Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:61
https://doi.org/10.28919/jmcs/7021
ISSN: 1927-5307

SOME RESULTS ON WEAKLY SEMI COMPATIBLE MAPPINGS IN FUZZY METRIC SPACE

V. SRINIVAS[1], K. SATYANNA[2,*]

[1]Department of Mathematics, University College of Science, Saifabad, Osmania University, Hyderbad, Telangana State, India

[2]Department of Mathematics, M.A.L.D. Govt. Degree College, Palamoor University, Mahaboobnagar,Telangana State, India

Abstract: The purpose of this paper is to generate two fixed point theorems in complete fuzzy metric space by using the concepts of weakly semi compatible mappings, sub sequentially continuous mappings and occasionally weakly compatible mappings. Further these results are validated by discussing suitable examples.

Keywords: self-mappings; sub sequentially continuous mappings; weakly semi compatible mappings; occasionally weakly compatible mappings; fuzzy metric space.

2010 AMS Subject Classification: 54H25, 47H10.

1. INTRODUCTION

The thirst area of present research in analysis is fixed point theory due to its novelty, innovation, usage, scope and application. The satiating topic of fixed point theory is the fuzzy set and was coined by Zadeh [1]. It is the cornerstone of fuzzy logic and this concept has numerous applications in neural network, stability process, mathematical programming, mathematical modelling, engineering science, medical field, game theory, decision making, genetics and image

*Corresponding author
E-mail address: satgjls@gmail.com
Received November 23, 2021

processing etc. In covid-19 situation many inferences, theories were made due to fuzzy logic like [2] and [3]. Kramosil and Michalek [4] defined fuzzy metric space and used this in the formation of Hausdorff topology. Further this notion was modified through continuous t-norm by George and Veeramani [5]. Afterwards some more refinements were made by Kaleva and Scikkala [6] in generating some fixed point results Further Singh and Jain [7] used the implicit relation to prove some of results in fuzzy metric space. The General conditions to establish fixed theorems we need commutativity, continuity and contraction or similar conditions. Chauhan et al. [8] coined the idea of compatible mappings in fuzzy metric space and established some theory regarding the generation of fixed points. Jungck and Rhodes [9] generalized the compatibility resulting in the formation of weakly compatible mappings and established some results. In fuzzy metric space, V. Srinivas and B.V. Reddy [10] presented six maps to establish fixed point theorem by employing the concepts of weakly compatible, semi- compatible and continuous t-norm. Some more theorems have been witnessed like [11], [12] and [13] in fuzzy metric space. In this paper we use the notion of weakly semi compatibility, sub sequentially continuous mappings and occasionally weakly compatible mappings and generate two results in fuzzy metric space without using the condition of continuity.

2. PRELIMINARIES

Definition 2.1 [10] A binary relation $*: [0,1] \to [0,1]$ is mentioned continuous t- norm if $*$ satisfying

- $*$ is associative
- $\mu * 1 = \mu \quad \forall \mu \in [0,1]$
- $*$ is commutative
- $*$ is continuous
- $\mu * \vartheta \leq \omega * \alpha$ whenever $\mu \leq \omega$ and $\vartheta \leq \alpha$, $\forall \mu, \vartheta, \omega, \alpha \in [0,1]$.

Example 2.2 If $*$ is defined by $\alpha * \beta = \min\{\alpha, \beta\}$ and $\alpha * \beta = \alpha\beta$ then $*$ is t- norm.

Definition 2.3 [10] A triplet $(\Omega, \mathcal{M}, *)$ is known as fuzzy metric space if Ω is an random set, $*$ is continuous t- norm and $\mathcal{M}$ is fuzzy set defined on $\Omega^2 \times (0, \infty)$ satisfying the following postulates $\forall \alpha, \beta, \gamma \in \Omega$ and $t_1, t_2 > 0$

- $\mathcal{M}(\alpha, \beta, 0) = 0$
- $\mathcal{M}(\alpha, \beta, t_1) = 1 \ \forall t_1 > 0$ if and only if $\alpha = \beta$
- $\mathcal{M}(\alpha, \beta, t_1) = \mathcal{M}(\beta, \alpha, t_1)$

- $\mathcal{M}(\alpha,\ \beta,\ t_1) * \mathcal{M}(\beta,\ \gamma,\ t_2) \leq \mathcal{M}(\alpha,\ \gamma,\ t_1 + t_2)$
- $\mathcal{M}(\alpha,\ \beta,\ .): [0,\ \infty) \rightarrow [0,1]$ continuous from left
- $\lim_{\eta\to\infty} \mathcal{M}(\alpha,\ \beta,\ t_1) = 1.$

Example 2.4 Let $(\Omega,\ \rho)$ be a metric space defined $\alpha * \beta = \min\{\alpha,\ \beta\}\ \forall\, \alpha,\ \beta \in \Omega$ and $t_2 > 0$, define

$$\mathcal{M}(\alpha,\ \beta,\ t_2) = \frac{t_2}{t_2 + \rho(\alpha,\ \beta)} \qquad (1).$$

Then $(\Omega,\ \mathcal{M},\ *)$ forms fuzzy metric space. Also it is mentioned as fuzzy metric space $\mathcal{M}$ induced by metric ρ or stated standard fuzzy metric. This shows that each metric generates a fuzzy metric but there exists no metric on Ω satisfying (1).

Definition 2.5 [10] In fuzzy metric space $(\Omega,\ \mathcal{M},\ *)$ a sequence (μ_m)

- **converges** to a some point $\mu \in \Omega$ if $\lim_{m\to\infty} \mathcal{M}(\mu_m,\ \mu,\ t_1) = 1\ \forall\ t_1 > 0.$
- **Cauchy** if $\lim_{m\to\infty} \mathcal{M}(\mu_m,\ \mu_{m+q},\ t_1) = 1\ \forall\ t_1 > 0$ and $q > 0.$
- Further fuzzy metric space $(\Omega,\ \mathcal{M},\ *)$ is **complete** if every cauchy sequence converges in Ω.

Lemma 2.6 Let $(\Omega,\ \mathcal{M},\ *)$ be a fuzzy metric space if $\exists\ k \in (0,1)$ in order for $\mathcal{M}(\alpha,\ \beta,\ kt_2) \geq \mathcal{M}(\alpha,\ \beta,\ t_2)\ \forall\ t_2 > 0$ implies $\alpha = \beta$.

Definition 2.7 A pair of self-mappings $\varphi,\ \psi$ of a fuzzy metric space $(\Omega,\ \mathcal{M},\ *)$ is said to

- **compatible [7]** if $\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m,\ \psi\varphi\mu_m,\ t_\Delta) = 1,\ \forall\ t_\Delta > 0,$ whenever sequence

  (μ_m) in Ω such that $\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = \partial$ for some $\partial \in \Omega$.
- **Weakly compatible [7]** if they are commuting at their coincidence points.
- **Occasionally weakly compatible (OWC)[11]** if there is a coincidence point at which the mappings are commuting.
- **Semi compatible[7]** if $\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m,\ \psi\partial,\ t_\Delta) = 1,\ \forall\ t_\Delta > 0,$ whenever a sequence

  (μ_m) in Ω such that $\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = \partial$ for some $\partial \in \Omega$.
- **weakly semi compatible [13]** if

  $\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m,\ \psi\partial,\ t_\Delta) = 1$ or $\lim_{m\to\infty} \mathcal{M}(\psi\varphi\mu_m,\ \varphi\partial,\ t_\Delta) = 1\ \ \forall\ t_\Delta > 0,$ whenever

  sequence (μ_m) in Ω such that $\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = \partial$ for some $\partial \in \Omega$.

- **Sub sequentially continuous [11]** if

  $\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m,\ \varphi\partial,\ t_\Delta) = 1$ and $\lim_{m\to\infty} \mathcal{M}(\psi\varphi\mu_m,\ \psi\partial,\ t_\Delta) = 1\ \ \forall\ \ t_\Delta > 0$, for some sequence (μ_m) in Ω such that

  $\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = \partial$ for some $\partial \in \Omega$.

Remark: It can be noted that semi compatible pair of mappings imply weakly semi compatible but not conversely as can be observed under.

Example 2.8 Let $\Omega = [-\frac{\pi}{3},\ \frac{\pi}{3}]$ and $(\Omega,\ \mathcal{M},\ t_\Delta)$ be the induced fuzzy metric space with

$$\mathcal{M}(\alpha,\ \beta,\ t_\Delta) = \frac{t_\Delta}{t_\Delta + \rho(\alpha,\ \beta)}. \tag{2.8.1}$$

Define the mappings $\varphi\ ,\ \psi\ :\ \Omega\ \to \Omega$ as

$$\varphi(x) = sin^2 x,\ \ x \in [-\frac{\pi}{3},\ \frac{\pi}{3}], \tag{2.8.2}$$

$$\psi(x) = \begin{cases} x^2, & x \in \left[-\frac{\pi}{3},\ \frac{\pi}{3}\right] - \{0\} \\ \frac{\pi}{2}, & x = 0. \end{cases} \tag{2.8.3}$$

Consider $(\mu_m) = -\frac{2}{m}\ \forall\, m \geq 1$. Then from (2.8.2), (2.8.3)

$$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \varphi\left(-\frac{2}{m}\right) = \lim_{m\to\infty} \sin^2\left(-\frac{2}{m}\right) = 0, \tag{2.8.4}$$

$$\lim_{m\to\infty} \psi\mu_m = \lim_{m\to\infty} \psi\left(-\frac{2}{m}\right) = \lim_{m\to\infty} (-\frac{2}{m})^2 = \lim_{m\to\infty} \frac{4}{m^2} = 0. \tag{2.8.5}$$

From (2.8.4) and (2.8.5) $\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = 0$.

$$\lim_{m\to\infty} \varphi\psi\mu_m = \lim_{m\to\infty} \varphi\left(\frac{4}{m^2}\right) = \lim_{m\to\infty} \sin^2\left(\frac{4}{m^2}\right) = 0 \neq \frac{\pi}{2} = \psi(0). \tag{2.8.6}$$

$$\lim_{m\to\infty} \psi\varphi\mu_m = \lim_{m\to\infty} \psi\left(\sin^2\left(-\frac{2}{m}\right)\right) = \lim_{m\to\infty} \left(\sin^2\left(-\frac{2}{m}\right)\right)^2 = \varphi(0). \tag{2.8.7}$$

From (2.8.6) and (2.8.7)

$\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m,\ \psi(0),\ t_\Delta) \neq 1$ and $\lim_{m\to\infty} \mathcal{M}(\psi\varphi\mu_m,\ \varphi(0),\ t_\Delta) = 1\ \ \forall\ \ t_\Delta > 0$.

Thus the mappings $\varphi,\ \psi$ are weakly semi compatible but are not semi compatible.

Example 2.9 Let $(\Omega,\ \mathcal{M},\ t_\Delta)$ be the as in (2.8.1) where $\Omega = \mathbb{R}$.

Define the mappings $\varphi\ ,\ \psi\ :\ \Omega\ \to \Omega$ as

$$\varphi(x) = \begin{cases} (\frac{3}{2}+x)^2, & x < \frac{3}{2} \\ 5, & x \geq \frac{3}{2} \end{cases} \tag{2.9.1}$$

$$\psi(x) = \begin{cases} (\frac{3}{2}-x)^2, & x < \frac{3}{2} \\ 7, & x \geq \frac{3}{2} \end{cases} \tag{2.9.2}$$

consider a sequence $(\mu_m) = \frac{\sqrt{3}}{m} \ \forall\, m \geq 1.$ Then

$$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \varphi\left(\frac{\sqrt{3}}{m}\right) = \lim_{m\to\infty} (\frac{3}{2}+\frac{\sqrt{3}}{m})^2 = \frac{9}{4}, \tag{2.9.3}$$

$$\lim_{m\to\infty} \psi\mu_m = \lim_{m\to\infty} \psi\left(\frac{\sqrt{2}}{m}\right) = \lim_{m\to\infty} (\frac{3}{2}-\frac{\sqrt{3}}{m})^2 = \frac{9}{4}. \tag{2.9.4}$$

From (2.9.3) and (2.9.4)

$$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = \frac{9}{4},$$

$$\lim_{m\to\infty} \varphi\psi\mu_m = \lim_{m\to\infty} \varphi\left((\frac{3}{2}-\frac{\sqrt{3}}{m})^2\right) = \lim_{m\to\infty} 5 = 5 = \varphi\left(\frac{9}{4}\right), \tag{2.9.5}$$

$$\lim_{m\to\infty} \psi\varphi\mu_m = \lim_{m\to\infty} \psi\left((\frac{3}{2}+\frac{\sqrt{3}}{m})^2\right) = \lim_{m\to\infty} 7 = 7 = \psi\left(\frac{9}{4}\right). \tag{2.9.6}$$

From (2.9.5) and (2.9.6)

$$\lim_{m\to\infty} \mathcal{M}\left(\varphi\psi\mu_m, \varphi\left(\frac{9}{4}\right), t_\Delta\right) = 1 \text{ and } \lim_{m\to\infty} \mathcal{M}\left(\psi\varphi\mu_m, \psi\left(\frac{9}{4}\right), t_\Delta\right) = 1 \quad \forall \quad t_\Delta > 0.$$

Thus both the mappings φ, ψ are sub sequentially continuous but are not continuous at x = $\frac{3}{2}$.

Example 2.10 Let $(\Omega, \mathcal{M}, t_\Delta)$ be the as in (2.8.1) where $\Omega = \mathbb{R}$.

Define the mappings $\varphi , \psi : \Omega \to \Omega$ as

$$\varphi(x) = \begin{cases} x^2, & x < 3 \\ 7, & x = 3 \\ 11, & x > 3 \end{cases} \tag{2.10.1}$$

$$\psi(x) = \begin{cases} (\sqrt{3})x, & x < 3 \\ 9, & x = 3 \\ 7, & x > 3 \end{cases} \tag{2.10.2}$$

consider a sequence $(\mu_m) = \frac{\sqrt{3}}{m} \ \forall\, m \geq 1.$ Then from (2.10.1) and (2.10.2)

$$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \varphi\left(\frac{\sqrt{3}}{m}\right) = \lim_{m\to\infty} (\frac{\sqrt{3}}{m})^2 = 0. \tag{2.10.3}$$

$$\lim_{m\to\infty} \psi\mu_m = \lim_{m\to\infty} \psi\left(\frac{\sqrt{3}}{m}\right) = \lim_{m\to\infty} = (\sqrt{3})(\frac{\sqrt{3}}{m}) = 0. \tag{2.10.4}$$

From (2.10.3) and (2.10.4)

$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = 0.$

$\lim_{m\to\infty} \varphi\psi\mu_m = \lim_{m\to\infty} \varphi(\frac{3}{m}) = \lim_{m\to\infty} (\frac{3}{m})^2 = 0 = \varphi(0).$ (2.10.5)

$\lim_{m\to\infty} \psi\varphi\mu_m = \lim_{m\to\infty} \psi\left(\frac{3}{m^2}\right) = \lim_{m\to\infty} (\sqrt{3})\left(\frac{3}{m^2}\right) = 0 = \psi(0).$ (2.10.6)

From (2.10.5) and (2.10.6)

$\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m, \varphi(0), t_\Delta) = 1$ and $\lim_{m\to\infty} \mathcal{M}(\psi\varphi\mu_m, \psi(0), t_\Delta) = 1 \quad \forall \quad t_\Delta > 0,$ (2.10.7)

$\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m, \psi(0), t_\Delta) = 1$ and $\lim_{m\to\infty} \mathcal{M}(\psi\varphi\mu_m, \varphi(0), t_\Delta) = 1 \quad \forall \quad t_\Delta > 0.$ (2.10.8)

Thus from (2.10.7) the mappings φ, ψ are sub sequentially continuous.

Consider a sequence $(\mu_m) = \sqrt{3} + \frac{1}{m} \; \forall \, m \geq 1$. Then from (2.10.1) and (2.10.2)

$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \varphi\left(\sqrt{3} + \frac{1}{m}\right) = \lim_{m\to\infty} (\sqrt{3} + \frac{1}{m})^2 = 3,$ (2.10.9)

$\lim_{m\to\infty} \psi\mu_m = \lim_{m\to\infty} \psi\left(\sqrt{3} + \frac{1}{m} \right) = \lim_{m\to\infty} \sqrt{3}\,(\sqrt{3} + \frac{1}{m}) = 3.$ (2.10.10)

From (2.10.9) and (2.10.10)

$\lim_{m\to\infty} \varphi\mu_m = \lim_{m\to\infty} \psi\mu_m = 3.$ Then

$\lim_{m\to\infty} \varphi\psi\mu_m = \lim_{m\to\infty} \varphi\left(3 + \frac{\sqrt{3}}{m}\right) = \lim_{m\to\infty} 11 \neq 9 = \psi(3)$ and (2.10.11)

$\lim_{m\to\infty} \psi\varphi\mu_m = \lim_{m\to\infty} \psi\left((\sqrt{3} + \frac{1}{m})^2 \right) = \lim_{m\to\infty} 7 = 7 = \varphi(3).$ (2.10.12)

From (2.10.11) and (2.10.12)

$\lim_{m\to\infty} \mathcal{M}(\varphi\psi\mu_m, \psi(3), t_\Delta) \neq 1$ and $\lim_{m\to\infty} \mathcal{M}(\psi\varphi\mu_m, \varphi(3), t_\Delta) = 1 \quad \forall \quad t_\Delta > 0.$ (2.10.13)

The mappings have coincidence points at x = 0, $\sqrt{3}$.

At x = $\sqrt{3}$, $\varphi(\sqrt{3}) = 3 = \psi(\sqrt{3})$, $\psi\,\varphi(\sqrt{3}) = \psi(3) = 9$ and $\varphi\psi(\sqrt{3}) = \varphi(3) = 7$ implies $\varphi\psi(3) \neq \psi\varphi(3).$ (2.10.14)

Thus from (2.10.7), (2.10.13) and (2.10.14) the mappings φ, ψ are are sub sequentially continuous and weakly semi compatible but not weakly compatible. Resulting that sub sequentially continuous and weakly semi compatible mappings are weaker than weakly compatible mappings.

Example 2.11 Let $(\Omega,\ \mathcal{M},\ t_\Delta)$ be the as in (2.8.1) where $\Omega = \mathbb{R}$.

Define the mappings $\varphi\ ,\ \psi\ :\ \Omega \to \Omega$ as $\varphi(x) = 10^x,\ \psi(x) = 10^{x^2}\ \ \forall\, x \in \Omega$.

Then x = 0, 1 are intersecting points there the mappings $\varphi,\ \psi$ are coincidence.
At x = 0, $\varphi(0) = 1 = \psi(0),\ \psi\,\varphi(0) = \psi(1) = 10$ and $\varphi\psi(0) = \varphi(1) = 10$ implies
$\varphi\psi(0) = \psi\varphi(0)$. (2.11.1)
But at x = 1, $\varphi(1) = 10 = \psi(1),\ \psi\,\varphi(1) = \psi(10) = 10^{100}$ and
$\varphi\psi(1) = \varphi(10) = 10^{10}$ implies
$\varphi\psi(1) \neq \psi\varphi(1)$. (2.11.2)
From (2.11.1) and (2.11.2) we can assert as the mappings $\varphi\ ,\ \psi$ are occasionally weakly compatible but are not weakly compatible.

A class of Implicit relation 2.12[7]

Let Φ be collection of all real valued continuous functions $\emptyset : (\mathbb{R}^+)^4 \to \mathbb{R}$ non decreasing in the 1st argument satisfying

(2.12.1) $\emptyset\,(\alpha,\ \beta,\ \beta,\ \alpha) \geq 0$ or $\emptyset\,(\alpha,\ \beta,\ \alpha,\ \beta) \geq 0 \Rightarrow \alpha \geq \beta$.

(2.12.2) $\emptyset\,(\alpha,\ \alpha,\ 1,\ 1) \geq 0 \ \Rightarrow \alpha \geq 1$.

Example 2.13

Defining $\emptyset\,(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4) = 3\alpha_1 - \frac{13}{5}\alpha_2 + \alpha_3 - \frac{7}{5}\alpha_4$ .Then

(i) $\emptyset\,(\alpha_1,\ \beta,\ \beta,\ \alpha_1) \geq 0 \Rightarrow 3\alpha_1 - \frac{13}{5}\beta + \beta - \frac{7}{5}\alpha_1 \geq 0 \Rightarrow \frac{8}{5}\alpha_1 - \frac{8}{5}\beta \geq 0 \Rightarrow \alpha_1 \geq \beta$.

(ii) $\emptyset\,(\alpha_1,\ \beta,\ \alpha_1,\ \beta) \geq 0 \Rightarrow 3\alpha_1 - \frac{13}{5}\beta + \alpha_1 - \frac{7}{5}\beta \geq 0 \Rightarrow 4\alpha_1 - 4\beta \geq 0 \Rightarrow \alpha_1 \geq \beta$.

(iii) $\emptyset\,(\alpha_1,\ \alpha_1,\ 1, 1) \geq 0 \Rightarrow 3\alpha_1 - \frac{13}{5}\alpha_1 + 1 - \frac{7}{5} \geq 0 \Rightarrow \frac{2}{5}\alpha_1 - \frac{2}{5} \geq 0 \Rightarrow \alpha_1 \geq 1$.

The following theorem was proved by Bijendra Singh and Shishir Jain [7].

Theorem (A)

Let $A,\ B,\ S$ and T be self -mappings of a complete fuzzy metric space $(X,\ \mathcal{M},\ *)$ satisfying that

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

(ii) the pair (A, S) is semi compatible and (B, T) is weakly compatible ;

(iii) one of A or S is continuous;

for some φ ∈ Φ there exists k ∈ (0, 1) such that for all x, y and t > 0,

(iv) $\varphi\,(\mathcal{M}(Ax,By,kt),\mathcal{M}\,(Sx,T\,y,t),\mathcal{M}\,(Ax,Sx,t),\mathcal{M}(By,T\,y,kt))\ \geq\ 0,$

(v) $\varphi\,(\mathcal{M}\,(Ax,By,kt),\mathcal{M}(Sx,T\,y,t),\mathcal{M}\,(Ax,Sx,kt),\mathcal{M}(By,T\,y,t))\ \geq\ 0.$

Then A,B,S and T have unique common fixed point in X.

Now we generalize the above theorem by using only contraction condition (iv) as under.

3. MAIN RESULTS

Theorem 3.1 Let A, B, S and T be self -mappings of a complete fuzzy metric space $(\Omega,\ \mathcal{M},\ *)$ satisfying that

(i) $A(\Omega)\ \subseteq\ T(\Omega),\ B(\Omega)\ \subseteq\ S(\Omega);$

(ii) the pair (A,S) is weakly semi compatible, sub sequentially continuous and (B,T) is occasionally weakly compatible ;

(iii) $\varphi\ (\mathcal{M}\ (Ax,\ By,\ kt_\Delta),\ \mathcal{M}\ (Sx,\ Ty,\ t_\Delta),\ \mathcal{M}\ (Ax,\ Sx,\ t_\Delta),\ \mathcal{M}(By,\ Ty,\ kt_\Delta)) \geq 0$

for some $\varphi\ \in\ \Phi$, $\exists$ k $\in$ (0, 1) such that $\forall$ x, y $\in \Omega$ and $t_\Delta > 0$.

Then A,B,S and T having unique common fixed point in Ω.

Proof:

The sub sequentially continuous of the pair $(A,\ S)$ implies existing a sequence $(\upsilon_\eta)\ \in\ \Omega$

with

$$\lim_{\eta\to\infty} A\upsilon_\eta = \lim_{\eta\to\infty} S\upsilon_\eta = \alpha \tag{3.1.1}$$

such that

$$\lim_{\eta\to\infty} \mathcal{M}\left(AS\upsilon_\eta, A\alpha, t_\Delta\right) = 1 \text{ and } \lim_{\eta\to\infty} \mathcal{M}\left(SA\upsilon_\eta, S\alpha, t_\Delta\right) = 1. \tag{3.1.2}$$

And also weakly semi compatibility of (A, S) implies

$$\lim_{\eta\to\infty} \mathcal{M}\left(AS\upsilon_\eta, S\alpha, t_\Delta\right) = 1 \text{ or } \lim_{\eta\to\infty} \mathcal{M}\left(SA\upsilon_\eta, A\alpha, t_\Delta\right) = 1. \tag{3.1.3}$$

Using (3.1.3) in (3.1.2) we get $\mathcal{M}\ (A\alpha\ ,\ S\alpha, t_\Delta) = 1$ this gives

$A\alpha = S\alpha$. By (i) $A\alpha\ \in\ A(\Omega)\ \subseteq\ T(\Omega)$ implies there exists $\beta\ \in\ \Omega$ in order that

$A\alpha = T\beta$. Thus

$$A\alpha = S\alpha\ = T\beta. \tag{3.1.4}$$

Claim $B\beta\ = T\beta$.

By using x = α, y = β in (iii)

$\varphi\ (\mathcal{M}\ (A\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (S\alpha,\ T\beta,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ S\alpha,\ t_\Delta),\ \mathcal{M}(B\beta,\ T\beta,\ kt_\Delta)) \geq 0.$

Using (3.1.4)

$\varphi\ (\mathcal{M}\ (T\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (T\beta,\ T\beta,\ t_\Delta),\ \mathcal{M}\ (T\beta,\ T\beta,\ t_\Delta),\ \mathcal{M}(B\beta,\ T\beta,\ kt_\Delta)) \geq 0.$

$\varphi\ (\mathcal{M}\ (\alpha,\ B\beta,\ kt_\Delta),\ 1,\ 1,\ \mathcal{M}(B\beta,\ \alpha,\ kt_\Delta)) \geq 0.$

By (2.12.2) $\mathcal{M}\ (T\beta,\ B\beta,\ kt_\Delta)\ \geq\ 1$ implies $B\beta\ = T\beta$. Therefore

$A\alpha = S\alpha\ = T\beta\ =\ B\beta.$ (3.1.5)

Claim $\alpha = B\beta$.

Then by (iii) substitute x = υ_η, y = β

$\varphi\ (\mathcal{M}\ (A\upsilon_\eta,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ \left(S\upsilon_\eta,\ T\beta,\ t_\Delta\right),\ \mathcal{M}\ (A\upsilon_\eta,\ S\upsilon_\eta,\ t_\Delta),\ \mathcal{M}(B\beta,\ T\beta,\ kt_\Delta)) \geq 0$

Using (3.1.1) and (3.1.5) and letting as $\eta\ \rightarrow\ \infty$

$\varphi\ (\mathcal{M}\ (\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ \alpha,\ t_\Delta),\ \mathcal{M}(B\beta,\ B\beta,\ kt_\Delta)) \geq 0$

$\varphi\ (\mathcal{M}\ (\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ 1,\ 1\ )\ ) \geq 0.$

As φ is increasing in the 1st argument

$\varphi\ (\mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ 1,\ 1)) \geq 0.$

Using (2.12.2) we get $\mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta)\ \geq\ 1$.

This gives $\alpha\ =\ B\beta$. Hence

$\alpha\ =\ B\beta\ =\ T\beta.$ (3.1.6)

Since the OWC of (B, T) is resulting $TB\beta\ = BT\beta$.

This gives

$T\alpha = B\alpha.$ (3.1.7)

Claim $\alpha = A\alpha$.

Then by substituting x = α, y = β in (iii)

$\varphi\ (\mathcal{M}\ (A\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (S\alpha,\ T\beta,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ S\alpha,\ t_\Delta),\ \mathcal{M}(B\beta,\ T\beta,\ kt_\Delta)) \geq 0.$

Using (3.1.6) and (3.1.7)

$\varphi\ (\mathcal{M}\ (A\alpha,\ \alpha,\ kt_\Delta),\ \mathcal{M}\ (A\alpha,\ \alpha,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ A\alpha,\ t_\Delta),\ \mathcal{M}(B\beta,\ B\beta,\ kt_\Delta)) \geq 0,$

$\varphi\ (\mathcal{M}\ (A\alpha,\ \alpha,\ kt_\Delta),\ \mathcal{M}\ (A\alpha,\ \alpha,\ t_\Delta),\ 1,\ 1\ ) \geq 0.$

As φ is non -decreasing in the 1st argument

$\varphi\ (\mathcal{M}\ (A\alpha,\ \alpha,\ kt_\Delta),\ \mathcal{M}\ (A\alpha,\ \alpha,\ t_\Delta),\ 1,\ 1\ ) \geq 0.$

Using (2.12.2) we get $\mathcal{M}\ (A\alpha,\ \alpha,\ t_\Delta)\ \geq\ 1$.

This gives $\alpha\ =\ A\alpha$. Hence

$\alpha = A\alpha = S\alpha.$ (3.1.8)

Claim $\alpha = B\alpha$

Then by substituting x = α, y = α in (iii)

$\varphi\ (\mathcal{M}\ (A\alpha,\ B\alpha,\ kt_\Delta),\ \mathcal{M}\ (S\alpha,\ T\alpha,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ S\alpha,\ t_\Delta),\ \mathcal{M}(B\alpha,\ T\alpha,\ kt_\Delta)) \geq 0.$

Using (3.1.7) and (3.1.8)

$\varphi\ (\mathcal{M}\ (\alpha,\ B\alpha,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ B\alpha,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ \alpha,\ t_\Delta),\ \mathcal{M}(B\alpha,\ B\alpha,\ kt_\Delta)) \geq 0,$

$\varphi\ (\mathcal{M}\ (\alpha,\ B\alpha,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ B\alpha,\ t_\Delta),\ 1,\ 1\ ) \geq 0.$

As φ is non -decreasing in the 1^{st} argument

$\varphi\ (\mathcal{M}\ (\alpha,\ B\alpha,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ B\alpha,\ t_\Delta),\ 1,\ 1\ ) \geq 0.$

Using (2.12.2) we get $\mathcal{M}\ (\alpha,\ B\alpha,\ t_\Delta) \geq 1.$

This gives $\alpha = B\alpha$. Hence

$\alpha = B\alpha = T\alpha.$ (3.1.9)

From (3.1.8), (3.1.9) conclude that

$\alpha = B\alpha = T\alpha = A\alpha = S\alpha.$ (3.1.10)

Uniqueness:

Assume γ be another point such that

$A\gamma = S\gamma = B\gamma = T\gamma = \gamma.$ (3.1.11)

Applying x = α, y = γ in (iii)

$\varphi\ (\mathcal{M}\ (A\alpha,\ B\gamma,\ kt_\Delta),\ \mathcal{M}\ (S\alpha,\ T\gamma,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ S\alpha,\ t_\Delta),\ \mathcal{M}(B\gamma,\ T\gamma,\ kt_\Delta)) \geq 0$

Using (3.1.10) and (3.1.11)

$\varphi\ (\mathcal{M}\ (\alpha,\ \gamma,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ \gamma,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ \alpha,\ t_\Delta),\ \mathcal{M}(\gamma,\ \gamma,\ kt_\Delta)) \geq 0$

$\varphi\ (\mathcal{M}\ (\alpha,\ \gamma,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ \gamma,\ t_\Delta),\ 1,\ 1) \geq 0$

As φ is non -decreasing in the 1^{st} argument consequences

$\varphi\ (\mathcal{M}\ (\alpha,\ \gamma,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ \gamma,\ t_\Delta),\ 1,\ 1) \geq 0$

By (2.12.2) $\mathcal{M}\ (\alpha,\ \gamma,\ t_\Delta) \geq 1$ implies $\alpha = \gamma$.

Thus the four maps A, S, B and T are having single common fixed point in Ω.

Example 3.2 Let $(\Omega,\ \mathcal{M},\ t_\Delta)$ be the induced fuzzy metric space as

$\mathcal{M}(\alpha,\ \beta,\ t_\Delta) = \frac{t_\Delta}{t_\Delta + \rho(\alpha,\ \beta)}$ where $\Omega = R$.

Define the mapping $T,\ S\ :\ \Omega\ \rightarrow \Omega$ as

$$T(x) = S(x) = \begin{cases} x^2, & x \leq \sqrt{2} \\ 1.7, & \sqrt{2} < x < 2 \\ 5, & x \geq 2 \end{cases} \tag{3.2.1}$$

Define the mappings A, B : $\Omega \rightarrow \Omega$ as

$$A(x) = B(x) = \begin{cases} (\sqrt{2}+1)x - \sqrt{2}, & x \leq \sqrt{2} \\ 1.9, & \sqrt{2} < x < 2 \\ 1.7, & x \geq 2. \end{cases} \tag{3.2.2}$$

Then from (3.2.1), (3.2.2), we have $T(\Omega) = S(\Omega)$ = [0, 2]U{5}, $A(\Omega) = B(\Omega)$ = $(-\infty$, 2]U{5} implies $A(\Omega) \subseteq T(\Omega)$, $B(\Omega) \subseteq S(\Omega)$.

The mappings B, T have coincidence points at $x = 1, \sqrt{2}$.

At $x = \sqrt{2}$, $B(\sqrt{2}) = 2$, $T(\sqrt{2}) = 2$ further $BT(\sqrt{2}) = B(2) = 1.7$,

$TB(\sqrt{2}) = T(2) = 5$ so that $B(\sqrt{2}) = T(\sqrt{2})$ and $BT(\sqrt{2}) \neq TB(\sqrt{2})$.

Thus the mappings are not weakly compatible but OWC.

Since at $x = 1$, $B(1) = T(1) = 1 \Rightarrow BT(1) = TB(1)$.

Consider a sequence $(\mu_m) = 1 + \frac{e}{m}$ $\forall\, m \geq 1$. Then from (3.2.1), (3.2.2)

$$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} A\left(1 + \frac{e}{m}\right) = \lim_{m\to\infty} (1 + \frac{e}{m})^2 = 1, \tag{3.2.3}$$

$$\lim_{m\to\infty} S\mu_m = \lim_{m\to\infty} S\left(1 + \frac{e}{m}\right) = \lim_{m\to\infty} (\sqrt{2}+1)(1 + \frac{e}{m}) - \sqrt{2} = \lim_{m\to\infty} (1 + \frac{\sqrt{2}+1}{m}) = 1. \tag{3.2.4}$$

From (3.2.3) and (3.2.4)

$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} S\mu_m = 1$ and

$$\lim_{m\to\infty} SA\mu_m = \lim_{m\to\infty} S\left((1 + \frac{e}{m})^2\right) = \lim_{m\to\infty} (\sqrt{2}+1)(1 + \frac{e}{m})^2 - \sqrt{2} = 1 = A(1). \tag{3.2.5}$$

Then from (3.2.1), (3.2.5) and

$$\lim_{m\to\infty} \mathcal{M}\,(AS\mu_m, A(1), t_\Delta) = 1. \tag{3.2.6}$$

$$\lim_{m\to\infty} AS\mu_m = \lim_{m\to\infty} A\left((1 + \frac{\sqrt{2}+1}{m})\right) = \lim_{m\to\infty} (1 + \frac{\sqrt{2}+1}{m})^2 = 1 = A(1), \tag{3.2.7}$$

$$\lim_{m\to\infty} SA\mu_m = \lim_{m\to\infty} S\left((1+\tfrac{e}{m})^2\right) = \lim_{m\to\infty} (\sqrt{2}+1)(1+\tfrac{e}{m})^2 - \sqrt{2} = 1 = S(1). \quad (3.2.8)$$

From (3.2.7), (3.2.8)

$$\lim_{m\to\infty} \mathcal{M}\,(AS\mu_m, A(1), t_\Delta) = 1 \text{ and } \lim_{m\to\infty} \mathcal{M}\,(SA\mu_m, S(1), t_\Delta) = 1. \quad (3.2.9)$$

Thus from (3.2.9) the mappings A, S are sub sequentially continuous.

Consider a sequence $(\mu_m) = \sqrt{2} - \frac{3}{m}\ \forall\, m \geq 1.$ Then

$$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} A\left(\sqrt{2}-\tfrac{3}{m}\right) = \lim_{m\to\infty} (\sqrt{2}-\tfrac{3}{m})^2 = 2 \quad \text{and} \quad (3.2.10)$$

$$\lim_{m\to\infty} S\mu_m = \lim_{m\to\infty} S\left(\sqrt{2}-\tfrac{3}{m}\right) = \lim_{m\to\infty} (\sqrt{2}+1)\left(\sqrt{2}-\tfrac{3}{m}\right) - \sqrt{2} = \lim_{m\to\infty} \left(2-\tfrac{3(\sqrt{2}+1)}{m}\right) = 2. \quad (3.2.11)$$

From (3.2.10) and (3.2.11)

$$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} S\mu_m = 2.$$

$$\lim_{m\to\infty} AS\mu_m = \lim_{m\to\infty} A\left(2-\tfrac{3(\sqrt{2}+1)}{m}\right) = \lim_{m\to\infty} 1.9 = 1.9 \neq 5 = S(2). \text{ Then} \quad (3.2.12)$$

$$\lim_{m\to\infty} SA\mu_m = \lim_{m\to\infty} S\left((\sqrt{2}-\tfrac{3}{m})^2\right) = \lim_{m\to\infty} 1.7 = 1.7 = A(2). \quad (3.2.13)$$

From (3.2.12) and (3.2.13)

$$\lim_{m\to\infty} \mathcal{M}\,(AS\mu_m, S(2), t_\Delta) \neq 1 \text{ but } \lim_{m\to\infty} \mathcal{M}\,(SA\mu_m, A(2), t_\Delta) = 1. \quad (3.2.14)$$

From (3.2.6), (3.2.9) and (3.2.14) we observe that the pair (A,S) satisfied weakly semi compatibility, sub sequentially continuity and (B,T) satisfies occasionally weakly compatible property. At $x = 1$, $A(1) = S(1) = B(1) = T(1) = 1.$

Further we can assert that $x = 1$ is the single common fixed point for the mappings A, S, B and T.

Moreover from (3.2.14) the pair (A,S) does not satisfy semi compatibility and the pair (B,T) is not weakly compatible there by fulfilling all the conditions of Theorem(3.1).

Now we present another theorem.

Theorem 3.3 Let A, B, S and T be self -mappings of a complete fuzzy metric space $(\Omega, \mathcal{M}, *)$ satisfying

(i) the pairs (A,S), (B,T) are weakly semi compatible and sub sequentially continuous

(ii) $\varphi\ (\mathcal{M}\,(Ax, By, kt_\Delta), \mathcal{M}\,(Sx, Ty, t_\Delta), \mathcal{M}\,(Ax, Sx, t_\Delta), \mathcal{M}(By, Ty, kt_\Delta)) \geq 0$

for some $\varphi \in \Phi$, $\exists$ k $\in$ (0, 1) such that $\forall$ x, y $\in \Omega$, $t_\Delta > 0$.

Then A, B, S and T having unique common fixed point in Ω.

Proof:

The pair $(A,\ S)$ is sub sequentially continuous implies there is a sequence $(\upsilon_\eta) \in \Omega$ with

$\lim_{\eta\to\infty} A\upsilon_\eta = \lim_{\eta\to\infty} S\upsilon_\eta = \alpha$

(3.3.1)

such that

$\lim_{\eta\to\infty} \mathcal{M}\left(AS\upsilon_\eta, A\alpha, t_\Delta\right) = 1$ and $\lim_{\eta\to\infty} \mathcal{M}\left(SA\upsilon_\eta, S\alpha, t_\Delta\right) = 1.$

(3.3.2) Also the pair (A, S) is weakly semi compatible implies

$\lim_{\eta\to\infty} \mathcal{M}\left(AS\upsilon_\eta, S\alpha, t_\Delta\right) = 1$ or $\lim_{\eta\to\infty} \mathcal{M}\left(SA\upsilon_\eta, A\alpha, t_\Delta\right) = 1.$ (3.3.3)

Using (3.3.3) in (3.3.2) we get $\mathcal{M}\ (A\alpha\ ,\ S\alpha, t_\Delta) = 1$ this gives

$A\alpha = S\alpha.$ (3.3.4)

The pair $(B,\ T)$ is also sub sequentially continuous implies there is a sequence$(\mu_\eta) \in \Omega$ with

$\lim_{\eta\to\infty} B\mu_\eta = \lim_{\eta\to\infty} T\mu_\eta = \beta$ (3.3.5)

such that

$\lim_{\eta\to\infty} \mathcal{M}\left(BT\mu_\eta,\ T\beta,\ t_\Delta\right) = 1$ and $\lim_{\eta\to\infty} \mathcal{M}\left(TB\mu_\eta, B\beta, t_\Delta\right) = 1.$ (3.3.6)

Also weakly semi compatible property of the pair (B, T) is implies

$\lim_{\eta\to\infty} \mathcal{M}\left(BT\mu_\eta,\ B\beta,\ t_\Delta\right) = 1$ and $\lim_{\eta\to\infty} \mathcal{M}\left(TB\mu_\eta, T\beta, t_\Delta\right) = 1.$

(3.3.7)

Using (3.3.7) and (3.3.6) we get $\mathcal{M}\ (B\beta\ ,\ T\beta, t_\Delta) = 1$ this gives

$B\beta\ =\ T\beta.$ (3.3.8)

Claim $A\alpha\ = B\beta.$

By using x = α, y = β in (ii)

$\varphi\ (\mathcal{M}\ (A\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (S\alpha,\ T\beta,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ S\alpha,\ t_\Delta),\ \mathcal{M}(B\beta,\ T\beta,\ kt_\Delta)) \geq 0.$

Using (3.3.4) and (3.3.8)

$\varphi\ (\mathcal{M}\ (A\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (A\alpha,\ B\beta,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ A\alpha,\ t_\Delta),\ \mathcal{M}(B\beta,\ B\beta,\ kt_\Delta)) \geq 0.$

$\varphi\ (\mathcal{M}\ (A\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (A\alpha,\ B\beta,\ t_\Delta),\ 1,\ 1)\ \geq 0.$

As φ is non -decreasing in the 1^{st} argument consequences

φ $(\mathcal{M}\ (A\alpha,\ B\beta,\ t_\Delta),\ \mathcal{M}\ (A\alpha,\ B\beta,\ t_\Delta),\ 1,\ 1) \geq 0.$

By (2.12.2) $\mathcal{M}\ (A\alpha,\ B\beta,\ t_\Delta) \geq 1$ implies

$$A\alpha = B\beta. \tag{3.3.9}$$

Claim $\alpha = \beta$.

By using x = υ_η, y = μ_η in (ii)

φ $(\mathcal{M}\ (A\upsilon_\eta,\ B\mu_\eta,\ kt_\Delta),\ \mathcal{M}\left(S\upsilon_\eta,\ T\mu_\eta,\ t_\Delta\right),\ \mathcal{M}\ (A\upsilon_\eta,\ S\upsilon_\eta,\ t_\Delta),\ \mathcal{M}(B\mu_\eta,\ T\mu_\eta,\ kt_\Delta)) \geq 0.$

As $\eta \to \infty$ implies from (3.3.1), (3.3.5)

φ $(\mathcal{M}\ (\alpha,\ \beta,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ \beta,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ \alpha,\ t_\Delta),\ \mathcal{M}(\beta,\ \beta,\ kt_\Delta)) \geq 0.$

φ $(\mathcal{M}\ (\alpha,\ \beta,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ \beta,\ t_\Delta),\ 1,\ 1) \geq 0.$

As φ is non -decreasing in the 1^{st} argument consequences

φ $(\mathcal{M}\ (\alpha,\ \beta,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ \beta,\ t_\Delta),\ 1,\ 1) \geq 0.$

By (2.12.2) $\mathcal{M}\ (\alpha,\ \beta,\ t_\Delta) \geq 1$ implies $\alpha = \beta$.

Combining all we get

$B\alpha = T\alpha = A\alpha = S\alpha$.

Claim $\alpha = B\beta$.

Then substituting x = υ_η, y = $B\beta$ by using(ii)

φ $(\mathcal{M}\ (A\upsilon_\eta,\ B\beta,\ kt_\Delta),\ \mathcal{M}\left(S\upsilon_\eta,\ T\beta,\ t_\Delta\right),\ \mathcal{M}\ (A\upsilon_\eta,\ S\upsilon_\eta,\ t_\Delta),\ \mathcal{M}(B\beta,\ T\beta,\ kt_\Delta)) \geq 0$

Using (3.3.1) and (3.3.8) and letting as $\eta \to \infty$

φ $(\mathcal{M}\ (\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ \alpha,\ t_\Delta),\ \mathcal{M}(B\beta,\ B\beta,\ kt_\Delta)) \geq 0.$

φ $(\mathcal{M}\ (\alpha,\ B\beta,\ kt_\Delta),\ \mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ 1,\ 1)) \geq 0.$

As φ is increasing in the first argument implies

φ $(\mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ \mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta),\ 1,\ 1)) \geq 0.$

Using (2.12.2) we get $\mathcal{M}\ (\alpha,\ B\beta,\ t_\Delta) \geq 1$.

This gives $\alpha = B\beta$.

Substitute $\alpha = \beta$ we get

$B\alpha = T\alpha = A\alpha = S\alpha = \alpha$.

Uniqueness follows as in Theorem (3.1).

Example 3.4

Let $(\Omega,\ \mathcal{M},\ t_\Delta)$ be the induced fuzzy metric space with

$\mathcal{M}(\alpha,\ \beta,\ t_\Delta) = \frac{t_\Delta}{t_\Delta + \rho(\alpha,\ \beta)}$ where $\Omega = R$.

Define the mappings $T,\ S\ :\ \Omega \to \Omega$ as

$$T(x) = S(x) = \begin{cases} 2^{1+2(x-2)}, & x \le 4 \\ 11, & 4 < x < 2^5 \\ 7, & x \ge 2^5 \end{cases} \tag{3.4.1}$$

Define the mappings A, $B\ :\ \Omega \to \Omega$ as

$$A(x) = B(x) = \begin{cases} 2^{1+(x-2)^2}, & x \le 4 \\ 10, & 4 < x < 2^5. \\ 11, & x \ge 2^5 \end{cases} \tag{3.4.2}$$

From (3.4.1) and (3.4.2) the mappings B, T have coincidence points at $x = 2,\ 4.$

At $x = 4,\ B(4) = 2^{1+(4-2)^2} = 2^5,\ T(4) = 2^{1+2(4-2)} = 2^5$

$BT(4) = B(2^5) = 11,\ TB(4) = T(2^5) = 7$ so that

$B(4) = T(4)$ but $BT(4) \neq TB(4).$ (3.4.3)

Thus from (3.4.3) the mappings are not weakly compatible but OWC.

Since at $x = 2,\ B(2) = T(2) = 2 \Rightarrow BT(2) = TB(2).$

Consider a sequence $(\mu_m) = 2 + \frac{5}{m}\ \forall\, m \ge 1.$ Then from (3.4.1), (3.4.2)

$$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} A\left(2 + \frac{5}{m}\right) = \lim_{m\to\infty} 2^{1+\left(2+\frac{5}{m}-2\right)^2} = \lim_{m\to\infty} 2^{1+\left(\frac{5}{m}\right)^2} = 2, \tag{3.4.4}$$

$$\lim_{m\to\infty} S\mu_m = \lim_{m\to\infty} S\left(2 + \frac{5}{m}\right) = \lim_{m\to\infty} 2^{1+2\left(2+\frac{5}{m}-2\right)} = \lim_{m\to\infty} 2^{\left(1+\frac{10}{m}\right)} = 2. \tag{3.4.5}$$

From (3.4.4) and (3.4.5)

$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} S\mu_m = 2$ and

$$\lim_{m\to\infty} SA\mu_m = \lim_{m\to\infty} S\left(2^{1+\left(\frac{5}{m}\right)^2}\right) = \lim_{m\to\infty} 2^{1+2\left(2^{1+\left(\frac{5}{m}\right)^2}-2\right)} = 2 = A(2). \tag{3.4.6}$$

From (3.4.6)

$$\lim_{m\to\infty} \mathcal{M}\,(SA\mu_m, A(2),\ t_\Delta) = 1. \tag{3.4.7}$$

$$\lim_{m\to\infty} AS\mu_m = \lim_{m\to\infty} A\left(2^{\left(1+\frac{10}{m}\right)}\right) = \lim_{m\to\infty} 2^{1+\left(2^{\left(1+\frac{10}{m}\right)}-2\right)^2} = 2 = A(2), \tag{3.4.8}$$

$$\lim_{m\to\infty} SA\mu_m = \lim_{m\to\infty} S\left(2^{1+\left(\frac{5}{m}\right)^2}\right) = \lim_{m\to\infty} 2^{1+2\left(2^{1+\left(\frac{5}{m}\right)^2}-2\right)} = 2 = S(2). \tag{3.4.9}$$

Thus from (3.4.8) and (3.4.9)

$\lim_{m\to\infty} \mathcal{M}\ (SA\mu_m, S(2),\ t_\Delta) = 1$ and $\lim_{m\to\infty} \mathcal{M}\ (AS\mu_m, A(2),\ t_\Delta) = 1.$ (3.4.10)

Consider a sequence $(\mu_m) = 4 - \frac{2}{m}\ \forall\, m \geq 1.$ Then

$$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} A\left(4 - \frac{2}{m}\right) = \lim_{m\to\infty} 2^{1+(4-\frac{2}{m}-2)^2} = \lim_{m\to\infty} 2^{1+(2-\frac{2}{m})^2} = 2^5 \qquad (3.4.11)$$

$$\lim_{m\to\infty} \mathrm{S}\mu_m = \lim_{m\to\infty} \mathrm{S}\left(4 - \frac{2}{m}\right) = \lim_{m\to\infty} 2^{1+2(4-\frac{2}{m}-2)} = \lim_{m\to\infty} 2^{(5-\frac{4}{m})} = 2^5. \qquad (3.4.12)$$

From (3.4.11) and (3.4.12)

$\lim_{m\to\infty} A\mu_m = \lim_{m\to\infty} \mathrm{S}\mu_m = 2^5.$ Also

$$\lim_{m\to\infty} AS\mu_m = \lim_{m\to\infty} A\left(2^{(5-\frac{4}{m})}\right) = \lim_{m\to\infty} 10 = 10 \neq 7 = S(2^5) \text{ and} \qquad (3.4.13)$$

$$\lim_{m\to\infty} \mathrm{S}A\mu_m = \lim_{m\to\infty} \mathrm{S}\left(2^{1+(2-\frac{2}{m})^2}\right) = \lim_{m\to\infty} 11 = 11 = A(2^5). \qquad (3.4.14)$$

Thus from (3.4.13) and (3.4.14)

$\lim_{m\to\infty} \mathcal{M}\ (SA\mu_m, A(2^5),\ t_\Delta) = 1$ but $\lim_{m\to\infty} \mathcal{M}\ (AS\mu_m, S(2^5),\ t_\Delta) \neq 1.$ (3.4.15)

Thus from (3.4.7), (3.4.10) and (3.4.15) we say that the joint pairs (A, S), (B, T) are weakly semi compatible and sub sequentially continuous. At $x = 2,\ A(2) = S(2) = B(2) = T(2) = 2.$ We can conclude that $x = 2$ is the single common fixed point for the mappings A, S, B and T. Moreover from (3.4.3), (3.4.15) the pairs (A, S), (B, T) are neither semi compatible nor weakly compatible.

CONCLUSION

In this paper we generalized Theorem (A) in two ways by using

(i) weaker conditions such as weakly semi compatibility, sub sequentially continuity and occasionally weakly compatible mappings instead of semi compatibility, continuity and weakly compatible mappings in Theorem (3.1) and dropped condition (v) of Theorem(A).

(ii) Further using the weaker conditions such as weakly semi compatibility and sub sequentially continuity instead of semi compatibility, continuity and weakly compatible maps in Theorem (3.2) without using the inclusion condition and also by dropping the condition(v) of Theorem(A).

Moreover these two results are validated by discussing the appropriate examples.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets Syst. 1 (1978) 3–28.

[2] A.P. Silva Artur, B.M. Paterline Novais Abreu, L.J.P. De Lima Oliveira, E.R. Nucci, Application of Fuzzy logic for the analysis of the pandemic (COVID-19), if there was no social isolation, Saúde (Sta. Maria). 46 (2020), 1-13.

[3] J.M. Brotons-Martínez, M.E. Sansalvador-Sellés, Proposal for a fuzzy model to assess cost overrun in healthcare due to delays in treatment, Mathematics. 9 (2021), 408.

[4] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces. Kybernetika 11 (1975), 336-344.

[5] A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets Syst. 90 (1997), 365–368.

[6] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets Syst. 12 (1984), 215–229.

[7] B. Singh, S. Jain, Semicompatibility and fixed point theorems in fuzzy metric space using implicit relation, Int. J. Math. Math. Sci. 2005 (2005), 2617–2629.

[8] S. Chauhan, S. Bhatnagar, S. Radenović, Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces, Le Matematiche 68 (2013), 87-98.

[9] G. Jungck, B. E. Rhoades, Fixed point theorems for occasionally weakly compatible mappings, Fixed Point Theory 7 (2006), 287-296.

[10] V. Srinivas, B. Vijayabasker Reddy, A common fixed point theorems on fuzzy metric space using weakly compatible and semi-compatible mappings, Int. J. Math. Stat. Invent 4 (2016), 27-31.

[11] K. Satyanna and V. Srinvas, Fixed point theorem using semi compatible and sub sequentially continuous mappings in Menger space, J. Math. Comput. Sci. 10 (2020), 2503-2515.

[12] A. Jain, V. K. Gupta, V. H. Badshah, R. Singh Chandelkar, Fixed point theorem in fuzzy metric space using semi-compatible mappings, Adv. Inequal. Appl., 2014 (2014), Article ID 19

[13] R. Sriramula, V. Srinivas, A fixed point theorem using weakly Semicompatible mappings in metric space, Int. J. Math. Anal. 11 (2017), 695-706.