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QUALITATIVE ANALYSIS OF A.P.A. SOLUTION FOR FRACTIONAL ORDER NEUTRAL STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY G-BROWNIAN MOTION

A. D. NAGARGOJE[1,*], V. C. BORKAR[2], R. A. MUNESHAWAR[1]

[1]P. G. Department of Mathematics N.E.S. Science College, Nanded - 431602, (MH), India

[2]Department of Mathematics and Statistics Yeshwant Mahavidyalaya, Nanded- 431605, Maharashtra, India

Abstract. In this paper, we will analyses the square mean almost pseudo automorphic mild solution for fractional order equation,

$$(1) \quad {}_0^cD_\gamma^\alpha[\aleph(\gamma) - D(\gamma,\aleph(\gamma))] = [A\aleph(\gamma) + \phi(\gamma,\aleph(\gamma))]d\gamma + \varphi(\gamma,\aleph(\gamma))d\langle B\rangle(\gamma) + \psi(\gamma,\aleph(\gamma))dB(\gamma), \gamma \in R$$

where $A(\gamma) : \mathscr{D}(A(\gamma)) \subset \mathfrak{L}_G^2(\mathscr{F}) \to \mathfrak{L}_G^2(\mathscr{F})$ is densely closed linear operator and the functions D, ϕ, φ and ψ : $\mathfrak{L}_G^2(\mathscr{F}) \to \mathfrak{L}_G^2(\mathscr{F})$ are jointly continuous. We drive square mean almost pseudo automorphic mild solution for fractional order neutral stochastic evolution equations driven by G-Brownian motion is obtain by using evolution operator theorem and fixed point theorem. Moreover, we prove that this mild solution of equation (1) is unique.

Keywords: fractional derivative and integral; existence and uniqueness; almost pseudo automorphic; G-Brownian motion; fixed point.

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1. Introduction

Some results on of existence and uniqueness of the square-mean almost pseudo almost automorphic mild solutions for fractional differential equation have been discussed by some authors

*Corresponding author
E-mail address: nagargojearun1993@gmail.com
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which can be found in [1, 2, 3, 4, 5, 6]. The aims of this article is to discussed the square mean almost pseudo automorphic mild solution for neutral stochastic evolution equations of fractional order driven by G-Brownian motion(G-NSEEF for short), which is given by equation (1). Now we will recall following definitions of fractional derivative,

Riemann–Liouville definition [5, 6]: For $\alpha \in [n-1,n)$ the α - derivative of f is

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n x}{dt^n} \int_a^\alpha \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

Caputo definition[5, 6]:For $\alpha \in (n-1,n)$ the α - derivative of f is

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

2. Preliminaries

Lemma[10] If $0 \le \mathfrak{T} < \infty$, then

(1) $\mathbb{E}\left[\mid \int_0^{\mathfrak{T}} \eta_\gamma d\langle B\rangle(\gamma) \mid\right] \ \sigma^{-2}\mathbb{E}\left[\int_0^{\mathfrak{T}} \mid \eta_\gamma \mid d\gamma\right]$, for any $\eta_\gamma \in \mathscr{M}_G^1([0,\mathfrak{T}])$.

(2) $\mathbb{E}\left[\left(\int_0^{\mathfrak{T}} \eta_\gamma dB(\gamma)\right)^2\right] = \mathbb{E}\left[\int_0^{\mathfrak{T}} \eta_\gamma^2 d\langle B\rangle(\gamma)\right]$, for any $\eta_\gamma \in \mathscr{M}_G^2([0,\mathfrak{T}])$.

(3) $\mathbb{E}\left[\left(\int_0^{\mathfrak{T}} \mid \eta_\gamma \mid^p d\gamma\right)\right] \le \int_0^{\mathfrak{T}} \left[\mathbb{E} \mid \eta_\gamma \mid^p\right] d\gamma$, for any $\eta_\gamma \in \mathscr{M}_G^p([0,\mathfrak{T}]), p \ge 1$.

Definition An $\mathscr{F}_\gamma$ progressively measurable process $\{\aleph(\gamma)\}_{\gamma \in R}$ is called a mild solution of the (1), if

(2)
$$\aleph(\gamma) - D(\gamma,\aleph(\gamma)) = \mho(\gamma,s)[\aleph(s) - D(s,\aleph(s))] + \frac{1}{\Gamma(-\alpha-n)} \int_s^\gamma \frac{\mho(\gamma,r)\phi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dr$$
$$+ \frac{1}{\Gamma(-\alpha-n)} \int_s^\gamma \frac{\mho(\gamma,r)\varphi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}} d\langle B\rangle(r) + \frac{1}{\Gamma(-\alpha-n)} \int_s^\gamma \frac{\mho(\gamma,r)\psi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dB(r)$$

for any $\gamma \ge s$ and $s \in R$.

For our convenience and further use we consider the following assumptions.

$(H1)$ $\exists\ \Omega > 0$ and $\mu > 0$ such that the evolution family $\mho(\gamma,s)$ generated by $A(\gamma)$ is exponentially stable,

$$\| \mho(\gamma,s) \| \le \Omega e^{-\mu(\gamma-s)}, \gamma \ge s.$$

$(H2)$ The coefficients $D(\gamma,x),\phi(\gamma,x),\varphi(\gamma,x)$ and $\psi(\gamma,x): R\times L^2_G(\mathscr{F})\to \mathfrak{L}^2_G(\mathscr{F})$ are functions of $SPAA(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$. Furthermore, $\exists\ \mathfrak{L}_D,\mathfrak{L}_\phi,\mathfrak{L}_\varphi,\mathfrak{L}_\psi \geq 0$ such that

$$\| D(\gamma,x)-D(\gamma,y) \|^2 \leq \mathfrak{L}_D \| x-y \|^2, \| \phi(\gamma,x)-\phi(\gamma,y) \|^2 \leq \mathfrak{L}_\phi \| x-y \|^2$$

and

$$\| \varphi(\gamma,x)-\varphi(\gamma,y) \|^2 \leq \mathfrak{L}_\varphi \| x-y \|^2, \| \psi(\gamma,x)-\psi(\gamma,y) \|^2 \leq \mathfrak{L}_\psi \| x-y \|^2$$

for $x,y\in \mathfrak{L}^2_G(\mathscr{F})$ and $\gamma\in R$.

$(H3)$ $D=D_1+D_2\in SPAA(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, where $D_1\in SAA(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, $D_2\in SBC_0(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$. $\phi=\phi_1+\phi_2\in SPAA(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, where $\phi_1\in SAA(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, $\phi_2\in SBC_0(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$. $\varphi=\varphi_1+\varphi_2\in SPAA(R\times L^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, where $\varphi_1\in SAA(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, $\varphi_2\in SBC_0(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$. $\psi=\psi_1+\psi_2\in SPAA(R\times L^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, where $\psi_1\in SAA(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$, $\psi_2\in SBC_0(R\times \mathfrak{L}^2_G(\mathscr{F}),\mathfrak{L}^2_G(\mathscr{F}))$.

3. Main Result

Theorem If the hypothesis $(H1)-(H3)$ are satisfied, and

$$4\mathfrak{L}_D+\frac{4\Omega^2\mathfrak{L}_\phi}{\mu^2}+\frac{4\sigma^{-4}\Omega^2\mathfrak{L}_\varphi}{\mu^2}+\frac{4\sigma^{-2}\Omega^2\mathfrak{L}_\psi}{\mu}<1.$$

then, the system (1) has a unique mild solution $\aleph\in SPAA(R,\mathfrak{L}^2_G(\mathscr{F}))$ and this solution can be expressed as

(3)

$$\aleph(\gamma)=D(\gamma,\aleph(\gamma))+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\phi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\varphi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{t}\frac{\mho(\gamma,r)\psi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r).$$

Proof Firstly we will discuss the Existence of the square mean almost pseudo automorphic mild solution for of equation (1).

Claim: For all $\gamma\geq s$ and at each $s\in R$, we will show that $\aleph(\gamma)$ defined by (3) satisfies the equation (2) and hence $\aleph(\gamma)$ will be a mild solution of (1).

For any $\aleph\in SPAA(R,\mathfrak{L}^2_G(\mathscr{F}))$, we define the operator $(\Phi\aleph)(\gamma)$ as follows,

(4)

$$(\Phi \aleph)(\gamma) = D(\gamma, \aleph(\gamma)) + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\phi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\varphi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r) + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\psi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r),$$

which is well defined and satisfies (2). From $(H3)$, we have

(5)

$$(\Phi \aleph)(\gamma) = \Big(D_1(\gamma, \aleph(\gamma)) + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r) + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\psi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r)\Big)$$
$$+\Big(D_2(\gamma, \aleph(\gamma)) + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\phi_2^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr + \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(t,r)\varphi_2^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(t,r)\psi_2^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r)\Big).$$
$$= (\Phi_1 \aleph)(\gamma) + (\Phi_2 \aleph)(\gamma).$$

Now we will show that $(\Phi_1 \aleph)(\gamma) \in SAA(R, \mathfrak{L}_G^2(\mathscr{F}))$ and $(\Phi_2 \aleph)(t) \in SBC_0(R, \mathfrak{L}_G^2(\mathscr{F}))$. We will illustrate the facts through following three steps.

Step 1. Sub Claim - I: $(\Phi_1 \aleph)(\gamma)$ is continuous.

By the definition of $(\Phi_1 \aleph)(\gamma)$, we have

(6)

$$\mathbb{E} \parallel (\Phi_1 \aleph)(\gamma+s) - (\Phi_1 \aleph)(\gamma) \parallel^2 = \mathbb{E} \parallel D_1(\gamma+s, \aleph(\gamma+s)) - D_1(\gamma, \aleph(\gamma))$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma+s} \frac{\mho(\gamma+s,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dr - \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma+s} \frac{\mho(\gamma+s,r)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}d\langle B\rangle(r) - \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{t} \frac{\mho(\gamma,r)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma+s} \frac{\mho(\gamma+s,r)\psi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dB(r) - \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\psi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r) \parallel^2 .$$

As $D_1(\gamma,x) \in SAA(R \times \mathfrak{L}_G^2(\mathscr{F}), \mathfrak{L}_G^2(\mathscr{F}))$, then we conclude that,

$$\lim_{s\to 0} \mathbb{E} \parallel D_1(\gamma+s, \aleph(\gamma+s)) - D_1(\gamma, \aleph(\gamma)) \parallel^2 = 0. \quad (7)$$

By means of the properties of evolution family $\mho(\gamma,r)$ and elementary inequality, we get

$$\begin{aligned}
&\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma+s}\frac{\mho(\gamma+s,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dr-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr\parallel^2\\
&=\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{(\mho(\gamma+s,\gamma)-I)\mho(\gamma,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr\\
&+\frac{1}{\Gamma(-\alpha-n)}\int_{\gamma}^{\gamma+s}\frac{\mho(\gamma+s,\gamma)\phi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dr\parallel^2\\
&\leq 2\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{(\mho(\gamma+s,\gamma)-I)\mho(\gamma,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr\parallel^2\\
&+2\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{\gamma}^{\gamma+s}\frac{\mho(\gamma+s,\gamma)\phi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dr\parallel^2.
\end{aligned}$$

By the dominated convergence theorem, we conclude that,

(8)

$$\lim_{s\to 0}\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma+s}\frac{\mho(\gamma+s,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dr-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\phi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr\parallel^2=0.$$

By using the properties of evolution family $\mho(\gamma,r)$ and Lemma 2, we conclude that,

$$\begin{aligned}
&\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma+s}\frac{\mho(\gamma+s,r)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}d\langle B\rangle(r)-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)\parallel^2\\
&=\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{(\mho(\gamma+s,\gamma)-I)\mho(\gamma,r)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)\\
&+\frac{1}{\Gamma(-\alpha-n)}\int_{\gamma}^{\gamma+s}\frac{\mho(\gamma+s,\gamma)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}d\langle B\rangle(r)\parallel^2\\
&\leq 2\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{(\mho(\gamma+s,\gamma)-I)\mho(\gamma,r)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)\parallel^2\\
&+2\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{\gamma}^{\gamma+s}\frac{\mho(\gamma+s,\gamma)\varphi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}d\langle B\rangle(r)\parallel^2\\
&\leq 2\sigma^{-4}\mathbb{E}\Big(\int_{-\infty}^{\gamma}\parallel \frac{(\mho(\gamma+s,\gamma)-I)\mho(\gamma,r)\varphi_1^{(n)}(r,\aleph(r))}{\Gamma(-\alpha-n)(\gamma-r)^{-\alpha+1-n}}\parallel dr\Big)^2\\
&+2\sigma^{-4}\mathbb{E}\Big(\int_{\gamma}^{\gamma+s}\frac{\mho(\gamma+s,\gamma)\parallel \varphi_1^{(n)}(r,\aleph(r))\parallel}{\parallel \Gamma(-\alpha-n)(\gamma+s-r)^{-\alpha+1-n}\parallel}dr\Big)^2.
\end{aligned}$$

And

$$\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma+s}\frac{\mho(\gamma+s,r)\psi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dB(r)-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\psi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r)\parallel^2$$

$$=\mathbb{E}\parallel \frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{(\mho(\gamma+s,\gamma)-I)\mho(\gamma,r)\psi_1^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r)$$

$$+\frac{1}{\Gamma(-\alpha-n)}\int_{\gamma}^{\gamma+s}\frac{\mho(\gamma+s,\gamma)\psi_1^{(n)}(r,\aleph(r))}{(\gamma+s-r)^{-\alpha+1-n}}dB(r)\parallel^2$$

$$\leq 2\sigma^{-2}\int_{-\infty}^{\gamma}\mathbb{E}\parallel \frac{(\mho(\gamma+s,\gamma)-I)\mho(\gamma,r)\psi_1^{(n)}(r,\aleph(r))}{\Gamma(-\alpha-n)(\gamma-r)^{-\alpha+1-n}}\parallel^2 dr$$

$$+2\sigma^{-2}\int_{\gamma}^{\gamma+s}\frac{\mho(\gamma+s,\gamma)\mathbb{E}\parallel \psi_1^{(n)}(r,\aleph(r))\parallel^2}{\parallel \Gamma(-\alpha-n)(\gamma+s-r)^{-\alpha+1-n}\parallel^2}dr.$$

Hence, it follows

$$\lim_{s\to 0}\mathbb{E}\parallel (\Phi_1\aleph)(\gamma+s)-(\Phi_1\aleph)(\gamma)\parallel^2=0.$$

Step 2. As $D(\gamma,x),\phi(\gamma,x),\varphi(\gamma,x)$ and $\psi(t,x)$ are the functions of $SAA(R\times\mathfrak{L}_G^2,\mathfrak{L}_G^2(\mathscr{F}))$, therefore, $\exists$ a subsequence $\{r_n\}$ of $\{r_n^{|}\}_{n\in N}$, for some stochastic process $\widetilde{D_1},\widetilde{\phi}_1,\widetilde{\varphi_1}$ and $\widetilde{\psi_1}$: $R\times\mathfrak{L}_G^2,\mathfrak{L}_G^2(\mathscr{F})$, such that

$$\lim_{n\to\infty}\mathbb{E}\parallel D_1(\gamma+r_n,\aleph(\gamma+r_n))-\widetilde{D_1}(\gamma,\aleph(\gamma))\parallel^2=0 \textit{ and } \lim_{n\to\infty}\mathbb{E}\parallel \widetilde{D_1}(\gamma-r_n,\aleph(\gamma-r_n))-D_1(\gamma,\aleph(\gamma))\parallel^2=0,$$

$$\lim_{n\to\infty}\mathbb{E}\parallel \phi_1(\gamma+r_n,\aleph(\gamma+r_n))-\widetilde{\phi}_1(\gamma,\aleph(\gamma))\parallel^2=0 \textit{ and } \lim_{n\to\infty}\mathbb{E}\parallel \widetilde{\phi}_1(\gamma-r_n,\aleph(\gamma-r_n))-\phi_1(\gamma,\aleph(\gamma))\parallel^2=0,$$

$$\lim_{n\to\infty}\mathbb{E}\parallel \varphi_1(\gamma+r_n,\aleph(\gamma+r_n))-\widetilde{\varphi_1}(\gamma,\aleph(\gamma))\parallel^2=0$$

and

$$\lim_{n\to\infty}\mathbb{E}\parallel \widetilde{\varphi_1}(\gamma-r_n,\aleph(\gamma-r_n))-\varphi_1(\gamma,\aleph(\gamma))\parallel^2=0,$$

$$\lim_{n\to\infty}\mathbb{E}\parallel h_1(\gamma+r_n,\aleph(\gamma+r_n))-\widetilde{h}_1(\gamma,\aleph(\gamma))\parallel^2=0 \textit{ and } \lim_{n\to\infty}\mathbb{E}\parallel \widetilde{\psi_1}(\gamma-r_n,\aleph(\gamma-r_n))-\psi_1(\gamma,\aleph(\gamma))\parallel^2=0,$$

$\forall\ \gamma \in R$ and $\aleph(\gamma) \in \mathfrak{L}_G^2(\mathscr{F})$.

To prove that $(\Phi_1 \aleph)(\gamma)$ is a square mean almost automorphic process, we consider the following operator $(\widetilde{\Phi_1} \aleph)(\gamma)$,

(9)

$$(\widetilde{\Phi_1} \aleph)(\gamma) = \widetilde{D_1}(\gamma, \aleph(\gamma)) + \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\phi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dr$$

$$+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\varphi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} d\langle B\rangle(r) + \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\psi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dB(r).$$

And hence we have,

$$\mathbb{E} \parallel (\Phi_1 \aleph)(\gamma+r_n) - (\widetilde{\Phi_1} \aleph)(\gamma) \parallel^2$$

$$= \mathbb{E} \parallel D_1(\gamma+r_n, \aleph(\gamma+r_n)) + \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\phi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} dr$$

$$+ \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\varphi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} d\langle B\rangle(r) + \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)h_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} dB(r)$$

$$- \widetilde{D_1}(\gamma, \aleph(\gamma)) - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\phi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dr - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\varphi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} d\langle B\rangle(r)$$

$$- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\psi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dB(r) \parallel^2$$

(10)

$$\leq 4\mathbb{E} \parallel D_1(\gamma+r_n, \aleph(\gamma+r_n)) - \widetilde{D_1}(\gamma, \aleph(\gamma)) \parallel^2 + 4\mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\phi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} dr$$

$$- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\phi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dr \parallel^2 + 4\mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\varphi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} d\langle B\rangle(r)$$

$$- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\varphi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} d\langle B\rangle(r) \parallel^2 + 4\mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\psi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} dB(r)$$

$$- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\psi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dB(r) \parallel^2 .$$

By using the Cauchy-Schwarz inequality, we obtain,

$$\mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\phi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} dr - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\phi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dr \parallel^2$$

$$= \mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{t} \frac{\mho(\gamma,r)\phi_1^{(n)}(r+r_n, \aleph(r+r_n))}{(\gamma+r_n-r)^{-\alpha+1-n}} dr - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\phi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dr \parallel^2$$

$$\leq \int_{-\infty}^{\gamma} \mho(\gamma,r)dr \int_{-\infty}^{\gamma} \mho(\gamma,r).\mathbb{E} \parallel \phi_1^{(n)}(r+r_n, \aleph(r+r_n)) - \widetilde{\phi_1^{(n)}}(r, \aleph(r)) \parallel^2 \frac{dr}{\xi^*}, \tag{11}$$

For shake of simplicity, we consider, $\Gamma(-\alpha-n)(\gamma-r)^{-\alpha+1-n} = \xi$ and $\parallel \Gamma(-\alpha-n)(\gamma-r)^{-\alpha+1-n} \parallel = \parallel \xi \parallel = \xi^*$, where the last estimate converges to zero as $n \to \infty$.
Note that, for any $\gamma \in R, \widetilde{\langle B \rangle}(\gamma)$ the difference $\langle B \rangle(\gamma+r_n) - \langle B \rangle(r_n)$ has the same distribution with $\langle B \rangle(\gamma)$ and by using the Cauchy-Schwarz inequality again, we have

$$\mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\varphi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} d\langle B \rangle(r) - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{t} \frac{\mho(\gamma,r)\widetilde{\varphi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} d\langle B \rangle(r) \parallel^2$$

$$= \sigma^{-4}\mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma,r)[\varphi_1^{(n)}(r+r_n, \aleph(r+r_n)) - \widetilde{\varphi_1^{(n)}}(r, \aleph(r))] d\langle B \rangle(r) \frac{1}{\xi} \parallel^2$$
$$\leq \mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma,r)[\varphi_1^{(n)}(r+r_n, \aleph(r+r_n)) - \widetilde{\varphi_1^{(n)}}(r, X(r))] dr \frac{1}{\xi} \parallel^2$$

$$\leq \sigma^{-4} \int_{-\infty}^{\gamma} \mho(\gamma,r)dr \int_{-\infty}^{\gamma} \mho(\gamma,r)\mathbb{E} \parallel \varphi_1^{(n)}(r+r_n, \aleph(r+r_n)) - \widetilde{\varphi_1^{(n)}}(r, \aleph(r)) \parallel^2 \frac{dr}{\xi^*}. \tag{12}$$

Therefor, we have

$$\lim_{n\to\infty} \mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\varphi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} d\langle B \rangle(r)$$
$$- \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\varphi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} d\langle B \rangle(r) \parallel^2 = 0.$$

Let $\tilde{B}(\gamma) = B(\gamma+r_n) - B(r_n)$ for each $\gamma \in R$, then $\tilde{B}(\gamma)$ is also a G-Brownian motion with the same distribution as $B(\gamma)$, we obtain

$$\mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma+r_n} \frac{\mho(\gamma+r_n,r)\psi_1^{(n)}(r, \aleph(r))}{(\gamma+r_n-r)^{-\alpha+1-n}} dB(r) - \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\widetilde{\psi_1^{(n)}}(r, \aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dB(r) \parallel^2$$

$$= \mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma,r)[\psi_1^{(n)}(r+r_n, \aleph(r+r_n)) - \widetilde{\psi_1^{(n)}}(r, \aleph(r))] d\tilde{B}(r) \frac{1}{\xi} \parallel^2$$

$$\leq \sigma^{-2} \int_{-\infty}^{\gamma} \parallel \mho(\gamma,r) \parallel \mathbb{E} \parallel \psi_1^{(n)}(r+r_n, \aleph(r+r_n)) - \widetilde{\psi_1^{(n)}}(r, \aleph(r)) \parallel^2 \frac{dr}{\xi^*}, \tag{13}$$

where the last estimate converges to zero as $n \to \infty$.

Therefore, we can conclude that

$$\lim_{n\to\infty} \mathbb{E} \parallel (\Phi_1 \aleph)(\gamma + r_n) - (\widetilde{\Phi_1} \aleph)(\gamma) \parallel^2 = 0.$$

By the same arguments as above, we obtain,

$$\lim_{n\to\infty} \mathbb{E} \parallel (\widetilde{\Phi_1} \aleph)(\gamma - r_n) - (\Phi_1 \aleph)(\gamma) \parallel^2 = 0.$$

From the Steps 1 and 2, we conclude that, $(\Phi_1 \aleph)(\gamma) \in SAA(R, \mathfrak{L}^2_G(\mathscr{F}))$.

Step 3. Sub Claim - II: $(\Phi_2 \aleph)(\gamma)$ is stochastically continuous process.

According to the functions D_2, F_2, G_2 and $H_2 \in SBC_0(R \times \mathfrak{L}^2_G(\mathscr{F}), \mathfrak{L}^2_G(\mathscr{F}))$, then it follows that $(\Phi_2 \aleph)(\gamma)$ is stochastically bounded. Now we aim to prove that

$$\lim_{\mathfrak{T}\to\infty} \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel (\Phi_2 \aleph)(\gamma) \parallel^2 d\gamma = 0.$$

From the definition of $(\Phi_2 \aleph)(\gamma)$, we have

$$\begin{aligned}
&\frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel (\Phi_2 \aleph)(\gamma) \parallel^2 d\gamma \\
&\leq 4\Big[\frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel D_2(\gamma, \aleph(\gamma)) \parallel^2 d\gamma + \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\phi_2^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dr \parallel^2 d\gamma \\
&+ \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\varphi_2^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}} d\langle B\rangle(r) \parallel^2 d\gamma \\
&+ \frac{1}{2T} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel \frac{1}{\Gamma(-\alpha-n)} \int_{-\infty}^{\gamma} \frac{\mho(\gamma,r)\psi_2^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}} dB(r) \parallel^2 d\gamma\Big] \\
&\leq 4\Big[\frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel D_2(\gamma, \aleph(\gamma)) \parallel^2 d\gamma + \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} [\int_{-\infty}^{\gamma} \mho(\gamma,r)dr \int_{-\infty}^{\gamma} \mho(\gamma,r)\mathbb{E} \parallel \phi_2^{(n)}(r,\aleph(r)) \parallel^2 dr \frac{1}{\xi^*}]d\gamma \\
&\qquad + \sigma^{-4} \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} [\int_{-\infty}^{\gamma} \mho(\gamma,r)dr \int_{-\infty}^{\gamma} \mho(\gamma,r)\mathbb{E} \parallel \varphi_2^{(n)}(r,\aleph(r)) \parallel^2 dr \frac{1}{\xi^*}]d\gamma \\
&\qquad + \sigma^{-2} \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \int_{-\infty}^{\gamma} \mho^2(\gamma,r)\mathbb{E} \parallel \psi_2^{(n)}(r,\aleph(r)) \parallel^2 \frac{1}{\xi^*} dr d\gamma\Big] \\
&\leq 4\Big[\frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \mathbb{E} \parallel D_2(\gamma, \aleph(\gamma)) \parallel^2 d\gamma + \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} [\int_{-\infty}^{\gamma} \mho(\gamma,r)dr \int_{-\infty}^{\gamma} \mho(\gamma,r)\mathbb{E} \parallel \phi_2^{(n)}(r,\aleph(r)) \parallel^2 dr \frac{1}{\xi^*}]d\gamma \\
&\qquad + \sigma^{-4} \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} [\int_{-\infty}^{\gamma} \mho(\gamma,r)dr \int_{-\infty}^{\gamma} \mho(\gamma,r)\mathbb{E} \parallel \varphi_2^{(n)}(r,\aleph(r)) \parallel^2 dr \frac{1}{\xi^*}]d\gamma \\
&\qquad + \sigma^{-2} \frac{1}{2\mathfrak{T}} \int_{-\mathfrak{T}}^{\mathfrak{T}} \int_{-\infty}^{\gamma} \mho^2(\gamma,r)\mathbb{E} \parallel \psi_2^{(n)}(r,\aleph(r)) \parallel^2 \frac{1}{\xi^*} dr d\gamma\Big]
\end{aligned}$$

$$\leq 4\Big[\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}\mathbb{E}\parallel D_2(\gamma,\aleph(\gamma))\parallel^2 d\gamma+\frac{\Omega^2}{\mu}\times\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}[\int_{-\infty}^{\gamma}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}]d\gamma$$
$$+\frac{\Omega^2\sigma^{-4}}{\mu}\times\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}[\int_{-\infty}^{\gamma}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\varphi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}]d\gamma$$
$$+\frac{\Omega^2\sigma^{-2}}{\mu}\times\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}\int_{-\infty}^{\gamma}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\psi_2^{(n)}(r,\aleph(r))\parallel^2\frac{1}{\xi^*}drd\gamma\Big]$$

As to the second part of the last inequality, it follows

$$\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}d\gamma\int_{-\infty}^{\gamma}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}$$
$$=\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}d\gamma\int_{-T}^{\gamma}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}+\frac{1}{2\mathfrak{T}}\int_{-\infty}^{-\mathfrak{T}}d\gamma\int_{-\infty}^{\mathfrak{T}}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}$$
$$=\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}dr\int_{r}^{\mathfrak{T}}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel^2 d\gamma\frac{1}{\xi^*}+\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}d\gamma\int_{-\infty}^{\mathfrak{T}}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}$$

$$\leq\frac{1}{\mu}\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}+\frac{1}{\mu^2}\frac{1}{2\mathfrak{T}}\mathbb{E}\parallel\phi_2^{(n)}(r,\aleph(r))\parallel_\infty^2\to 0 \tag{14}$$

as $\mathfrak{T}\to\infty$.

Similarly, we have

$$\lim_{\mathfrak{T}\to\infty}\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}[\int_{-\infty}^{\gamma}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\varphi_2^{(n)}(r,\aleph(r))\parallel^2 dr\frac{1}{\xi^*}]dt=0$$

and

$$\lim_{\mathfrak{T}\to\infty}\frac{1}{2\mathfrak{T}}\int_{-\mathfrak{T}}^{\mathfrak{T}}\int_{-\infty}^{\gamma}e^{-\mu(\gamma-r)}\mathbb{E}\parallel\psi_2^{(n)}(r,\aleph(r))\parallel^2\frac{1}{\xi^*}drdt=0.$$

Thus, we conclude that, $(\Phi_2\aleph)(\gamma)\in SBC_0(R,\mathfrak{L}_G^2(\mathscr{F}))$. According to the above three steps, finally we could demonstrate $(\Phi\aleph)(\gamma)\in SPAA(R,\mathfrak{L}_G^2(\mathscr{F}))$.

Uniqueness: For a unique fixed point, suppose that $\aleph(\gamma)$ and $Y(\gamma)$ are the solutions of (1). Now consider,

$\mathbb{E}\parallel(\Phi\aleph)(\gamma)-(\Phi Y)(\gamma)\parallel^2$

$$=\mathbb{E}\parallel D(t,\aleph(\gamma))+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\phi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}dr+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\varphi^{(n)}(r,\aleph(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)$$
$$+\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\psi^{(n)}(r,\aleph(r))}{(t-r)^{-\alpha+1-n}}dB(r)-D(\gamma,Y(\gamma))-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\phi^{(n)}(r,Y(r))}{(\gamma-r)^{-\alpha+1-n}}dr$$
$$-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\varphi^{(n)}(r,Y(r))}{(\gamma-r)^{-\alpha+1-n}}d\langle B\rangle(r)-\frac{1}{\Gamma(-\alpha-n)}\int_{-\infty}^{\gamma}\frac{\mho(\gamma,r)\psi^{(n)}(r,Y(r))}{(\gamma-r)^{-\alpha+1-n}}dB(r)\parallel^2$$

$$\leq 4\mathbb{E} \parallel D(\gamma, \aleph(\gamma)) - D(\gamma, Y(\gamma)) \parallel^2 + 4\mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma, r)[\phi^{(n)}(r, \aleph(r)) - \phi^{(n)}(r, Y(r))]dr\frac{1}{\xi} \parallel^2$$
$$+ 4\mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma, r)[\varphi^{(n)}(r, \aleph(r)) - \varphi^{(n)}(r, Y(r))]d\langle B\rangle(r)\frac{1}{\xi} \parallel^2$$
$$+ 4\mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma, r)[\psi^{(n)}(r, \aleph(r)) - \psi^{(n)}(r, Y(r))]dB(r)\frac{1}{\xi} \parallel^2$$
$$= 4\sum_{i=1}^{4} \Pi_i(\gamma)$$

By using the assumption $(H2)$, we get

$$\Pi_1(\gamma) = \mathbb{E} \parallel D(\gamma, \aleph(\gamma)) - D(\gamma, Y(\gamma)) \parallel^2 \leq \mathfrak{L}_D \sup_{\gamma \in R} \mathbb{E} \parallel \aleph(\gamma) - Y(\gamma) \parallel^2 . \tag{15}$$

By using Cauchy-Schwarz inequality, hypothesis $(H1)$ and $(H2)$, we conclude that,

$$\Pi_2(\gamma) = \mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma, r)[\phi^{(n)}(r, \aleph(r)) - \phi^{(n)}(r, Y(r))]dr\frac{1}{\xi} \parallel^2$$
$$\leq \int_{-\infty}^{\gamma} \mho(\gamma, r)dr \int_{-\infty}^{\gamma} \mho(\gamma, r)\mathbb{E} \parallel \phi^{(n)}(r, \aleph(r)) - \phi^{(n)}(r, Y(r)) \parallel^2 dr\frac{1}{\xi^*}$$
$$\leq \frac{\Omega^2 \mathfrak{L}_\phi}{\mu} \int_{-\infty}^{\gamma} e^{-\mu(\gamma - r)}\mathbb{E} \parallel \aleph(r) - Y(r) \parallel^2 dr\frac{1}{\xi^*}$$

$$\leq \frac{\Omega^2 \mathfrak{L}_\phi}{\mu^2} \sup_{\gamma \in R} \parallel \aleph(\gamma) - Y(\gamma) \parallel^2 . \tag{16}$$

Now by using Lemma 2, Cauchy-Schwarz inequality, hypothesis $(H1)$ and $(H2)$, we have,

$$\Pi_3(\gamma) = \mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma, r)[\varphi^{(n)}(r, \aleph(r)) - \varphi^{(n)}(r, Y(r))]d\langle B\rangle(r)\frac{1}{\xi} \parallel^2$$
$$\leq \sigma^{-4}\mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma, r)[\varphi^{(n)}(r, \aleph(r)) - \varphi^{(n)}(r, Y(r))]dr\frac{1}{\xi} \parallel^2$$
$$\leq \sigma^{-4} \int_{-\infty}^{\gamma} \mho(\gamma, r)dr \int_{-\infty}^{\gamma} \mho(\gamma, r)\mathbb{E} \parallel \varphi^{(n)}(r, \aleph(r)) - \varphi^{(n)}(r, Y(r)) \parallel^2 dr\frac{1}{\xi^*}$$

$$\leq \frac{\sigma^{-4}\Omega^2 \mathfrak{L}_\varphi}{\mu^2} \sup_{\gamma \in R} \parallel \aleph(\gamma) - Y(\gamma) \parallel^2 . \tag{17}$$

From Lemma 2, $(H1)$ and $(H2)$, we can easily verify

$$\Pi_4(\gamma) = \mathbb{E} \parallel \int_{-\infty}^{\gamma} \mho(\gamma,r)[\psi^{(n)}(r,\aleph(r)) - h\psi^{(n)}(r,Y(r))]dB(r)\frac{1}{\xi} \parallel^2$$
$$= \mathbb{E}\int_{-\infty}^{\gamma} \parallel \mho(\gamma,r)[\psi^{(n)}(r,\aleph(r)) - \psi^{(n)}(r,Y(r))] \parallel^2 d\langle B\rangle(r)\frac{1}{\xi^*}$$
$$\leq \sigma^{-2}\Omega^2\mathfrak{L}_\psi \int_{-\infty}^{\gamma} e^{-2\mu(\gamma-r)}\mathbb{E} \parallel \aleph(r) - Y(r) \parallel^2 dr\frac{1}{\xi^*}$$

$$\leq \frac{\sigma^{-2}\Omega^2\mathfrak{L}_\psi}{2\mu} \sup_{\gamma\in R} \parallel \aleph(\gamma) - Y(\gamma) \parallel^2 . \tag{18}$$

Now by using equation (15) in (18), we deduce

(19)
$$\mathbb{E} \parallel (\Phi\aleph)(\gamma) - (\Phi Y)(\gamma) \parallel^2 \leq \left[4\mathfrak{L}_D + \frac{4\Omega^2\mathfrak{L}_\phi}{\mu^2} + \frac{\sigma^{-4}\Omega^2\mathfrak{L}_\varphi}{\mu^2} + \frac{2\sigma^{-2}\Omega^2\mathfrak{L}_\psi}{\mu}\right] \sup_{\gamma\in R} \parallel \aleph(\gamma) - Y(\gamma) \parallel^2 .$$

So,

(20)
$$\mathbb{E} \parallel (\Phi\aleph)(\gamma) - (\Phi Y)(\gamma) \parallel^2_{SPAA} \leq \left[4\mathfrak{L}_D + \frac{4\Omega^2\mathfrak{L}_\phi}{\mu^2} + \frac{\sigma^{-4}\Omega^2\mathfrak{L}_\varphi}{\mu^2} + \frac{2\sigma^{-2}\Omega^2\mathfrak{L}_\psi}{\mu}\right] \parallel \aleph(\gamma) - Y(\gamma) \parallel^2_{SPAA} .$$

Consequently, Φ has a unique fixed point in $SPAA(R, \mathfrak{L}^2_G(\mathscr{F}))$, which shows that (1) has unique square mean pseudo almost automorphic mild solution.

4. Conclusion

In this paper, we analysed square mean almost pseudo automorphic mild solution for fractional order neutral stochastic evolution equations driven by G-Brownian motion is obtain by using evolution operator theorem and fixed point theorem. Moreover, we proved this mild solution of equation (1) is unique.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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