PROPERTIES OF MULTI ANTI L-FUZZY QUOTIENT GROUP $\tilde{\alpha}$ OF A GROUP $G$

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Abstract: In this Paper, the notion of multi anti L–fuzzy quotient group $\tilde{\alpha}$ of a group $G$ determined by $A$ and $K$ is introduced and discussed its properties.

Keywords: fuzzy set; multi-L-fuzzy subgroup; homomorphism of multi L-fuzzy group; anti homomorphism of multi L-fuzzy group; quotient subgroup; multi L-fuzzy quotient subgroup.

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1. INTRODUCTION

L. A. Zadeh [19] introduced the notion of a fuzzy subset $A$ of a set $X$ as a function from $X$ into $I = [0, 1]$. Rosenfeld [3] and Kuroki [12] applied this concept in group theory and semi group theory, and developed the theory of fuzzy subgroups and fuzzy sub semi groupoids respectively. The concept of anti – fuzzy subgroup was introduced by Biswas [5]. The Concept

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of multi fuzzy subgroups was introduced by Souriar Sebastian and S. Babu Sundar [17]. In all these studies, the closed unit interval [0, 1] is taken as the Membership lattice. We introduce the notion of a multi L-fuzzy sub group G and discussed some of its properties. The characterizations of a Multi L-fuzzy subgroup under homomorphism and anti homomorphism on multi L-fuzzy quotient subgroup on a group is discussed

2. PRELIMINARIES

In this Section, we review some definitions and some results of Multi L-fuzzy subgroups which will be used in the later sections. Throughout this section we mean that (G,*) is a group, e is the identity of G and xy as x*y.

2.1 Definition:

Let X be any nonempty set. A fuzzy set A of X is A: X → [0, 1].

2.2 Definition:

Let (G, .) be a group. A fuzzy subset A of G is said to be a fuzzy subgroup (FSG) of G if the following conditions are satisfied:

i. A(xy) ≥ min{A(x), A(y)},

ii. A(x⁻¹) = A(x), for all x and y ∈ G.

2.3 Definition:

Let (G, .) be a group. A fuzzy subgroup A of G is said to be a normal fuzzy subgroup of G if A(xy) = A(yx), for all x and y ∈ G.

2.4 Definition:

A fuzzy subset A of G is said to be a anti fuzzy group of G, if for all x, y ∈ G

i. A(xy) ≤ max{A(x), A(y)}

ii. A(x⁻¹) = A(x).

2.5 Definition:

An anti fuzzy subgroup A of G is called a anti fuzzy normal subgroup (AFNS) of G if for every x, y ∈ G, A(xy⁻¹) ≤ A(y).
2.6 Definition:
Let $X$ be a non-empty set. A multi $L$-fuzzy set $A$ in $X$ is defined as a set of ordered sequences, $A = \{(x, A_1(x), A_2(x), \ldots, A_i(x), \ldots); x \in X\}$, where $A_i : X \rightarrow L$ for all $i$.

2.7 Definition:
A multi $L$-fuzzy subset $A$ of $G$ is called a multi $L$-fuzzy subgroup (MLFS) of $G$ if for every $x, y \in G$,

i. $A(xy) \geq A(x) \wedge A(y)$

ii. $A(x^{-1}) = A(x)$.

2.8 Definition:
A multi $L$-fuzzy subset $A$ of $G$ is called a multi anti $L$-fuzzy subgroup (MALFS) of $G$ if for every $x, y \in G$,

i. $A(xy) \leq A(x) \vee A(y)$

ii. $A(x^{-1}) = A(x)$.

2.9 Definition
The function $f: G \rightarrow G'$ is said to be a homomorphism if $f(xy) = f(x)f(y)$ $\forall x, y \in G$.

2.10 Definition
The function $f: G \rightarrow G'$ ($G$ and $G'$ are not necessarily commutative) is said to be an anti homomorphism if $f(xy) = f(y)f(x)$ $\forall x, y \in G$.

2.11 Definition
Let $f$ be any function from a set $X$ to a set $Y$, and let $A$ be any $L$-fuzzy subset of $X$. Then $A$ is called $f$-invariant if $f(x) = f(y)$ implies $A(x) = A(y)$, where $x, y \in X$.

2.12 Definition:
Let $(G, \cdot)$ be a group. A multi $L$-fuzzy subgroup $A$ of $G$ is said to be a multi $L$-fuzzy normal subgroup of $G$ if $A(xy) = A(yx)$, for all $x$ and $y \in G$.

2.13 Definition:
Let $(G, \cdot)$ be a group. A multi anti $L$-fuzzy subgroup $A$ of $G$ is said to be a multi anti $L$-fuzzy normal subgroup of $G$ if $A(xy) = A(yx)$, for all $x$ and $y \in G$. 
2.14 Definition:

Let $A$ be a multi L-fuzzy normal subgroup of $G$ with identity $e$. Let $K = \{ x \in G / A(x) = A(e) \}$. Consider the map $\bar{A} : G/K \rightarrow \mathbb{L}^k$ defined by $\bar{A}(xK) = \vee A(xk)$ for all $k \in K$ and $x \in G$. Then, the multi L-fuzzy subgroup $\bar{A}$ of $G/K$ is called a multi L-fuzzy quotient group of $A$ by $K$.

Remarks:

i. $\bar{A}$ is not a multi L-fuzzy normal quotient group of $G/K$, Since, $\bar{A}(xKyK) \neq \bar{A}(yKxK)$.

ii. Consider the map, $\bar{A} : G/K \rightarrow \mathbb{L}$ defined by $\bar{A}(xK) = A(x)$ for all $k \in K$ and $x \in G$. Then, $\bar{A}$ is a multi L-fuzzy normal quotient group of $G/K$.

3. Properties of Multi Anti L-Fuzzy Quotient Group $\bar{A}$ Determined by $A$ and $K$

In this section, the properties of multi L-fuzzy quotient group $\bar{A}$ determined by $A$ and $K$ are discussed.

3.1 Theorem:

Let $A$ be a multi anti L-fuzzy normal subgroup of $G$ with identity $e$. Let $K = \{ x \in G / A(x) = A(e) \}$. Consider the map $\bar{A} : G/K \rightarrow \mathbb{L}^k$ defined by $\bar{A}(xK) = \wedge A(xk)$ for all $k \in K$ and $x \in G$. Then, $K$ is a normal subgroup of $G$.

i. The map $\bar{A}$ is well defined.

ii. $\bar{A}$ is a multi L-fuzzy subgroup of $G$.

Proof:

Given $A$ is a multi anti L-fuzzy normal subgroup of $G$ and

i. $K = \{ x \in G / A(x) = A(e) \}$. Let $x \in G$ and $y \in K$, then $A(y) = A(e)$.

Now, $A(xyx^{-1}) = A(y) = A(e)$, since $A$ is a normal subgroup of $G$.

Hence, $xyx^{-1} \in K$. 

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Hence, $K = \{ x \in G / A(x) = A(e) \}$ is a normal subgroup of $G$.

ii. Consider the map, $\overline{A} : G / K \rightarrow L$ defined by

\[ \overline{A}(xK) = \land A(xK) \text{ for all } k \in K \text{ and } x \in G. \]

Let $Kx = Ky$ for some $x, y \in G$. Then $xy^{-1} \in K$. That is, $A(xy^{-1}) = A(e)$.

That is, $A(xK) = A(yK)$

That is, $\overline{A}(xK) = \overline{A}(yK)$.

Hence, the map $\overline{A}$ is well defined.

iii. Now, $\overline{A}(xKyK) = \overline{A}(xyK) = \land A(xyK)$, for all $k \in K$ and $x, y \in G$.

\[ \leq \land (A(xk_1) \lor A(yk_2)) , k_1, k_2 \in K. \]

\[ \leq (\land A(xk_1)) \lor (\land A(yk_2)) , k_1, k_2 \in K. \]

\[ \leq \overline{A}(xK) \lor \overline{A}(yK). \]

\[ \overline{A}(xKyK) \leq \overline{A}(xK) \lor \overline{A}(yK). \]

\[ \overline{A}((xK)^{-1}) = \overline{A}(x^{-1}K) \]

\[ = \land A(x^{-1}K) \text{ for all } k \in K \text{ and } x \in G. \]

\[ = \land A(xK) \text{ for all } k \in K \text{ and } x \in G. \]

\[ = \overline{A}(xK). \]

\[ \overline{A}((xK)^{-1}) = \overline{A}(xK). \]

Hence, $\overline{A}$ is a multi anti L-fuzzy subgroup of $G / K$.

3.2 Definition

Let $A$ be a multi anti L-fuzzy normal subgroup of $G$ with identity $e$. Let $K = \{ x \in G / A(x) = A(e) \}$. Consider the map $\overline{A} = G / K \rightarrow L^k$ defined by $\overline{A}(xK) = \land A(xk)$ for all $k \in K$ and $x \in G$. Then, the multi anti L-fuzzy subgroup $\overline{A}$ of $G / K$ is called a multi anti L-fuzzy quotient group of $A$ by $K$. 
Remarks:

i. $\overline{A}$ is not a multi anti L-fuzzy normal quotient group of $G/K$.

Since, $\overline{A}(xKyK) \neq \overline{A}(yKxK)$.

ii. Consider the map, $\overline{A}: G/K \rightarrow L$ defined by $\overline{A}(xK) = A(x)$ for all $k \in K$ and $x \in G$.

Then, $\overline{A}$ is a multi anti L-fuzzy normal quotient group of $G/K$.

3.3 Theorem:

Let $\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ is a multi anti L-fuzzy quotient group of a group of $G/K$, iff $\overline{A}_i, i = 1, 2, \ldots k$, is an anti L-fuzzy quotient group of a group $G/K$.

Proof:

Let $\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ is a multi anti L-fuzzy quotient group of a group of $G/K$. Then,

$\Leftrightarrow \overline{A}(xy) \leq \overline{A}(x) \lor \overline{A}(y)$ and $\overline{A}(x^{-1}) = \overline{A}(x)$.

$\Leftrightarrow \overline{A}_i(xy) \leq \overline{A}_i(x) \lor \overline{A}_i(y)$ and $\overline{A}_i(x^{-1}) = \overline{A}_i(x)$ for all $i = 1, 2, \ldots k$.

$\Leftrightarrow \overline{A}_i, i = 1, 2, \ldots k$, is an anti L-fuzzy quotient group of a group $G/K$.

Remark:

If $\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ is not a multi anti L-fuzzy quotient group of a group $G/K$, then there is at least one $\overline{A}_i, i = 1, 2, \ldots k$, is not an anti L-fuzzy quotient group of a group $G/K$.

3.4 Theorem:

If $\overline{A}$ is a multi anti L-fuzzy quotient group of a group $G/K$, then $\overline{A}(xK) \geq \overline{A}(eK)$, for $x \in G$, where $e \in G$ is the identity element of $G$. 
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**Proof:**

Let the element $x \in G$, where $e \in G$ is the identity element of $G$.

Now,

$$\overline{A}(e) = \overline{A}(xx^{-1}K)$$

$$\leq \overline{A}(xK) \lor \overline{A}(x^{-1}K)$$

$$= \overline{A}(xK).$$

Therefore, $\overline{A}(eK) \leq \overline{A}(xK)$.

**3.5 Theorem:**

$\overline{A}$ is a multi anti L-fuzzy quotient group of a group $G/K$ if and only if

$$\overline{A}(xKy^{-1}K) \leq \overline{A}(xK) \lor \overline{A}(yK),$$

for all $x$ and $y$ in $G$.

**Proof:**

Assume that $\overline{A}$ is a multi anti L-fuzzy quotient group of a group $G/K$.

We have,

$$\overline{A}(xKy^{-1}K) \leq \overline{A}(xK) \lor \overline{A}(y^{-1}K)$$

$$\leq \overline{A}(xK) \lor \overline{A}(yK)$$

Therefore, $\overline{A}(xKy^{-1}K) \leq \overline{A}(xK) \lor \overline{A}(yK)$, for all $x$ and $y$ in $G$.

Conversely, if $\overline{A}(xKy^{-1}K) \leq \overline{A}(xK) \lor \overline{A}(yK)$, then

$$\overline{A}(x^{-1}K) = \overline{A}(ex^{-1}K)$$

$$\leq \overline{A}(eK) \lor \overline{A}(xK)$$

$$= \overline{A}(xK).$$

Therefore, $\overline{A}(x^{-1}) \leq \overline{A}(x)$, for all $x$ in $G$.

Hence, $\overline{A}(x^{-1}K) \leq \overline{A}(x^{-1}K)$ and $\overline{A}(xK) \geq \overline{A}(x^{-1}K)$.

Therefore, $\overline{A}(x^{-1}K) = \overline{A}(xK)$, for all $x$ in $G$.

Now, replace $y$ by $y^{-1}$, then
\[ \bar{A}(xyK) = \bar{A}(x(y^{-1})^{-1}K) \]
\[ \leq \bar{A}(xK) \lor \bar{A}(y^{-1}K) \]
\[ = \bar{A}(xK) \lor \bar{A}(yK), \text{ for all } x \text{ and } y \text{ in } G. \]

Hence, \( \bar{A} \) is a multi anti L-fuzzy quotient group of a group \( \mathbb{G}/K \).

3.6 Theorem:

If \( \bar{A} \) and \( \bar{B} \) are two multi anti L-fuzzy quotient groups of a group \( \mathbb{G}/K \), then \( \bar{A} \cap \bar{B} \) is a multi L-fuzzy quotient group of \( \mathbb{G}/K \).

Proof:

It is trivial.

Remark:

The intersection of a family of multi anti L-fuzzy quotient groups of a group \( \mathbb{G}/K \), is a multi anti L-fuzzy quotient group of a group \( \mathbb{G}/K \).

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES


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