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SYMMETRY CONSIDERATIONS FOR DIFFERENTIAL EQUATION FORMULATIONS FROM CLASSICAL AND FRACTIONAL LAGRANGIANS

UCHECHUKWU M. OPARA[1,2,*], FESTUS I. ARUNAYE[2]

[1]Department of Pure & Applied Physics, Veritas University, Abuja, Nigeria

[2]Department of Mathematics, Delta State University (DELSU), Abraka—Delta State, Nigeria

Abstract. The utility of Noether's classical theorem on differential equations extended to a generalized non-classical theorem is the focus of this paper. After addressing a couple of standard related Partial Differential Equation (P.D.E.) formulations from classical Lagrangians, it culminates into a non-classical formulation of the diffusion equation in one spatial dimension from a fractional Lagrangian. Comparisons and contrasts between techniques for the classical and fractional formulations, as done here, facilitate the basic computational methods required for building analytical results. A noteworthy interface between Distribution theory, Trace theory and Lie symmetry theory is a key point of interest in this study.

Keywords: calculus of variations; classical P.D.E's; fractional Lagrangians; infinitesimal symmetries; diffusion equation.

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1. Introduction

Optimization of regular functionals on Banach spaces leads to formulation settings for differential equations, from which symmetry considerations can often reveal much about possible

*Corresponding author

E-mail address: ucmiop@yahoo.com

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solutions. This observation is particularly useful in physical systems governed by such differential equations, as the qualitative and quantitative properties of equilibrium states of these systems can be inferred from admitted symmetry groups and the associated group invariant solutions. Much classical work has been dedicated to this profound concept, with Noether's theorem at the hub of its efficacy and development. Indeed, the concern about which differential equations can be formulated via optimization in the calculus of variations, remains very cogent in several modern computational scientific endeavors. Fractional calculus, being required for extrapolation to generalized non-classical versions of Noether's theorem, has paved the way for inclusion of 'conservation laws' for certain dissipative systems. Needless to mention, this has immensely expanded the utility of Noether's theorem in extensive analysis of differential equations, particularly in the 21st century. In this study, we shall visit the formulation of Laplace's equation and Poisson's equation, the latter which is often regarded as the static version of the diffusion equation. The transition across these formulations is a vital concept highlighted in [4] in appreciable detail. In conjunction with modern methods from the fractional calculus of variations and some key classical methods summarized in the reference by Olver, substantive symmetry considerations of the diffusion equation are examined from the vantage point of its fractional Lagrangian. Reconciliation between partial differential equations that can be formulated from the calculus of variations and those equations which possibly admit infinitesimal symmetries is pointed out as a vital link being sought.

2. Formulation of Laplace's Equation and Poisson's Equation

Classical related equations and their formulation techniques shall be elucidated in this section. We may begin with the task of optimizing a given functional defined on an improper subset of some Banach functional space. In this event, we have the optimization theorem below as a guarantor of existence of an extremal, given fulfillment of the included criteria.

Optimization Theorem (**1**) [[3], pp.198] - Let E be a real reflexive Banach space, and the functional $f : E \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semi-continuous and proper. Then,

i.) for any non-empty $K \subset E$ that is weakly compact (closed, convex and norm-bounded) $\exists\, \overline{v} \in K$ such that $f(\overline{v}) = \min_{v \in K} f(v)$;

ii.) if in addition f is coercive, $\exists\, \overline{v} \in E$ such that $f(\overline{v}) = \underset{v \in E}{min} f(v)$.
If in addition f is strictly convex, then the minimizer $\overline{v}$ would exist uniquely in either case mentioned above.

There are certain settings whereby this elegant optimization theorem is not quite relevant, such as when the Banach space of reference cannot be made reflexive without losing other crucial properties (such as continuous differentiability) of the test functions and candidates for the optimal solution $\overline{v}$. Moreover, although this theorem can be tweaked by negating f to speak about local maxima, it does not reckon with criteria for seeking saddle points, which are evident legitimate critical points. By differentiation in the Banach spaces of reference and appropriate incorporation of the fundamental lemma of the calculus of variations, we are able to formulate differential equations which any solution $\overline{v}$ must satisfy [8]. There is providence for a reverse check via the Lax-Milgram theorem [[2], pp.140] that a solution to the formulated differential equation is indeed an extremal of the function f in the above given optimization theorem (1).
Lax-Milgram Theorem (**2**)*:* Assume that $a(u,v)$ is a continuous and coercive bilinear form on a Hilbert Space H. Then given any $\varphi \in H^*$, there exists a unique $u \in H$ such that

$$a(u,v) = \langle \varphi, v \rangle \quad \forall v \in H.$$

Moreover, if a is symmetric, then u is characterized by

$$\frac{1}{2}a(u,u) - \langle \varphi, u \rangle = \underset{v \in H}{min} \left\{ \frac{1}{2}a(v,v) - \langle \varphi, v \rangle \right\}.$$

In this event, we may begin with the boundary-constrained differential equation and then attempt to identify an associated optimization problem which its weak solution(s) must solve. Hence, the Lax-Milgram theorem is a means to identify certain PDE's which can be formulated from optimization via the calculus of variations, with aid of standard multivariate integration formulas. As for equations of classically dissipative systems that are constructed from fractional Lagrangians, this particular scheme is not sufficient.

Intricacies of determination of admitted infinitesimal symmetries either starting from infinitesimal criteria of the PDE or the Lagrangian of its associated optimization problem shall be compared for the equations of choice here in due course. It is worthy of note that although sufficiently regular solutions of class $C^k : k \in \mathbb{N}$ are often desired, the peculiarities of structures available in reflexive functional spaces compel us to first formulate equations weakly in larger Sobolev spaces ($W^{k,p}(\Omega) : k \in \mathbb{N}, p \geq 1$). Eventually, after establishing (unique) existence of weak solutions, we may check whether weak solutions are sufficiently regular as desirable. These are the rigors of classically identified procedures for confirming existence and uniqueness of solutions to an appreciable class of partial differential equations. Hence, before even reckoning with explicit solution techniques, the links to variational problems already become evident in core P.D.E. analysis.

Formulation of Laplace's Equation: Now, Laplace's equation is famously formulated via optimization of Dirichlet's energy functional. The Dirichlet energy functional on Ω is given by

$$\begin{array}{rccc} f: & K \subset C^1(\overline{\Omega}) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \displaystyle\int_\Omega ||\nabla v||^2 \, d\mu \end{array}$$

for $\Omega \subset \mathbb{R}^n$ an open, bounded set with C^1 topological boundary, where

$$K = \{v \in C^1(\overline{\Omega}) : v|_{\partial\Omega} = h\}$$

and h is a particular differentiable function defined on the compact set $\partial\Omega$.

The domain K' of f in the weak setting is the pre-image of the singleton $h \in L^2(\partial\Omega)$ under the continuous trace operator;

$$\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$$

so K' is (norm-) closed in $W^{1,2}(\Omega) := H^1(\Omega)$. Moreover, the set K' is convex because $\lambda u + (1-\lambda)v \in K' \quad \forall \lambda \in [0,1]$ and every $u, v \in K'$. The functional f is continuous, coercive and strictly convex on K'. For any function v_0 in K', it is easy to check that the set $B = \{v \in K' : f(v) \leq f(v_0)\}$ is bounded due to the coercivity of f, giving us existence of a minimizer for f on B and thus also on K'. The critical function $\bar{v}$ will exist uniquely due to strict convexity of f [8].

Given the minimizer $\overline{v} \in H^1(\Omega)$, the weak formulation for this problem is the following boundary value PDE:

$$\left.\begin{array}{c} \sum_{i=1}^{n} \frac{\partial^2 \overline{v}}{\partial x_i^2} = 0 \text{ in } \Omega; \\ \overline{v} = h \text{ on } \partial\Omega \end{array}\right\} \quad \cdots (3).$$

Let $x = (x_i)_{i=1}^{n}$ be the coordinates on Ω and $d\mu$ the volume element on Ω. Summarily, the technique of formulation of the associated differential equation via optimization of a sufficiently regular functional

$$v \mapsto \int_{\Omega} F(x, v, \nabla v) d\mu \cdots (4)$$

on a subset of $L^1(\Omega)$ reveals that

$$F_v(x, \overline{v}, \nabla\overline{v}) = div\ F_{\nabla v}(x, \overline{v}, \nabla\overline{v}) \cdots (5)$$

at any critical point $\overline{v}$ [8]. If Ω is an open subset of $\mathbb{R}^n$, then we may replace the divergence (div) operator above to express the formulation in (5) simply as

$$F_v(x, \overline{v}, \nabla\overline{v}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{v_{x_i}}(x, \overline{v}, \nabla\overline{v}) \ .$$

These are the multivariate Euler-Lagrange equations, and we have that any critical point $\overline{v}$ of (4) must be a solution of (5).

The symmetries admitted by the Laplace equation (3) are identified as the conformal Lie groups and they include the infinitesimal rotations, translations and scalings; which may be relevantly utilized to simplify or explicitly solve the equation provided they leave the boundary constraint $[\overline{v} = h \ \text{ on } \partial\Omega]$ of (3) invariant. More generic, engaging observations linking variational procedures and admitted symmetries in associated systems of differential equations shall be elucidated in each of the succeeding sections.

Formulation of Poisson's Equation: Let u be a particular element of $L^2(\Omega)$ for Ω in $\mathbb{R}^n$ an open, bounded set with C^1 topological boundary. In the weak setting, Poisson's equation:

$$\left.\begin{array}{c} \sum_{i=1}^{n} \frac{\partial^2 \overline{v}}{\partial x_i^2} = u \text{ in } \Omega; \\ \overline{v} = 0 \text{ on } \partial\Omega \end{array}\right\} \quad \cdots (6),$$

is formulated by optimizing the functional

$$\begin{array}{rccl} f: & H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \int_{\Omega}\left(\frac{1}{2}||\nabla v||^2 - u.v\right)d\mu. \end{array}$$

Arguments using the given optimization theorem (1) above yield existence of a unique minimizer. Moreover, engagement of the multivariate Euler-Lagrange equations in the Lagrangian reveals that any critical point $\bar{v}$ satisfies the Poisson equation: $\sum_{i=1}^{n}\frac{\partial^2\bar{v}}{\partial x_i^2} = u$ in Ω; formulated on the reflexive Hilbert space $H_0^1(\Omega)$.

3. Infinitesimal Symmetries Determined from Lagrangians

Consider a functional $I = \int_{\Omega} F(x,v,\nabla v)d\mu$, perturbed by one-parameter infinitesimal variations $(\tilde{x},\tilde{v})$ of the independent variables $(x_i)_{i=1}^n = x$ and the dependent variable v:

$$(\tilde{x},\tilde{v}) = (x,v) + \varepsilon(\xi,\eta).$$

The above variations also induce a variation of the gradient ∇v, which we denote

$$\tilde{v}_{\tilde{x}} = \nabla v + \varepsilon\eta^x + o(\varepsilon^2).$$

We require those variations which leave I invariant at its critical point(s) with respect to the parameter ε at $\varepsilon = 0$. This is to say, we require

$$\frac{d}{d\varepsilon}\left[\int_{\tilde{\Omega}} F[(x,v,\nabla v) + \varepsilon(\xi,\eta,\eta^x)]d\tilde{\mu}\right]|_{\varepsilon=0} = 0.$$

The higher order terms $o(\varepsilon^2)$ in the variation [$\tilde{v}_{\tilde{x}} = \nabla v + \varepsilon\eta^x + o(\varepsilon^2)$] are dropped because their derivatives evaluated at $\varepsilon = 0$ vanish, being a countable sum of zeros. Hence,

$$\begin{aligned} & \frac{d}{d\varepsilon}\left[\int_{\tilde{\Omega}} F[(x,v,\nabla v) + \varepsilon(\xi,\eta,\eta^x)]d\tilde{\mu}\right] \\ = & \int_{\tilde{\Omega}} \frac{d}{d\varepsilon}[F[(x,v,\nabla v) + \varepsilon(\xi,\eta,\eta^x)]d\tilde{\mu}] \\ = & \int_{\tilde{\Omega}}\left[(\xi.F_x + \eta.F_v + \eta^x.F_{\nabla v})d\tilde{\mu} + (F[(x,v,\nabla v) + \varepsilon(\xi,\eta,\eta^x)])\frac{d}{d\varepsilon}[d\tilde{\mu}]\right] \\ = & \int_{\tilde{\Omega}}\left[\left(\xi_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial v} + \eta^{x_i}\frac{\partial}{\partial v_{x_i}}\right)Fd\tilde{\mu} + (F[(x,v,\nabla v) + \varepsilon(\xi,\eta,\eta^x)])\frac{d}{d\varepsilon}[det[J]]d\mu\right]. \end{aligned}$$

In the above expression, $[J]$ is the Jacobian for the coordinate transformation from the system (x,v) to $(\tilde{x},\tilde{v})$, and the unperturbed Lagrangian is $F = F(x,v,\nabla v)$. According to Olver [[7], pp.254], we have

$$\frac{d}{d\varepsilon}[det[J]]|_{\varepsilon=0} = div(\xi) := \sum_{i=1}^{n} D_{x_i}\xi_i \ ,$$

which leads to the result:

$$\begin{aligned} & \frac{d}{d\varepsilon}\left[\int_{\tilde{\Omega}} F[(x,v,\nabla v)+\varepsilon(\xi,\eta,\eta^x)]d\tilde{\mu}\right]|_{\varepsilon=0} \\ = & \int_{\Omega}\left[\left(\xi_i\frac{\partial}{\partial x_i}+\eta\frac{\partial}{\partial v}+\eta^{x_i}\frac{\partial}{\partial v_{x_i}}\right)\circ F + F.div(\xi)\right]d\mu. \end{aligned}$$

For the above integral to give the desired zero result, we require a pointwise null result for the integrand:

$$\left(\xi_i\frac{\partial}{\partial x_i}+\eta\frac{\partial}{\partial v}+\eta^{x_i}\frac{\partial}{\partial v_{x_i}}\right)\circ F + F.div(\xi) \equiv 0 \ \cdots (7).$$

At this juncture, we reckon that the vector field $\left(\xi_i\frac{\partial}{\partial x_i}+\eta\frac{\partial}{\partial v}+\eta^{x_i}\frac{\partial}{\partial v_{x_i}}\right)$ is the first prolongation $[pr^{(1)}\mathbf{v}]$ of an infinitesimal symmetry $[\mathbf{v} = \xi_i\frac{\partial}{\partial x_i}+\eta\frac{\partial}{\partial v}]$ admitted by critical points of the functional I. Infinitesimal symmetries computed in this manner are identified as variational symmetries of I, with the prolongation vector coefficient(s) obtained as

$$\xi_i = \frac{d}{d\varepsilon}[\tilde{x}_i]|_{\varepsilon=0} \ , \ \eta = \frac{d}{d\varepsilon}[\tilde{v}]|_{\varepsilon=0} \ , \ \eta^{x_i} = \frac{d}{d\varepsilon}[\tilde{v}_{\tilde{x}_i}]|_{\varepsilon=0} \ .$$

All the above are consistent with the definitions in formally established literature, and the opportunity to reconcile infinitesimal symmetries with variational techniques on the Lagrangian from the first principles presents us with a viable platform for symmetry considerations for differential equations formulated from Lagrangians. It is also well-known that infinitesimal symmetries from Lagrangians are admitted by the associated Euler-Lagrange equations. The result in (7) is relevant for computing symmetries of both equations (3) and (6), being formulated from classical Lagrangians with maximum first-order derivatives. An extension of the formulation to Lagrangians with higher order integer derivatives can be naturally done to address differential equation formulations in other cases. For our scope, we eventually aim to extrapolate this technique to computing infinitesimal symmetries from fractional Lagrangians, in order to bring formulations for dissipative systems (such as the diffusion equation) into consideration. Before expounding this concept in further detail, it is needful to discuss the relevance of Noether's theorem in analysis of classical cases, with prospects for Noether's generalized theorem extended to fractional Lagrangians. The generalized theorem is a means to determine analogous conservation laws at play in classically dissipative systems, which creates a more robust framework to formulate previously elusive physical quantities such as heat and friction.

Noether's classical theorem (1918) reckons with the qualitative correspondence between symmetries and conservation laws in physical systems, as well as the quantitative correspondence in their associated differential equations. More specifically, this theorem states that a conservation law in variational mechanics follows whenever the Lagrangian function is invariant under a one-parameter continuous group of transformations, that transform dependent and/or independent variables [[6], pp.56]. In particular, if the integral in the Lagrangian is *nondegenerate*, there is a one-to-one correspondence between equivalence classes of nontrivial conservation laws of the Euler-Lagrange equations and equivalence classes of variational symmetries of the functional [[7], pp.334]. This theorem has been referenced ubiquitously since its establishment, but still remains indispensable in analysis of standard phenomena in general modern mechanics, such as energy and momentum. Other phenomena in variational mechanics such as friction, which are non-conservative (dissipative), may be addressed by formulations of the generalized theorem; reliant on transitioning from classical to fractional calculus techniques [[6], pp.25]. In quantitative terms, for a system of differential equations $\Delta(x,u^{(n)}) = 0$, a conservation law is simply defined as a divergence expression

$$Div\, P = 0$$

which vanishes for all solutions $u = f(x)$ of the given system. We have $P = (P_1(x,u^{(n)}),\cdots,P_m(x,u^{(n)}))$ as an m-tuple of smooth functions of x,u and the derivatives of u, while $Div\, P$ represents the total divergence. [[7], pp.261]

4. Formulation of the Diffusion Equation

The diffusion equation

$$u_t = u_{xx}\ ,$$

otherwise called the heat equation, is a common evolution equation describing a dissipative physical process. It is well-known that this equation cannot be formulated from a classical Lagrangian, owing partly to its non-conservative nature. However, the available access to non-classical fractional variational techniques provides us with a valuable platform to adapt this equation into the formulation mode of its static counterpart - the Poisson equation, as displayed previously. Before doing so, it is vital to go through a few details of requisite fractional calculus for fractional variational formulation.

Fundamental Tools of Fractional Calculus: In solutions $u(t,x)$, we differentiate fractionally with respect to time (t) in certain stages, while we maintain integer derivatives with respect to the spatial variable (x) all through. We hereby have to make reference to the Riemann-Liouville and Caputo fractional derivatives, as they relate to the Riemann-Liouville fractional Integral. The left Riemann-Liouville fractional integral of order α $(0<\alpha<1)$ of a function $u(t)$ with respect to t is given as:

$$ {}_aI_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}u(\tau)d\tau\ . $$

Note that Γ denotes the hypergeometric Gamma function, which extends the usual factorial from the natural numbers to $\mathbb{R}-(\mathbb{Z}^-\cup\{0\})$, such that

$$ \Gamma(n+1)=n!\ \ \forall n\in\mathbb{N}\cup\{0\};\quad \Gamma(p+1)=p\Gamma(p)\ \ \forall p\in\mathbb{R}-(\mathbb{Z}^-\cup\{0\})\ . $$

Hence, the left Riemann-Liouville fractional derivative of order α $(0<\alpha<1)$ of $u(t)$ with respect to t is given as:

$$ {}_aD_t^{\alpha}[u](t)=\frac{d}{dt}[{}_aI_t^{1-\alpha}u(t)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t\frac{u(\tau)}{(t-\tau)^{\alpha}}d\tau\ . $$

The operators ${}_aI_t^{1-\alpha}$ and $\frac{d}{dt}$ do not commute, and switching their order from the Riemann-Liouville definition to ${}_aI_t^{1-\alpha}\circ\frac{d}{dt}$ gives us the left Caputo fractional differential operator, denoted ${}_a^CD_t^{\alpha}$. Importantly, we have that

$$ {}_a^CD_t^{\alpha}[u](t)={}_aD_t^{\alpha}[u](t)-\frac{u(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)}\ . $$

As for the right fractional differential operators, we have the right Riemann-Liouville derivative of order α :

$$ {}_tD_b^{\alpha}[u](t)=-\frac{d}{dt}[{}_tI_b^{1-\alpha}u(t)]=\frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b\frac{u(\tau)}{(\tau-t)^{\alpha}}d\tau\ , $$

and the right Caputo derivative of order α:

$$ {}_t^C D_b^{\alpha}[u](t) = -{}_tI_b^{1-\alpha}\left[\frac{du}{dt}\right] = -{}_tD_b^{\alpha}[u](t) + \frac{u(b)}{(b-t)^{\alpha}\Gamma(1-\alpha)} \ . $$

We shall feature only left fractional differential operators in the Lagrangians of this study, since we only consider evolutions toward the future and not the past. In [4], there is an effort by the authors to motivate development of a doubled phase space (u_+, u_-) as an aspect of dynamical modeling of the future and past of an evolution process.

We shall impose the boundary value constraint $[u(0,x) = 0]$ a-priori in any case, so that the left Caputo and Riemann-Liouville derivatives coincide. We shall hence employ the more convenient notations $[D_-^{\alpha}u]$ and $[D_+^{\alpha}u]$ to stand for $[{}_0^C D_t^{\alpha}[u](t) = {}_0D_t^{\alpha}[u](t)]$ and $[{}_tD_b^{\alpha}[u](t)]$ respectively. With these observations, we have that the diffusion equation in one spatial dimension is formulated via optimization of the following fractional Lagrangian functional [4]:

$$ I = \int_{x_0}^{x_1}\int_0^b \left[(u_x)^2 - (D_-^{0.5}u)^2\right]\, dtdx \cdots (8). $$

We may attempt to assess optimality conditions of the functional in (8) by checking if the criteria in Theorem (1) are satisfied. Consider this functional to be set on a reflexive Sobolev space $W^{1,p}([0,b]\times[x_0,x_1]) :\ 1 < p < \infty$, wherein the constituent terms of its Lagrangian:

$$ u \mapsto (u_x)^2 \quad \text{and} \quad u \mapsto -(D_-^{0.5}u)^2 \ , $$

are continuous. However, this function lacks convexity and coercivity. Moreover, use of the Lax-Milgram theorem starting from the boundary-constrained diffusion equation still does not guarantee existence and uniqueness of solution, despite its adaptable relevance in coining the associated integral (8) to be optimized. In this event, we have to work in the Hilbert space $W^{1,2}([0,b]\times[x_0,x_1]) := H^1(\Omega)$. For every $v \in H^1(\Omega)$, we have that:

$$ u_{xx} = u_t \quad \Rightarrow \quad \int_{\Omega} u_{xx}.v = \int_{\Omega} u_t.v \quad \Rightarrow \quad \int_{\partial\Omega}\frac{\partial h}{\partial N} v d\sigma - \int_{\Omega} u_x.v_x = \int_{\Omega} u_t.v. $$

In the final implication above, we have engaged Green's integration formula with $[\ u|_{\partial\Omega} = h\ ]$ as the boundary constraint and N as the Gauss map on $\partial\Omega$. Reverting to the statement of Theorem (2), we have the continuous bilinear form $[a(u,v) = \int_{\Omega} u_x.v_x + \int_{\Omega} u_t.v]$ and the bounded linear form $\langle \varphi, v\rangle = \int_{\partial\Omega}\frac{\partial h}{\partial N} v d\sigma$, as crucial features. However, the form $a(u,v)$ here is neither coercive nor symmetric, thwarting prospects of the diffusion equation's link to optimization of a *classical* Lagrangian. Realization of the fractional Lagrangian functional (8) requires the fractional integration by parts formula [1], [5] to make the term $[\int_\Omega u_t.v]$ in the given expression for $a(u,v)$ into a symmetric form with fractional derivatives:

$$\int_\Omega u_t.v = \int_\Omega -[D_+^{0.5} \circ D_-^{0.5}](u).v = -\int_\Omega D_-^{0.5} u.D_-^{0.5} v,$$

while overlooking the trace boundary linear form. (Consequences of taking this latter action of dropping the boundary term are considered in the following 'Computational Results' section.) Lax-Milgram theorem is suited to establishing well posed-ness in elliptic PDE's but the diffusion equation is parabolic, which is why the theorems cited hitherto do not give definitive results on solution existence and uniqueness here, although they did for the prior elliptic examples of Laplace's equation and Poisson's equation. By all indications, the critical point(s) of the functional in (8) are saddle points instead of extremals, which is another deviation from the prior examples. To establish existence and uniqueness of solution to the diffusion equation, we have the version of Lax-Milgram theorem for parabolic P.D.E's, called J.L. Lion's theorem [[2], pp.340-341].

Speaking now of symmetries from the fractional Lagrangian, we have a similar setting to the classical case, outlined as follows. Consider a functional $I = \int_{x_0}^{x_1} \int_0^b F(t,x,u,D_t^\alpha u,u_x)\ dtdx$, perturbed by one-parameter infinitesimal variations $(\tilde{t},\tilde{x},\tilde{u})$ of the independent variables (t,x) and the dependent variable u:

$$(\tilde{t},\tilde{x},\tilde{u}) = (t,x,u) + \varepsilon(\tau,\xi,\eta)\ .$$

The above variations also induce variations of the fractional derivative $D_-^\alpha u := u_t^\alpha$ and the integer derivative u_x, which we respectively denote

$$D_{\tilde{t}}^\alpha \tilde{u} = D_t^\alpha u + \varepsilon \eta^{\alpha,t} + o(\varepsilon^2) \quad \text{and} \quad \tilde{u}_{\tilde{x}} = u_x + \varepsilon \eta^x + o(\varepsilon^2).$$

We require those variations which leave I invariant at its critical point(s) with respect to the parameter ε at $\varepsilon = 0$ [5]. This is to say, we require

$$\frac{d}{d\varepsilon}\left[\int_{\tilde{\Omega}} F[(t,x,u,D_t^\alpha u,u_x) + \varepsilon(\tau,\xi,\eta,\eta^{\alpha,t},\eta^x)]d\tilde{\mu}\right]|_{\varepsilon=0} = 0. \quad \text{Hence,}$$

$$\begin{aligned}
&\frac{d}{d\varepsilon}\left[\int_{\tilde{\Omega}} F[(t,x,u,D_t^{\alpha}u,u_x)+\varepsilon(\tau,\xi,\eta,\eta^{\alpha,t},\eta^{x})]d\tilde{\mu}\right] \\
=& \int_{\tilde{\Omega}} \frac{d}{d\varepsilon}\left[F[(t,x,u,D_t^{\alpha}u,u_x)+\varepsilon(\tau,\xi,\eta,\eta^{\alpha,t},\eta^{x})]d\tilde{\mu}\right] \\
=& \int_{\tilde{\Omega}} \left[(\tau.F_t+\xi.F_x+\eta.F_u+\eta^{\alpha,t}F_{u_t^{\alpha}}+\eta^{x}.F_{u_x})d\tilde{\mu}\right]+ \\
& \int_{\tilde{\Omega}} \left[F[(t,x,u,D_t^{\alpha}u,u_x)+\varepsilon(\tau,\xi,\eta,\eta^{\alpha,t},\eta^{x})]\frac{d}{d\varepsilon}[d\tilde{\mu}]\right] \\
=& \int_{\tilde{\Omega}} \left[(\tau.\frac{\partial}{\partial t}+\xi.\frac{\partial}{\partial x}+\eta.\frac{\partial}{\partial u}+\eta^{\alpha,t}.\frac{\partial}{\partial u_t^{\alpha}}+\eta^{x}.\frac{\partial}{\partial u_x})F d\tilde{\mu}\right]+ \\
& \int_{\tilde{\Omega}} \left[F[(t,x,u,D_t^{\alpha}u,u_x)+\varepsilon(\tau,\xi,\eta,\eta^{\alpha,t},\eta^{x})]\frac{d}{d\varepsilon}[det[J]]d\mu\right].
\end{aligned}$$

In the above expression, $[J]$ is the Jacobian for the coordinate transformation from the system (t,x,u) to $(\tilde{t},\tilde{x},\tilde{u})$, and the unperturbed Lagrangian is $F = F(t,x,u,D_t^{\alpha}u,u_x)$. As such, we have the result:

$$\begin{aligned}
&\frac{d}{d\varepsilon}\left[\int_{\tilde{\Omega}} F[(t,x,u,D_t^{\alpha}u,u_x)+\varepsilon(\tau,\xi,\eta,\eta^{\alpha,t},\eta^{x})]d\tilde{\mu}\right]|_{\varepsilon=0} \\
=& \int_{x_0}^{x_1}\int_0^b \left[(\tau.\frac{\partial}{\partial t}+\xi.\frac{\partial}{\partial x}+\eta.\frac{\partial}{\partial u}+\eta^{\alpha,t}.\frac{\partial}{\partial u_t^{\alpha}}+\eta^{x}.\frac{\partial}{\partial u_x})F+F.div(\tau,\xi)\right] dt\wedge dx.
\end{aligned}$$

For the above integral to give the desired zero result, we require a pointwise null result for the integrand:

$$\left(\tau.\frac{\partial}{\partial t}+\xi.\frac{\partial}{\partial x}+\eta.\frac{\partial}{\partial u}+\eta^{\alpha,t}.\frac{\partial}{\partial u_t^{\alpha}}+\eta^{x}.\frac{\partial}{\partial u_x}\right)\circ F+F.div(\tau,\xi)\equiv 0\cdots(\mathbf{9})$$

Here, the vector field $\left(\tau.\frac{\partial}{\partial t}+\xi.\frac{\partial}{\partial x}+\eta.\frac{\partial}{\partial u}+\eta^{\alpha,t}.\frac{\partial}{\partial u_t^{\alpha}}+\eta^{x}.\frac{\partial}{\partial u_x}\right)$ is the fractional prolongation $[pr^{(\alpha,1)}\mathbf{v}]$ of an infinitesimal symmetry $[\mathbf{v}=\tau\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial u}]$ admitted by critical points of the functional I. The fractional prolongation coefficient in the above vector field $[\eta^{\alpha,t}]$ is obtained via the generalized chain and Leibniz rules of fractional calculus [9]. Its explicit expression relevant in our formulation for $[\alpha=0.5]$ is hereby given as follows:

$$\begin{aligned}
\eta^{0.5,t} =\;& \partial_t^{0.5}\eta+(\eta_u-\tfrac{1}{2}D_t\tau)\partial_t^{0.5}u-u\partial_t^{0.5}(\eta_u)+\mu-\sum_{k=1}^{\infty}\begin{pmatrix}0.5\\k\end{pmatrix}D_t^k\xi.\partial_t^{0.5-k}(u_x) \\
&+\sum_{k=1}^{\infty}\left[\begin{pmatrix}0.5\\k\end{pmatrix}\partial_t^k(\eta_u)-\begin{pmatrix}0.5\\k+1\end{pmatrix}D_t^{k+1}\tau\right]\partial_t^{0.5-k}u\;,
\end{aligned}$$

where μ in the above expression is given as

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\begin{pmatrix} 0.5 \\ n \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix}\begin{pmatrix} k \\ r \end{pmatrix}\frac{1}{k!}\cdot\frac{t^{n-0.5}(-u)^r}{\Gamma(n+0.5)}\cdot\frac{\partial^m(u^{k-r})}{\partial t^m}\cdot\frac{\partial^{n-m}}{\partial t^{n-m}}\left(\frac{\partial^k\eta}{\partial u^k}\right) .$$

We refer the reader to [10] for details of how the above coefficient is determined.

5. Computational Results

Importantly, we reckon that each variational symmetry must also be a symmetry of the Euler-Lagrange equations. For this reason, the set of simultaneous equations determined while observing the infinitesimal criterion

$$pr(\mathbf{v})[E] = 0 \text{ whenever } E = 0;$$

for the Euler-Lagrange equations $[E = 0]$ must also be valid for the equations of the variational symmetries (**7**) and (**9**): for the classical and fractional Lagrangian cases above respectively. In any event, an Euler-Lagrange equation is a formulated P.D.E without imposed boundary value constraints. For the Poisson equation, we find by computation that if the P.D.E:

$$\sum_{i=1}^{n}\frac{\partial^2\overline{v}}{\partial x_i^2} = u \text{ in } \Omega ,$$

admits an infinitesimal symmetry $[\mathbf{v} = \xi_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial v}]$ then

$$pr^{(1)}\mathbf{v}[F_1] + F_1(D_x\xi + D_t\tau) + k.F_1 = 0 ,$$

where $[F_1 = \frac{1}{2}||\nabla v||^2 - u.v]$ is the integrand of the Lagrangian associated to the Poisson equation, and the arbitrary constant k is $\frac{\partial\eta}{\partial v}$ for this case. At first glance, notice that the term $k.F_1$ above appears to be anomalous to the infinitesimal criterion (**7**) derived previously, while beginning symmetry computations from the Lagrangian.

For the diffusion equation, we find by way of computation that if $[u_t = u_{xx}]$ admits an infinitesimal symmetry $[\mathbf{v} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}]$ then

$$pr^{(0.5,1)}\mathbf{v}\,[F_2] + F_2.div(\tau,\xi) + Q(x).F_2 = 0 ,$$

where $[F_2 = (u_x)^2 - (D_-^{0.5}u)^2]$ is the integrand of the fractional Lagrangian (**8**) associated to the diffusion equation, and the coefficient in the apparently anomalous term $[Q(x).F_2]$ for this case is given as

$$Q(x) = 2\eta_u + \xi_x.$$

Terminating our findings at this juncture would tend to suggest dearth of variational symmetries for each case. However, further considerations from the proposition below suggest a possible total or partial correspondence between existing variational symmetries and symmetries of the Euler-Lagrange equations, given appropriate imposition of the trace boundary constraints.

As the proposition, a compelling cue from the above computational results is that the boundary value constraints of well-posed P.D.E's must be suitably incorporated in the associated Lagrangian functionals in order to fully realize the correspondence between variational symmetries and symmetries of the Euler-Lagrange equations. When attempting to obtain the associated optimization problems from the boundary value P.D.E's using the mechanism available in the stated Lax-Milgram theorem (and its alternative versions) in each case, there is always a trace boundary term, which none of the herein identified Lagrangians actually incorporate. It is feasible and agreeable to concisely incorporate these boundary terms without leaving a second integral in the Lagrangian by way of Stoke's theorem for manifolds of finite volume:

$$\int_{\partial\Omega} \rho = \int_{\Omega} d\rho \ ,$$

for an n-dimensional open set Ω and an $(n-1)$ differential form ρ. We have to also assume relevance of the given Stoke's theorem for weak exterior derivatives of ρ instead of just the classically analytic functional cases, because existence and uniqueness theorems for P.D.E's cannot be extrapolated to spaces $C^{\infty}(\Omega)$. The formulation of Poisson's equation is worth being given a final thought at this point. Because its functional space of formulation is $H_0^1(\Omega)$, there should be zero trace contribution to the Lagrangian, when starting formulation from the Euler-Lagrange equation (that is, the P.D.E without imposed boundary value constraint). However, we must recall that we have only equivalence classes of functions in the setting of Sobolev spaces, meaning that $v \equiv 0$ almost everywhere on $\partial\Omega$ in (**6**). Taking the weak exterior differential of the trace value under this consideration would then cause a non-zero contribution to the Lagrangian.

When imposed boundary values interact with the hypersurface $\partial\Omega$ appropriately, then we have a greater chance of realizing more variational symmetries in P.D.E's formulated from the calculus of variations. As such, the observations made in this paper present a prospective frontier for a meaningful and interesting interface between distribution theory, trace theory and infinitesimal symmetry theory.

6. CONCLUSION

The factors of existence and uniqueness of solution ('well posed-ness'), calculus compatible with ambient functional solution spaces in the weak setting, imposed trace boundary values, Euler-Lagrange and variational symmetries of P.D.E's and their qualitative scientific interpretations; are apparently tied together in the hereby included direction of mathematical research. Pertaining to the question of which equations can be formulated from the calculus of variations, an occasional answer is made available by engaging the Lax-Milgram theorem, as pointed out in the paper. The extension of this prospect from elliptic P.D.E's to parabolic ones via the modifications of J.L. Lion's theorem provides a handy expansion of the scope of this endeavour, just as the extrapolation from classical calculus of variations to fractional calculus of variations does. Considering the often profound scientific implications of availability of group invariant equilibrium states in variational mechanics and beyond, the addressed underlying analytical nuances in the requisite mathematical framework are vividly unearthed and formally documented.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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