TO SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULAR POINTS BY ADOMIAN DECOMPOSITION METHOD

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Abstract. The Adomin Decomposition Method (ADM) is used to solve differential equations, so in this study we used (ADM) to solve second order ordinary differential equations with singular initial value problem then, the equation was given a generalization.

1. INTRODUCTION

Second-order ordinary differential equations are one of the most widely studied classes of differential equations in mathematics, physical science, and engineering [5]. Ordinary differential equations with singular points may have solutions which are not analytic at those points, so series solution might not exist there [6]. This is because the solution may not be analytic at point and hence without having a series expansions about the point. In stead, we must use a more general series expansions. A differential equations may only have few singular points, but solution behavior near these singular points is important. The Adomian decomposition method has been paid much attention in the recent years in applied mathematics, and in the field of

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series solution particularly. Moreover, it is a fact that this method is powerful, effective, as well easily solves many types of linear or nonlinear ordinary or partial differential equations, and integral equations [8, 2, 3, 4]. Many researchers are used this method to solve many kind of the differential equations such as Emden Flower Equation [7], first order ordinary differential equation [8]. We suppose the second ordinary differential equation with singular points as form:

$$(1) \qquad y'' + (2 + \frac{2}{x})y' + (1 + \frac{2}{x})y = z(x, y).$$

With initial conditions $y(0) = A, y'(0) = B$, where A, B are constants and $z(x, y)$ is known function.

2. DESCRIBE THE USER'S WAY

The equation (1) can be written as follow :

$$(2) \qquad Ly = z(x, y),$$

where

$$(3) \qquad L(.) = x^{-1} e^{-x} \frac{d^2}{dx^2} x e^x (.).$$

And we have the inverse operator L^{-1}

$$(4) \qquad L^{-1}(.) = x^{-1} e^{-x} \int_0^x \int_0^x x e^x (.) dx dx.$$

when we take L^{-1} for both sides of equation (2) we get

$$(5) \qquad y = \phi(x) + L^{-1} z(x, y),$$

with conditions

$$y(0) = A, y'(0) = B.$$

The (ADM) is given the solution as series

$$(6) \qquad y(x) = \sum_0^\infty y_n(x),$$

and

$$(7) \qquad z(x, y) = \sum_0^\infty A_n,$$

the Adomian polynomials A_n are first constructed by Adomian, it gives general formula to determine the values of A_n which gives the terms as:

$$A_0 = V(y_0),$$

$$A_1 = y_1 V'(y_0),$$

$$A_2 = y_2 V'(y_0) + \frac{1}{2!} y^2 V''(y_0), \dots$$

Now in compensation (6) and (7) in to (5), we get

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) + L^{-1} \sum_{n=0}^{\infty} A_n,$$

the $y_n(x)$ can by found as following :

(8)
$$y_0 = \phi(x)$$

$$y_{n+1} = -L^{-1} A_n, n \geq 0,$$

which gives

$$y_0 = \phi(x),$$

$$y_1 = -L^{-1} A_0,$$

$$y_2 = -L^{-1} A_1,$$

$$y_3 = -L^{-1} A_2.$$

[4, 2].

Example(1): Consider the following equation

$$y'' + (2 + \frac{2}{x})y' + (1 + \frac{2}{x})y = x^2 + 6x + 6 + e^{x^2} - e^y,$$

$y(0) = 0$ and $y'(0) = 0$.

Now the equation write it as

$$Ly = x^2 + 6x + 6 + e^{x^2} - e^y,$$

when we take L^{-1} to the last equation we get

$$y = \phi(x) + L^{-1}(x^2 + 6x + 6 + e^{x^2}) - L^{-1}(e^y),$$

and the $\phi(x) = 0$. Now the first value of y is

$$y_0(x) = L^{-1}(x^2 + 6x + 6 + e^{x^2}),$$

the non-linear part is

$$y_{n+1}(x) = -L^{-1}(A_n), n \geq 0.$$

The Adomian polynomials for non-linear part e^y are

$$A_0 = e^{y_o},$$

$$A_1 = y_1 e^{y_0},$$

Now we give the first terms

(9)
$$y_0 = \frac{7x^2}{6} - \frac{x^3}{12} + \frac{3x^4}{40} - \frac{x^5}{45},$$

(10)
$$y_1 = -L^{-1}(A_0) = -\frac{x^2}{6} + \frac{x^3}{12} - \frac{x^4}{12} - \frac{x^5}{36},$$

(11)
$$y_2 = -L^{-1}(A_1) = \frac{x^4}{120} - \frac{x^5}{180},$$

from(9), (10) and (11) we obtain the solution in a series form

(12)
$$y(x) = y_0 + y_1 + y_2 = x^2 - \frac{x^4}{60} + \frac{x^5}{90}.$$

Table: The comparision between exact solution $y(x) = x^2$ and ADM.

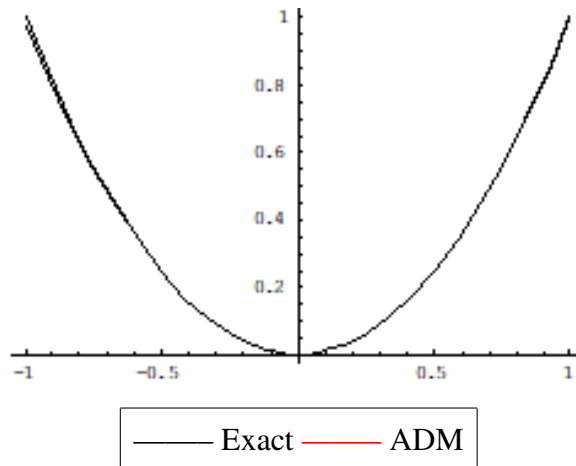| x	Exact	ADM	Absolut error
0.0	0.000	0.00000000	0.00000000
0.1	0.01	0.00999844	0.00000156
0.2	0.04	0.0399769	0.0000231
0.3	0.09	0.089892	0.000108
0.4	0.16	0.159687	0.000313
0.5	0.25	0.249306	0.000694
0.6	0.36	0.350704	0.009296
0.7	0.49	0.487866	0.002134
0.8	0.64	0.636814	0.003186
0.9	0.81	0.805626	0.004374
1.0	0.1	0.994444	0.005556

——— Exact ——— ADM

FIGURE 1. The exact solution $y = x^2$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$.

3. GENERALIZATION

In this section, we will genralize equation (1) to the following form

(13)
$$\sum_{r=0}^{n} \binom{n}{r}\left(\frac{r}{x}+1\right)y^{(n-r)} = z(x,y).$$

With initial conditions $y(0) = A, y'(0) = B, y''(0) = C, ... y^{(n)}(0) = D$. Where A, B, C, D are constants and $z(x,y)$ is a known function.

3.1. Theorem: We have the equation (13) if $m \in N$

$$x^{-1}e^{-x}\frac{d^m}{dx^m}xe^x(y) = \sum_{r=0}^{m}\binom{m}{r}(\frac{r}{x}+1)y^{(m-r)}.$$

Proof : We prove that by using mathematical induction :

When $m = 1$ then the equation is written as

$$x^{-1}e^{-x}\frac{d}{dx}xe^x(y) = \sum_{r=0}^{1}\binom{1}{r}(\frac{r}{x}+1)y^{(1-r)},$$

then both sides give the same equations

$$y' + (1+\frac{1}{x})y = y' + (1+\frac{1}{x})y,$$

then the equation is hold.

Now we must prove the following formula

$$x^{-1}e^{-x}\frac{d^{m+1}}{dx^{m+1}}xe^x(y) = \sum_{r=0}^{m+1}\binom{m+1}{r}(\frac{r}{x}+1)y^{(m+1-r)}.$$

Suppose that

$$x^{-1}e^{-x}\frac{d^m}{dx^m}xe^x(y) = \sum_{r=0}^{m}\binom{m}{r}(\frac{r}{x}+1)y^{(m-r)},$$

then

$$x^{-1}e^{-x}\frac{d^{m+1}}{dx^{m+1}}xe^x(y) = x^{-1}e^{-x}\frac{d^m}{dx^m}(e^x y + xe^x y + xe^x y'),$$

$$= x^{-1}e^{-x}\frac{d^m}{dx^m}(e^x y) + x^{-1}e^{-x}\frac{d^m}{dx^m}(xe^x y) + x^{-1}e^{-x}\frac{d^m}{dx^m}(xe^x y'),$$

$$= \frac{1}{x}\sum_{r=0}^{m}\binom{m}{r}y^{(m-r)} + \sum_{r=0}^{m}\binom{m}{r}(\frac{r}{x}+1)y^{(m-r)} + \sum_{r=0}^{m}\binom{m}{r}(\frac{r}{x}+1)y^{(m+1-r)},$$

$$= \frac{1}{x}\sum_{r=1}^{m}\binom{m}{r-1}y^{(m+1-r)} + \frac{(-1)}{x}\sum_{r=1}^{m}\binom{m}{r-1}y^{(m+1-r)} + \sum_{r=1}^{m}\binom{m}{r-1}(\frac{r}{x}+1)y^{(m+1-r)}$$

$$+y^{(m+1)} + \sum_{r=1}^{m}\binom{m}{r}(\frac{r}{x}+1)y^{(m+1-r)}$$

,

$$= y^{(m+1)} + \sum_{r=1}^{m}[\binom{m}{r-1}+\binom{m}{r}](\frac{r}{x}+1)y^{(m+1-r)},$$

$$= y^{(m+1)}) + \sum_{r=1}^{m}\binom{m+1}{r}(\frac{r}{x}+1)y^{(m+1-r)},$$

$$= \sum_{r=0}^{m}\binom{m+1}{r}(\frac{r}{x}+1)y^{(m+1-r)}.$$

Therefore

$$x^{-1}e^{-x}\frac{d^{m+1}}{dx^{m+1}}xe^x(y) = \sum_{r=0}^{m}\binom{m+1}{r}(\frac{r}{x}+1)y^{(m+1-r)}.$$

4. DESCRIBE THE USER'S WAY

The equation (13) can be written as follow:

(14)
$$Ly = z(x,y),$$

by using differential operator

(15)
$$L(.) = x^{-1}e^{-x}\frac{d^n}{dx^n}xe^x(.).$$

And we have the inverse operator L^{-1}

(16)
$$L^{-1}(.) = x^{-1}e^{-x}\int_0^x\int_0^x\int_0^x\dots\int_0^x xe^x(.)dxdxdx...dx.$$

when we take L^{-1} for both sides of equation (15) we get

(17)
$$y = \phi(x) + L^{-1}z(x,y).$$

[4, 2]

Example(2). Consider the following equation

$$y'''' + (4+\frac{4}{x})y''' + (6+\frac{12}{x})y'' + (4+\frac{12}{x})y' + (1+\frac{4}{x}y) = \frac{16e^x}{x}(2+lny).$$

And the condtions $y(0) = 1, y'(0) = 1$ and $y''(0) = 1$.

The equation can be written as

$$Ly = \frac{16e^x}{x}(2+lny).$$

When we take L^{-1} to both sides we get

$$y = \phi(x) + L^{-1}(\frac{16e^x}{x}(2+lny)).$$

And $\phi(x) = e^{-x}(1+2x+2x^2)$. We obtain

$$y_0(x) = \phi(x) + L^{-1}(\frac{32e^x}{x})$$

$$y_{n+1}(x) = -L^{-1}(A_n), n \geq 0.$$

The Adomian polynomials for non-linear part $Z(x,y) = \frac{16e^x}{x} Ln(y)$ are

$$A_0 = \frac{16e^x}{x} Ln(y_0),$$

$$A_1 = \frac{y_1}{y_0}\left(\frac{16e^x}{x}\right),$$

which gives the first term

(18) $$y_0 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{11x^4}{120} + \frac{19x^5}{360} - \frac{73x^6}{5040} + \frac{17x^7}{5040},$$

(19) $$y_1 = -L^{-1}(A_0) = \frac{2x^4}{15} - \frac{2x^5}{45},$$

(20) $$y_2 = -L^{-1}(A_1) = \frac{2x^7}{1575},$$

from(18), (19) and (20) we obtain the solution in a series form

(21) $$y(x) = y_0 + y_1 + y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{73x^6}{5040} + \frac{13x^7}{2800}.$$

Table: The comparision between exact solution $y(x) = e^x$ and ADM .

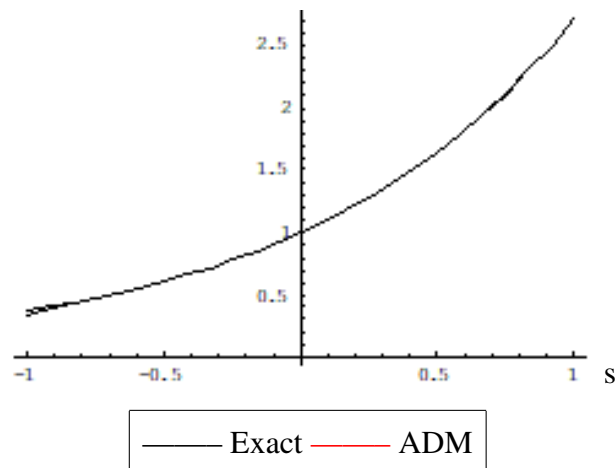| x	Exact	ADM	Absolut error
0.0	0.000	0.000000000	0.000000000
0.1	1.10517	1.10517	0.00000
0.2	1.2214	1.2214	0.0000
0.3	1.34986	1.34985	0.00001
0.4	1.49182	1.49177	0.000005
0.5	1.64872	1.64851	0.00021
0.6	1.82212	1.8215	0.00062
0.7	2.01375	2.01225	0.0015
0.8	2.22554	2.22231	0.00323
0.9	2.4596	2.45328	0.00632

FIGURE 2. The exact solution $y = e^x$ and the ADM solution $y = \sum_{n=0}^{2} y_n(x)$.

5. CONCLUSION

In this work, we noticed the easy way for finding the approximate solutions to exact solutions, and we found its generalization and proved the generalization by using mathematical induction. we discussed some examples to understand the method.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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