ and exact solutions of Example 2 with absolute errors for $t = 1.0$ and $\alpha = 0.5$.

x	Proposed method	Exact	Abs. error
-11.0	-0.580040	-0.580040	0.000000
-10.0	-0.588006	-0.588002	3.56791×10^{-6}
-9.0	-0.598195	-0.598187	7.48208×10^{-6}
-8.0	-0.611387	-0.611376	1.13829×10^{-5}
-7.0	-0.628748	-0.628733	1.55904×10^{-5}
-6.0	-0.652104	-0.652083	2.06986×10^{-5}
-5.0	-0.684502	-0.684476	2.66876×10^{-5}
-4.0	-0.731499	-0.731468	3.11716×10^{-5}
-3.0	-0.804599	-0.804572	2.71837×10^{-5}
-2.0	-0.932997	-0.932990	6.32454×10^{-6}
-1.0	-1.226149	-1.226149	0.000000

Table 6: The comparison between the approximate $W(x,\ t)$ and exact solutions of Example 2 with absolute errors for $t = 1.0$ and $\alpha = 0.7$.

x	**Proposed method**	**Exact**	**Abs. Error**
-11.0	-0.580265	-0.580265	0.000000
-10.0	-0.588289	-0.588288	2.74142×10^{-7}
-9.0	-0.598555	-0.598555	5.54152×10^{-7}
-8.0	-0.611856	-0.611855	7.8279×10^{-7}
-7.0	-0.629371	-0.629370	9.58215×10^{-7}
-6.0	-0.652953	-0.652952	1.02611×10^{-6}
-5.0	-0.685704	-0.685703	6.48945×10^{-7}
-4.0	-0.733299	-0.733300	1.39269×10^{-6}
-3.0	-0.807547	-0.807555	8.07072×10^{-6}
-2.0	-0.9386713	-0.938692	2.16472×10^{-5}
-1.0	-1.242120	-1.242120	0.000000

Table 7: The comparison between the approximate $W(x,\ t)$ and exact solutions of Example 2 with absolute errors for $t = 1.0$ and $\alpha = 0.9$.

N	**t = 1.0**		**t = 2.0**	
	l_2 **Error Norm**	**Max. Abs. Error**	l_2 **Error Norm**	**Max. Abs. Error**
50	$3.20717" \times 10^{-4}$	3.01521×10^{-4}	3.27737×10^{-4}	2.78368×10^{-4}
100	8.26284×10^{-5}	7.74564×10^{-5}	8.13397×10^{-5}	7.29719×10^{-5}
200	5.51329×10^{-5}	2.86132×10^{-5}	4.14158×10^{-5}	2.24487×10^{-5}
400	5.78373×10^{-5}	3.11596×10^{-5}	4.18175×10^{-5}	2.13544×10^{-5}

Table 8: Error norms for $k = 0.01,\ c = 0.1,\ \nu = 1,\ r = 1.5$ and $\alpha = 0.7$ for Example 2.

6. CONCLUSION

In this study, a numerical treatment for the nonlinear fractional modified Burgers' equation is proposed using a collocation method with cubic non-polynomial spline functions. Applying the Von Neumann stability analysis, the proposed method is shown to be conditionally stable. In comparison to exact solutions, the derived approximate solutions have good accuracy. Therefore, we can conclude that the proposed method is a very effective and dependable tool for fractional differential equations that arise in a variety of physics and engineering fields.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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