Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:83
https://doi.org/10.28919/jmcs/7077
ISSN: 1927-5307

APPROXIMATION OF BEST PROXIMITY PAIR FOR NONCYCLIC RELATIVELY ρ-NONEXPANSIVE MAPPINGS IN MODULAR SPACES ENDOWED WITH A GRAPH

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Abstract. In this work, at first we prove an existence result of best proximity pair for noncyclic relatively ρ-nonexpansive mapping in the setting of modular spaces endowed with a convex directed graph. Furthermore, we study the convergence of a pair of sequences $((x_n, x'_n))_n$ generated by a new iterative scheme for noncyclic relatively (ρG)-nonexpansive mapping in uniformly convex modular spaces equipped with a convex directed graph.

Keywords: best proximity pair; uniformly convex Banach space; noncyclic relatively nonexpansive mapping; Mann iteration; graph theory; modular space; uniformly convex modular; directed graph.

2010 AMS Subject Classification: 32H50.

1. Introduction

The notion of modular spaces was firstly initiated in 1950 by Nakano [11] in connection with the theory of ordered spaces. These spaces was developed and generalized by Orlicz and Musielak [10].

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Received December 13, 2021

Let A and B be two nonempty subsets of a modular space X_ρ. A self-mapping $T : A \cup B \longrightarrow A \cup B$ is said to be noncyclic provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. For such mapping, a minimization problem can be considered such that it solution is called a best proximity pair of the mapping T, that is, a pair $(p,q) \in A \times B$ such that $Tp = p$, $Tq = q$ and $\rho(p-q) = dist_\rho(A,B)$.

The existence of best proximity pair was firstly considered and studied in 2005 by Eldred et al. [4]. They proved that if (A,B) is a nonempty weakly compact pair of a strictly convex Banach space X such that (A,B) has a geometric property called proximal normal structure, then every noncyclic relatively nonexpansive mapping defined on $A \cup B$ has a best proximty pair. Afterwords, many authors have studied and developed many results on the existence problem of best proximity pair for noncyclic mapping under various contractive condition in different type of spaces. In the last fifteen years, the question of the existence and convergence to a best proximity pair was investigated by many authors and found extensions and generalizations for different class of mappings and spaces; for more related works and results, we refer readers to [1, 3, 5, 6, 7, 8].

In the current paper, we prove an existence result for noncyclic relatively nonexpansive mapping in the sitting of modular spaces endowed with a directed graph. Moreover, we established a convergence result for a new iterative process to a best proximity pair for such mapping in modular spaces equipped with a directed graph.

2. Preliminaries

Throughout this work, X stands for a linear vector space on the field $\mathbb{R}$. Let us start with some preliminaries and notations.

Definition 1. [2] *A function $\rho : X \longrightarrow [0,+\infty]$ is called a modular if the following holds:*

(1) $\rho(x) = 0$ *if and only if* $x = 0$*;*
(2) $\rho(-x) = \rho(x)$*;*
(3) $\rho(\alpha x + (1-\alpha)y) \leq \rho(x) + \rho(y)$*, for any* $\alpha \in [0,1]$ *and for any* x*,* y *in* X*.*

If (3) is replaced by (3') $\rho(\alpha x + (1-\alpha)y) \leq \alpha\rho(x) + (1-\alpha)\rho(y)$ for any $\alpha \in [0,1]$ and x, y in X, then ρ is called a convex modular.

The modular space is defined as $X_\rho = \left\{ x \in X : \lim\limits_{\lambda \to 0} \rho(\lambda x) = 0 \right\}$. Throughout this paper, we will assume that the modular ρ is convex.

The Luxemburg norm in X_ρ is defined as

$$\|x\|_\rho = \inf \left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

Associated to a modular, we introduce some basic notions needed throughout this work.

Definition 2. [1] *Let ρ be a modular defined on a vector space X.*

(1) *We say that a sequence $(x_n)_{n\in\mathbb{N}} \subset X_\rho$ is ρ-convergent to $x \in X_\rho$ if and only if $\lim\limits_{n\to\infty} \rho(x_n - x) = 0$. Note that the limit is unique.*
(2) *A sequence $(x_n)_n \subset X_\rho$ is called ρ-Cauchy if $\rho(x_n - x_m) \longrightarrow 0$ as $n, m \longrightarrow +\infty$.*
(3) *We say that X_ρ is ρ-complete if and only if any ρ-Cauchy sequence is ρ-convergent.*
(4) *A subset C of X_ρ is said ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belong to C.*
(5) *A subset C of X_ρ is said ρ-bounded if we have $diam_\rho(C) = \sup\{\rho(x-y) : x, y \in C\} < \infty$.*
(6) *A subset K of X_ρ is said ρ-sequentially compact if any sequence $(x_n)_n$ of C has a subsequence ρ-convergent to a point $x \in C$.*
(7) *We say that ρ satisfy the Fatou property if $\rho(x-y) \leq \liminf\limits_n \rho(x_n - y_n)$ whenever $(x_n)_n$ ρ-converges to x and $(y_n)_n$ ρ-converges to y, for any x_n, x, y, y_n in X_ρ.*

Let us note that ρ-convergence does not imply ρ-Cauchy condition. Also, $x_n \xrightarrow{\rho} x$ does not imply in general $\lambda x_n \xrightarrow{\rho} \lambda x$, for every $\lambda > 1$.

Let A, B be nonempty subsets of a modular space X_ρ. We adopt the notations:

$$dist_\rho(A, B) = \inf\{\rho(x-y) : x \in A\ ,\ y \in B\},$$

$$\delta_\rho(A, B) = \sup\{\rho(x-y)\ :\ x \in A\,,\ y \in B\},$$

$$\delta_x(B) = \delta_\rho(\{x\}, B) = \{\rho(x-y)\ :\ y \in A\}.$$

A pair (A,B) is said to satisfy a property if both A and B satisfy that property. For instance, (A,B) is ρ-closed (resp. convex, ρ-bounded) if and only if A and B are ρ-closed (resp. convex, ρ-bounded). A pair (A,B) is not reduced to one point means that A and B are not singletons.

Recall the definition of the modular uniform convexity.

Definition 3. [2] *Let ρ be a modular and $r>0$, $\varepsilon>0$. Define, for $i\in\{1,2\}$,*

$$D_i(r,\varepsilon)=\left\{(x,y)\in X_\rho\times X_\rho : \rho(x)\leq r\,,\ \rho(y)\leq r\,,\ \rho\left(\frac{x-y}{i}\right)\geq r\varepsilon\right\}.$$

If $D_i(r,\varepsilon)\neq\emptyset$, let

$$\delta_i(r,\varepsilon)=\inf\left\{1-\frac{1}{r}\rho\left(\frac{x+y}{2}\right) : (x,y)\in D_i\right\}.$$

If $D_i(r,\varepsilon)=\emptyset$, we set $\delta_i(r,\varepsilon)=1$. For $i\in\{1,2\}$, we say that

(i) *ρ is uniformly convex (UCi) if for every $r>0$ and $\varepsilon>0$, we have $\delta_i(r,\varepsilon)>0$.*

(ii) *ρ is unique uniformly convex (UUCi) if for all $s\geq 0$ and $\varepsilon>0$, there exists $\eta(s,\varepsilon)>0$ such that $\delta_i(r,\varepsilon)>\eta(s,\varepsilon)$, for $r>s$.*

(iii) *ρ is strictly convex (SC), if for every x, $y\in X_\rho$ such that $\rho(x)=\rho(y)$ and $\rho\left(\frac{x+y}{2}\right)=\frac{\rho(x)+\rho(y)}{2}$, we have $x=y$.*

The following proposition characterize the relationship between the above notions:

Proposition 1. [2]

(a) *$(UUCi)$ implies (UCi) for $i=1,2$;*

(b) *$\delta_1(r,\varepsilon)\leq\delta_2(r,\varepsilon)$ for $r>0$ and $\varepsilon>0$;*

(c) *$(UC1)$ implies $(UC2)$ implies (SC);*

(d) *$(UUC1)$ implies $(UUC2)$.*

Definition 4. [9] *Let ρ be a modular. We say that the modular space X_ρ satisfies the property (R) if and only if for every decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of nonempty ρ-closed convex and ρ-bounded subsets of X_ρ has a nonempty intersection.*

Lemma 1. [2] *Let ρ be a convex modular satisfying the Fatou property. Assume that X_ρ is ρ-complete and ρ is (UUC2). Then X_ρ satisfies the property (R).*

Proposition 2. [2] *Let ρ be a convex modular. Assume that X_ρ is ρ-complete and ρ is (UUC2). Let C be a ρ-closed convex and ρ-bounded nonempty subset of X_ρ. Let $(C_i)_{i\in I}$ be a family of ρ-closed convex nonempty subsets of C such that $\bigcap\limits_{i\in F} C_i$ is nonempty, for any finite subset F of I. Then, $(C_i)_{i\in I}$ has a nonempty intersection.*

Let us finish this section with a few terminology and basic definitions of graph theory. Let C be a nonempty subset of a modular space X_ρ and $\Delta = \{(x,x) : x \in C\}$ the loops set. Consider a directed graph G such that the set of vertices $V(G)$ coincides with C and the set of its edges $E(G)$ contains all loops, i.e. $\Delta \subset E(G)$. Assume that G has no parallel edges, so it can be identified to the pair $(V(G),E(G))$.

Let x and y be vertices of a graph G. A path from x to y of length $N \in \mathbb{N}$ is a finite sequence (x_n) of $N+1$ elements for which $x_0 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$, for $i = 0, \dots, N-1$.

A graph G is said to be connected if there is a path between any two vertices of the graph G. A directed graph $G = (V(G),E(G))$ is said to be transitive if, for any x, y and z in $V(G)$ such that (x,y) and (y,z) are in $E(G)$, then $(x,z) \in E(G)$. Moreover, the conversion of a graph G, denoted G^{-1}, is the graph obtained by reversing the direction of the edges of the graph G. Thus, we have $E(G^{-1}) = \{(y,x) \in X \times X : (x,y) \in E(G)\}$.

Definition 5. *Let X_ρ be a modular space. A graph G is said to be convex if and only if for any x, y, z, w in X_ρ and $\lambda \in [0,1]$, we have*

$$(x,y) \in E(G) \text{ and } (z,w) \in E(G) \text{ leads to } ((1-\lambda)x + \lambda z, (1-\lambda)y + \lambda w) \in E(G).$$

Definition 6. *Let C be a nonempty subset of a modular space X_ρ, and let $G = (V(G),E(G))$ be a directed graph such that $V(G) = C$. We say that C have the property (OSC):*
if each sequence $(x_n)_n \subset C$ ρ-converges to x and $(x_n, x_{n+1}) \in E(G)$, then $(x_n, x) \in E(G)$, for all $n \geq 0$.

3. Main Results

Definition 7. *Let (A,B) be a nonempty pair of a modular space X_ρ. A mapping $T : A \cup B \longrightarrow A \cup B$ is said to be noncyclic relatively (ρG)-nonexpansive mapping if the following conditions hold:*

i) *T is noncyclic, i.e. $T(A) \subset A$ and $T(B) \subset B$;*

ii) *T preserves edges, i.e. for all $x \in A$ and $b \in B$, if $(x,y) \in E(G)$ then $(Tx,Ty) \in E(G)$;*

iii) *$\rho(Tx - Ty) \leq \rho(x-y)$, for all $(x,y) \in A \times B$ such that $(x,y) \in E(G)$.*

In the sequel, we assume that the modular space X_ρ is equipped with a convex transitive graph G identified by the pair $(V(G),E(G))$.

Definition 8. *We will say that a pair of sequences $((x_n,y_n))_n$ ρ-converges to a pair (x,y), if the sequences $(x_n)_n$ and $(y_n)_n$ ρ-converge to x and y, respectively.*

Definition 9. *Let C be a nonempty subset of a modular space X_ρ, and let $G = (V(G),E(G))$ be a directed graph such that $V(G) = C$. We say that G have the property (F), if $(x_n,y_n) \in E(G)$ and $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$, then $(x,y) \in E(G)$.*

Definition 10. *Let C be a nonempty subset of a modular space X_ρ, and let $G = (V(G),E(G))$ be a directed graph such that $V(G) = C$. We say that G have the property ($\mathscr{L}$) if the graph G does not have parallel edges, that is, if $(a,b) \in E(G)$ and $(b,a) \in E(G)$, then $a = b$.*

Definition 11. *We say that a nonempty pair (A,B) of a modular space is proximal ρ-compactness provided that every generalized sequence $((x_\alpha,y_\alpha))_{\alpha\in I}$ of $A\times B$ satisfying the following condition $\rho(x_\alpha - y_\alpha) = dist_\rho(A,B)$ has a ρ-convergent subsequence in (A,B).*

Property 1. *Let (A,B) be a nonempty pair of a modular space X_ρ such that A_0 is nonempty. If (A_0,B_0) is proximal ρ-compactness then (A_0,B_0) is ρ-closed.*

Proof. Let $(x_n)_n$ be a sequence in A_0 which ρ-converges to $x \in X_\rho$. For all $n \in \mathbb{N}$, there exists $y_n \in B_0$ such that $\rho(x_n - y_n) = dist_\rho(A,B)$.
Since (A_0,B_0) is proximal ρ-compactness, then there exists a subsequence $\left((x_{\varphi(n)},y_{\varphi(n)})\right)_n$ of the sequence $((x_n,y_n))_n$ which ρ-converges to a pair $(x',y') \in A_0 \times B_0$. Thus, $x = x'$. Since ρ satisfies the Fatou property, then

$$\rho(x-y') \leq \liminf_n \rho\left(x_{\varphi(n)} - y_{\varphi(n)}\right) = dist_\rho(A,B).$$

Therefore, $x \in A_0$. In the same way we show that B_0 is ρ-closed. $\square$

Theorem 1. *Let ρ be a convex modular satisfying Fatou property and (UUC1). Let (A,B) be a nonempty convex and ρ-bounded pair of a ρ-complete modular space X_ρ. Assume that A_0 is nonempty and (A_0,B_0) is proximal ρ-compactness, and $(x,y) \in E(G)$ for all $(x,y) \in A_0 \times B_0$. Suppose that $T : A \cup B \longrightarrow A \cup B$ is noncyclic relatively (ρG)-nonexpansive. Then, T has a best proximity pair (x,y) in $A \cup B$ such that $(x,y) \in E(G)$.*

Proof. The theorem is trivial if we assume that $A \cap B \neq \emptyset$, so we assume that $A \cap B = \emptyset$. Suppose that $\mathscr{F}$ denotes the collection of all nonempty ρ-closed convex and ρ-bounded pairs (E,F) of (A_0,B_0) such that T is noncyclic on (E,F) and $\rho\,(x-y) = d_\rho\,(A,B)$, for some $(x,y) \in E \times F$. Since A_0 is nonempty, ρ-closed, convex and ρ-bounded, and T is noncyclic on $A_0 \cup B_0$, then $(A_0,B_0) \in \mathscr{F}$. Hence, $\mathscr{F}$ is nonempty. Moreover, $\mathscr{F}$ is partially ordered by a reverse inclusion, that is, $(E_1,F_1) \leq (E_2,F_2)$ if and only if $(E_2,F_2) \subseteq (E_1,F_1)$. Let $(E_\alpha,F_\alpha)_{\alpha\in I}$ be a decreasing chain in $\mathscr{F}$. Let $E = \bigcap_\alpha E_\alpha$ and $F = \bigcap_\alpha F_\alpha$. By the fact that X_ρ satisfies the property (R), the pair (E,F) is nonempty ρ-closed and convex. Since $(E,F) \subseteq (A_0,B_0)$, then $(x,y) \in E(G)$ for all $(x,y) \in E \times F$. Moreover, T is noncyclic on $E \cup F$. Indeed, let $x \in E$, so $x \in E_\alpha$, for all $\alpha \in I$. Since T is noncyclic on $E_\alpha \cup F_\alpha$ for all $\alpha \in I$, then $Tx \in E_\alpha$ for all $\alpha \in I$. Therefore, $Tx \in E = \bigcap_\alpha E_\alpha$. Using the same arguments, one has $T\,(F_\alpha) \subseteq F_\alpha$.

Let $(x_\alpha,y_\alpha) \in E_\alpha \times F_\alpha$ such that $\rho\,(x_\alpha - y_\alpha) = dist_\rho\,(A,B)$. We have $(x_\alpha,y_\alpha) \in E(G)$, for all $\alpha \in I$. Since (A_0,B_0) is proximal ρ-compactness, then there exists a subsequence $(x_{\alpha_i},y_{\alpha_i})$ of the sequence (x_α,y_α) such that $x_{\alpha_i} \xrightarrow{\rho} x \in E$ and $y_{\alpha_i} \xrightarrow{\rho} y \in F$. Using the Fatou property, one has

$$\rho\,(x-y) \leq \liminf_i \rho\,(x_{\alpha_i} - y_{\alpha_i}) = dist_\rho(A,B).$$

Therefore, there exists a pair $(x,y) \in E \times F$ such that $\rho\,(x-y) = dist_\rho\,(A,B)$. So every increasing chain in $\mathscr{F}$ is bounded above with respect to the reverse inclusion. Hence, Zorn's lemma implies that $\mathscr{F}$ has a minimal element denoted by (K_1,K_2). Thus, $(x,y) \in E(G)$ for all $(x,y) \in K_1 \times K_2$, and $dist_\rho\,(K_1,K_2) = dist_\rho\,(A,B)$. Since T is noncyclic, then

$$T\,(\overline{conv}\,(K_1)) \subseteq T\,(K_1) \subseteq \overline{conv}\,(T\,(K_1)) \quad \text{and} \quad T\,(\overline{conv}\,(K_2)) \subseteq T\,(K_2) \subseteq \overline{conv}\,(T\,(K_2))\,,$$

where the notation $\overline{conv}\,(K_i)$ describes the ρ-closed convex hull of K_i, for $i \in \{1,2\}$. In fact, using the definition of the ρ-closed convex hull of a set, it is quite easy to see that

$T(K_1) \subseteq \overline{conv}(T(K_1))$.

Now, let us prove that $T(\overline{conv}(T(K_1))) \subseteq T(K_1)$. We have $T(K_1) \subseteq K_1$, then $\overline{conv}(T(K_1)) \subseteq K_1$. Hence, $T(\overline{conv}(T(K_1))) \subseteq T(K_1)$. Therefore, $T(\overline{conv}(K_1)) \subseteq T(K_1) \subseteq \overline{conv}(T(K_1))$. As the same way, we show that $T(\overline{conv}(K_2)) \subseteq T(K_2) \subseteq \overline{conv}(T(K_2))$. Thus, T is non-cyclic on $\overline{conv}(T(K_1)) \cup \overline{conv}(T(K_2))$. The minimality of the pair (K_1, K_2) in $\mathscr{F}$ leads to $K_1 = \overline{conv}(T(K_1))$ and $K_2 = \overline{conv}(T(K_2))$. If $dist_\rho(K_1, K_2) = 0$, then T has a fixed point on $A \cap B$. Consequently, we assume that $dist_\rho(K_1, K_2) > 0$.

Case one: $\min\left\{diam_\rho(K_1), diam_\rho(K_2)\right\} = 0$. We assume that $K_1 = \{x^*\}$. By the fact that (K_1, K_2) is proximal, there exists $y^* \in K_2$ such that $\rho(x^* - y^*) = dist_\rho(A, B)$.

Moreover, $(x^*, y^*) \in E(G)$. Since T is noncyclic relatively (ρG)-nonexpansive on $K_1 \cup K_2$, one has

$$\rho(T(x^*) - T(y^*)) = \rho(x^* - T(y^*)) \leq \rho(x^* - y^*) = dist_\rho(A, B).$$

Thus, $\rho(x^* - T(y^*)) = dist_\rho(A, B)$. Otherwise,

$$dist_\rho(A, B) \leq \rho\left(x^* - \frac{y^* + Ty^*}{2}\right) \leq \frac{1}{2}\rho(x^* - y^*) + \frac{1}{2}\rho(x^* - Ty^*) = dist_\rho(A, B).$$

Since ρ is (UUC1), then ρ is strict convexity (SC). Hence, $x^* - y^* = x^* - Ty^*$. Thus, $y^* = Ty^*$. Therefore, T has a best proximity pair $(x^*, y^*) \in A \times B$ such that $(x^*, y^*) \in E(G)$.

Case two: If $\min\left\{diam_\rho(K_1), diam_\rho(K_2)\right\} > 0$.

Assume that T does not have a best proximity pair. Let $(p, q) \in K_1 \times K_2$ such that $\rho(p - q) = dist_\rho(A, B)$. Moreover, we have $(p, q) \in E(G)$. Since T is noncyclic relatively (ρG)-nonexpansive, one has

$$dist_\rho(A, B) \leq \rho(Tp - Tq) \leq \rho(p - q) = dist_\rho(A, B).$$

Hence, $\rho(Tp - Tq) = dist_\rho(A, B)$. Thus, we must have $p \neq Tp$ and $q \neq Tq$. Therefore,

$$\rho\left(\frac{p + Tp}{2} - \frac{q + Tq}{2}\right) = dist_\rho(A, B).$$

Set $\varepsilon_0 = \min\{\rho(p - Tp), \rho(q - Tq)\}$. We have $\varepsilon_0 > 0$ and $R = \delta_\rho(K_1, K_2) > 0$, since $p \neq Tp$ and $q \neq Tq$. For all $y \in K_2$, one has $(p, y) \in E(G)$ since $(K_1, K_2) \subseteq (A_0, B_0)$ and $(x, y) \in E(G)$, for all $(x,y) \in A_0 \times B_0$. Thus, for all $y \in K_2$ one has

$$\rho(p-y) \leq R \text{ and } \rho(Tp-y) \leq R.$$

Moreover, $\rho(Tp-p) \geq \varepsilon_0 \geq R\dfrac{\varepsilon_0}{2R}$. Therefore, using the fact that ρ is (UUC1) we get

$$\rho\left(y-\frac{p+Tp}{2}\right) \leq R\left(1-\eta_1\left(\frac{R}{2},\frac{\varepsilon_0}{2R}\right)\right) < R.$$

Hence, $\delta_{\frac{p+Tp}{2}}(K_2) < R$. As the same way, one can prove that $\delta_{\frac{q+Tq}{2}}(K_1) < R$.
Set $x^* = \dfrac{p+Tp}{2}$ and $y^* = \dfrac{q+Tq}{2}$. Therefore, $(x^*,y^*) \in K_1 \times K_2$ such that $(x^*,y^*) \in E(G)$, $\rho(x^*-y^*) = dist_\rho(A,B)$ and $\max\left\{\delta_{x^*}(K_2), \delta_{y^*}(K_1)\right\} < \delta_\rho(K_1,K_2)$. Suppose that there exists $\lambda \in (0,1)$ such that $\max\left\{\delta_{x^*}(K_2), \delta_{y^*}(K_1)\right\} \leq \lambda\delta_\rho(K_1,K_2)$. Set

$$G_1 = \left\{x \in K_1 \,:\, \delta_x(K_2) \leq \lambda\delta_\rho(K_1,K_2)\right\} \text{ and } G_2 = \left\{x \in K_2 \,:\, \delta_x(K_1) \leq \lambda\delta_\rho(K_1,K_2)\right\}.$$

Thus, $(x^*,y^*) \in G_1 \times G_2$. Moreover, (G_1,G_2) is a ρ-closed convex pair.
In fact, let $\gamma \in (0,1)$ and $x, x' \in G_1$, and prove that $z = \gamma x + (1-\gamma)x' \in G_1$. For all $y \in K_2$, one has

$$\begin{aligned}
\rho(z-y) &= \rho\left(\gamma x + (1-\gamma)x' - y\right) \\
&\leq \gamma\rho(x-y) + (1-\gamma)\rho\left(x'-y\right) \\
&\leq \gamma\lambda\delta_\rho(K_1,K_2) + (1-\gamma)\lambda\delta_\rho(K_1,K_2) \\
&\leq \lambda\delta_\rho(K_1,K_2).
\end{aligned}$$

Therefore, $\delta_z(K_2) \leq \lambda\delta_\rho(K_1,K_2)$, that is, $z \in G_1$. Therefore, G_1 is ρ-closed.
Let $(x_n)_n$ be a sequence in G_1 such that $(x_n)_n$ ρ-converges to $x \in K_1$. Using the Fatou property for all $y \in K_2$, one has $\rho(x-y) \leq \liminf_n \rho(x_n-y)$. Since $x_n \in G_1$, for all $n \geq 0$, then $\rho(x_n-y) \leq \lambda\delta_\rho(K_1,K_2)$ for all $n \geq 0$.
Hence, $\liminf_n \rho(x_n-y) \leq \lambda\delta_\rho(K_1,K_2)$. Thus, $\rho(x-y) \leq \lambda\delta_\rho(K_1,K_2)$ for all $y \in K_2$. Therefore, $\delta_x(K_2) \leq \lambda\delta_\rho(K_1,K_2)$. This implies that $x \in G_1$, that is, G_1 is ρ-closed. Following the same arguments one can prove that G_2 is ρ-closed convex.
Now, let us prove that T is noncyclic on $G_1 \cup G_2$. Let $x \in G_2$, for all $y \in K_2$ we have $(x,y) \in E(G)$

and

$$\rho\left(Tx - Ty\right) \leq \rho\left(x - y\right) \leq \lambda \delta_{\rho}\left(K_1, K_2\right).$$

Then, $Ty \in B_{\rho}\left(Tx, \lambda \delta_{\rho}\left(K_1, K_2\right)\right)$, for all $y \in K_2$. Thus, $T\left(K_2\right) \subseteq B_{\rho}\left(Tx, \lambda \delta_{\rho}\left(K_1, K_2\right)\right) \cap K_2$. Since $K_2 = \overline{conv}\left(T\left(K_2\right)\right)$, we have $K_2 \subseteq B_{\rho}\left(Tx, \lambda \delta_{\rho}\left(K_1, K_2\right)\right)$.
Therefore, $\delta_{Tx}\left(K_2\right) \leq \lambda \delta_{\rho}\left(K_1, K_2\right)$. Hence, $Tx \in G_1$, that is, $T\left(G_1\right) \subseteq G_1$. As the same way, one can prove that $T\left(G_2\right) \subseteq G_2$. The minimality of the pair $\left(K_1, K_2\right)$ in the collection $\mathscr{F}$, implies that $G_1 = K_1$ and $G_2 = K_2$. Therefore, $x \in K_1$ and $\delta_x\left(K_1\right) \leq \lambda \delta_{\rho}\left(K_1, K_2\right)$. Thus,

$$\delta_{\rho}\left(K_1, K_2\right) = \sup_{x \in K_1} \delta_x\left(K_2\right) \leq \lambda \delta_{\rho}\left(K_1, K_2\right).$$

Hence, $\delta_{\rho}\left(K_1, K_2\right) = 0$, contradiction. Therefore, T has a best proximity pair $\left(x^*, y^*\right) \in A \times B$ such that $\left(x^*, y^*\right) \in E(G)$. $\square$

Proposition 3. *Let ρ be a convex modular (UUC1) such that for all $r > 0$ and $\varepsilon > 0$ we have $\rho\left(x\right) \leq r$, $\rho\left(y\right) \leq r$ and $\rho\left(x - y\right) \geq r\varepsilon$. Then,*

$$\rho\left(tx + (1-t)y\right) \leq r\left(1 - 2t(1-t)\eta_1\left(s, \varepsilon\right)\right),$$

for all $s < r$ and $t \in [0,1]$.

Proof. Let $r > 0$, $\varepsilon > 0$ and x, $y \in X_{\rho}$ such that

$$\rho\left(x\right) \leq r,\ \rho\left(y\right) \leq r \ \text{ and } \ \rho\left(x - y\right) \geq r\varepsilon.$$

Since ρ is (UUC1), there exists $\eta_1(s, \varepsilon) > 0$ such that $\delta_1(r, \varepsilon) \geq \eta_1(s, \varepsilon) > 0$, for $s < r$.
• If $t = \dfrac{1}{2}$, there is nothing to prove.

- If $t \in \left[0, \frac{1}{2}\right)$, then

$$\begin{aligned}
\rho\left(tx+(1-t)y\right) &= \rho\left(t(x+y)+(1-2t)y\right) \\
&= \rho\left(2t\frac{x+y}{2}+(1-2t)y\right) \\
&\leq 2t\rho\left(\frac{x+y}{2}\right)+(1-2t)\rho\left(y\right) \\
&\leq 2tr\left(1-\eta_1(s,\varepsilon)\right)+r(1-2t) \\
&\leq 2rt-2rt\eta_1\left(s,\varepsilon\right)+r-2rt \\
&\leq r\left(1-2t\eta_1\left(s,\varepsilon\right)\right).
\end{aligned}$$

Thus,

$$\rho\left(tx+(1-t)y\right) \leq r\left(1-2t\eta_1\left(s,\varepsilon\right)\right). \tag{1}$$

- If $t \in \left(\frac{1}{2}, 1\right]$, then

$$\begin{aligned}
\rho\left(tx+(1-t)y\right) &= \rho\left(tx-(1-t)x+(1-t)x+(1-t)y\right) \\
&= \rho\left((2t-1)x+2(1-t)\frac{x+y}{2}\right) \\
&\leq (2t-1)\rho\left(x\right)+2(1-t)\rho\left(\frac{x+y}{2}\right) \\
&\leq (2t-1)r+2r(1-t)\left(1-\eta_1(s,\varepsilon)\right) \\
&\leq 2rt-r+2r-2rt-2r(1-t)\eta_1(s,\varepsilon) \\
&\leq r\left(1-2(1-t)\eta_1\left(s,\varepsilon\right)\right).
\end{aligned}$$

Thus,

$$\rho\left(tx+(1-t)y\right) \leq r\left(1-2(1-t)\eta_1\left(s,\varepsilon\right)\right). \tag{2}$$

Otherwise, for all $t \in [0,1]$, we have $2t \geq 2t(1-t)$ and $2(1-t) \geq 2t(1-t)$. Therefore,

$$\rho\left(tx+(1-t)y\right) \leq r\left(1-2t(1-t)\eta_1\left(s,\varepsilon\right)\right),$$

for all $s < r$. $\square$

Consider the following two sequences given by: for all $n \in \mathbb{N}$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, z_n + \alpha_n y_n \\ y_n = (1-\beta_n)\, z_n + \beta_n T z_n \\ z_n = (1-\delta_n)\, x_n + \delta_n T x_n \end{cases} \quad \text{and} \quad \begin{cases} x'_{n+1} = (1-\alpha_n)\, z'_n + \alpha_n y'_n \\ y'_n = (1-\beta_n)\, z'_n + \beta_n T z'_n \\ z'_n = (1-\delta_n)\, x'_n + \delta_n T x'_n \end{cases} \tag{3}$$

where $(x_0, x'_0) \in A_0 \times B_0$ such that $\rho\,(x_0 - y_0) = dist_\rho\,(A,B)$, and $(\alpha_n)_n$, $(\beta_n)_n$ and $(\delta_n)_n$ are sequences in $(0,1)$ such that

$C_1)$ $\lim\limits_n \alpha_n = 0$;

$C_2)$ $0 < a \leq \delta_n \leq b < 1$ and $\lim\limits_n \delta_n = \theta$.

We will say that $((x_n)_n\,; (x'_n)_n)$ ρ-converges to a best proximity pair (x,x'), if (x_n) ρ-converges to x and (x'_n) ρ-converges to x' such that $Tx = x$, $Tx' = x'$ and $\rho\,(x - x') = d_\rho\,(A,B)$.

Lemma 2. *Let ρ be a convex modular and X_ρ be a modular space endowed with a directed transitive and convex graph $G = (V(G), E(G))$, where $V(G) = A \cup B$ is non empty ρ-closed convex and ρ-bounded, and $E(G)$ is convex and $\Delta \subseteq E(G)$. Let $T : A \cup B \longrightarrow A \cup B$ be mapping which preserve edges. Assume that there exists $(x_0, y_0) \in A \times B$ such that $(x_0, Tx_0) \in E(G)$ and $(x'_0, Tx'_0) \in E(G)$. Then, the pair of sequences $((x_n, x'_n))_n$ is G-monotone, that is, (x_n, x_{n+1}) and (x'_n, x'_{n+1}) are in $E(G)$, for all $n \geq 0$. Moreover, (x_n, Tx_n) and (x'_n, Tx'_n) are in $E(G)$, for all $n \geq 0$.*

Proof. By the convexity of the G-interval $[x_0, Tx_0]$, one has (x_0, z_0) and (z_0, Tx_0) are in $E(G)$. Since T preserves edges, then (Tx_0, Tz_0) in $E(G)$. Transitivity of the graph G leads to (z_0, Tz_0) in $E(G)$. The convexity of the G-interval $[z_0, Tz_0]$ implies that (z_0, y_0) and (y_0, Tz_0) are in $E(G)$. The convexity of the G-interval $[z_0, y_0]$ leads to (z_0, x_1) and (x_1, y_0) are in $E(G)$. Therefore, $(x_0, x_1) \in E(G)$. Moreover, since T preserves edges one has $(Tz_0, Tx_1) \in E(G)$. Transitivity of the graph G for (x_1, y_0), (y_0, Tz_0) and (Tz_0, Tx_1) leads to $(x_1, Tx_1) \in E(G)$. Thus, (x_0, x_1) and (x_1, Tx_1) are in $E(G)$.

By induction, let us prove that $(x_n, x_{n+1}) \in E(G)$. By the convexity of the G-interval $[x_n, Tx_n]$, one has (x_n, z_n) and (z_n, Tx_n) are in $E(G)$. Since T preserves edges, one has (Tx_n, Tz_n) in $E(G)$. Transitivity of the graph G for (z_n, Tx_n) and (Tx_n, Tz_n) implies that $(z_n, Tz_n) \in E(G)$. The convexity of the G-interval $[z_n, Tz_n]$ implies that (z_n, y_n) and (y_n, Tz_n) are in $E(G)$. Convexity of the G-interval $[z_n, y_n]$ leads to (z_n, x_{n+1}) and (x_{n+1}, y_n) are in $E(G)$. Transitivity of the graph G for (x_n, z_n) and (z_n, x_{n+1}) implies that $(x_n, x_{n+1}) \in E(G)$. Moreover, (x_{n+1}, y_n), (y_n, Tz_n) and (Tz_n, Tx_{n+1}) leads to $(x_{n+1}, Tx_{n+1}) \in E(G)$. Thus, (x_n, x_{n+1}) and (x_{n+1}, Tx_{n+1}) are in $E(G)$. Therefore, the sequence $(x_n)_n \subset A$ is G-monotone. Following the same argument, we prove that the sequence $(x'_n)_n \subset B$ is also G-monotone. Hence, the pair of sequences $((x_n, x'_n))_n$ is G-monotone. $\square$

Theorem 2. *Let ρ be a convex modular satisfying Fatou property and (UUC1). Let (A,B) be a nonempty convex and ρ-bounded pair of a ρ-complete modular space X_ρ. Assume that (A_0, B_0) is proximal ρ-compactness, and $(x,x') \in E(G)$ for all $(x,x') \in A_0 \times B_0$ and G satisfies the property (OSC). Let $T : A \cup B \longrightarrow A \cup B$ be a noncyclic relatively (ρG)-nonexpansive mapping. Assume that there exists $x_0 \in A_0$ such that $(x_0, Tx_0) \in E(G)$. Then,*

i) *the pair of sequences $((x_n, x'_n))_n$ of (3) has a subsequence $\left(\left(x_{\varphi(n)}, x'_{\varphi(n)}\right)\right)_n$ which ρ-converges to a best proximity pair (x,x') of T;*
ii) *Moreover, if G satisfies the property ($\mathscr{L}$) then sequences $((x_n, x'_n))_n$ of (3) ρ-converge to a best proximity pair of T.*

Proof. The theorem is trivial if $A \cap B \neq \emptyset$, so we assume that $A \cap B = \emptyset$. It follows from the proof of Theorem 1 that T has a best proximity pair $(p,q) \in A \times B$, then T has fixed point $q \in B_0$. Therefore, $(x_0, q) \in E(G)$, so $(Tx_0, q) \in E(G)$ and for all $n \in \mathbb{N}$, $(x_{n+1}, q) \in E(G)$, and

$$\begin{aligned}
\rho\left(x_{n+1} - q\right) &\leq \alpha_n \rho\left(y_n - q\right) + (1-\alpha_n)\rho\left(z_n - q\right) \\
&\leq \alpha_n\left(\beta_n \rho\left(Tz_n - q\right) + (1-\beta_n)\rho\left(z_n - q\right)\right) + (1-\alpha_n)\rho\left(z_n - q\right) \\
&\leq \alpha_n \rho\left(z_n - q\right) + (1-\alpha_n)\rho\left(z_n - q\right) \\
&\leq \rho\left(z_n - q\right) \\
&\leq \delta_n \rho\left(Tx_n - q\right) + (1-\delta_n)\rho\left(x_n - q\right) \\
&\leq \rho\left(x_n - q\right).
\end{aligned}$$

Hence, $(\rho(x_n-q))_n$ is a decreasing sequence. Therefore, $\lim_n \rho(x_n-q)=r\geq 0$ exists. Now, let us prove that $\lim_n \rho(x_n-Tx_n)=0$. Assume that there exist $\varepsilon_0>0$ and a subsequence $(x_{n_k})_k$ of the sequence $(x_n)_n$ such that $\rho(x_{n_k}-Tx_{n_k})\geq\varepsilon_0$.
For $\varepsilon\in(0,1)$, there exists $k_0\in\mathbb{N}$ such that $\rho\left(x_{n_{k_0}}-q\right)\leq r+2\varepsilon$, so $\rho\left(Tx_{n_{k_0}}-q\right)\leq r+2\varepsilon$. Since $(\rho(x_n-q))_n$ is a decreasing sequence, then for all $k\geq k_0$ we have $\rho(x_{n_k}-q)\leq r+2\varepsilon$ and $\rho(Tx_{n_k}-q)\leq r+2\varepsilon$.
Since ρ is (UUC1) and $\rho(x_{n_k}-Tx_{n_k})\geq\varepsilon_0>\dfrac{\varepsilon_0}{r+2}(r+2\varepsilon)$ and by the condition (C_1) and Proposition 3, one has

$$
\begin{aligned}
\rho\left(x_{n_k+1}-q\right) &\leq \alpha_{n_k}\rho(y_{n_k}-q)+(1-\alpha_{n_k})\rho(z_{n_k}-q)\\
&\leq \rho(z_{n_k}-q)\\
&= \rho\left(\delta_{n_k}(Tx_{n_k}-q)+(1-\delta_{n_k})(x_{n_k}-q)\right)
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&\leq (r+2\varepsilon)\left(1-2\delta_{n_k}(1-\delta_{n_k})\eta_1\left(r,\frac{\varepsilon_0}{r+2}\right)\right)\\
&\leq (r+2\varepsilon)\left(1-2a(1-b)\eta_1\left(r,\frac{\varepsilon_0}{r+2}\right)\right).
\end{aligned}
$$

By the condition (C_2) and limit as k goes to infinity, one has

$$
r=\lim_k \rho\left(x_{n_k+1}-q\right)\leq (r+2\varepsilon)\left(1-2a(1-b)\eta_1\left(r,\frac{\varepsilon_0}{r+2}\right)\right).
$$

Since ε is arbitrary chosen, we let ε goes to 0 and then

$$
r\leq r\left(1-2a(1-b)\eta_1\left(r,\frac{\varepsilon_0}{r+2}\right)\right)<r,
$$

which leads to a contradiction. Therefore, $\lim_{n\to\infty}\rho(x_n-Tx_n)=0$. In the same way we show that $\lim_n \rho\left(x'_n-Tx'_n\right)=0$.
Otherwise, we have $\rho\left(x_0-x'_0\right)=dist_\rho(A,B)$ then by the definition of the sequences $(x_n)_n$ and $(x'_n)_n$, and the convexity of ρ one has $\rho(x_n-x'_n)=dist_\rho(A,B)$ and $(x_n,x'_n)\in E(G)$ for all $n\geq 0$. Since (A_0,B_0) is proximal ρ-compactness, then the sequence $(x_n,x'_n)_n$ has a subsequence $\left(x_{\varphi(n)},x'_{\varphi(n)}\right)_n$ which ρ-converges to $(x,x')\in A_0\times B_0$. Let us prove that (x,x') is a best proximity pair of T. We have $\left(x_{\varphi(n)}\right)_n$ and $\left(x'_{\varphi(n)}\right)_n$ ρ-converge to x and x' respectively, and $\rho\left(x_{\varphi(n)}-x'_{\varphi(n)}\right)=dist_\rho(A,B)$, then using Fatou property we get

$$
\rho\left(x-x'\right)\leq\liminf_n \rho\left(x_{\varphi(n)}-x'_{\varphi(n)}\right)=dist_\rho(A,B).
$$

Therefore, $\rho\left(x-x'\right)=dist_{\rho}\left(A,B\right)$.

Let us prove that the sequence $\left(x_n\right)_n$ ρ-converges to x. By contradiction, we assume that $\left(x_n\right)_n$ does not ρ-converges to x. Thus, there exists $V\in\mathscr{V}(x)$ a neighbourhood of x and a subsequence $\left(x_{\psi(x)}\right)_n$ such that $x_{\psi(x)}\notin V$ for all $n\in\mathbb{N}$. Since A_0 is ρ-sequentially compact, then the sequence $\left(x_{\psi(n)}\right)_n$ has a subsequence $\left(x_{\psi\circ\sigma(n)}\right)_n$ ρ-converges to $y\in A_0$. We have the sequence $\left(x_{\psi\circ\sigma(n)}\right)_n$ is G-monotone and satisfies the property (OSC). Thus, $\left(x_{\psi\circ\sigma(n)},y\right)\in E(G)$. Also we have $\left(x_{\varphi(x)},x\right)\in E(G)$. For $x_{\psi\circ\sigma(n)}$, we construct a sequence $\left(k_n\right)_n$ such that for all $x_{\psi\circ\sigma(n)}$ one has $\left(x_{\psi\circ\sigma(k_n)},x_{\varphi(k_n)}\right)\in E(G)$. Property (F) implies that $(y,x)\in E(G)$. Otherwise, for $x_{\psi\circ\sigma(n)}$, we construct a sequence $\left(h_n\right)_n$ such that for all $x_{\psi\circ\sigma(n)}$ one has $\left(x_{\varphi(h_n)},x_{\psi\circ\sigma(h_n)}\right)\in E(G)$. The property (F) leads to $(x,y)\in E(G)$. We have (x,y) and (y,x) are in $E(G)$. By the property ($\mathscr{L}$), one has $x=y$. Therefore, the sequence $\left(x_{\psi\circ\sigma(n)}\right)_n$ ρ-converges to x. Hence, there exists $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$, $x_{\psi\circ\sigma(n)}\in V$ which a contradiction. Thus, $\left(x_n\right)_n$ ρ-converges to x.

Now, let us prove that $x=Tx$ and $x'=Tx'$. Consider the subsets

$$C_n=\left\{y\in B_0\ :\ \left(x_n,y\right)\in E(G)\ \text{ and }\ \rho\left(x_n-y\right)=dist_{\rho}\left(A,B\right)\right\},$$

for all $n\in\mathbb{N}$. It is quite easy to see that C_n is nonempty convex and ρ-bounded, for all $n\geq 0$. Moreover, let $\left(z_p\right)_p$ be a sequence of C_n which ρ-converges to a point $z\in B_0$. Let us prove that $z\in C_n$. Since $x_n\in A_0$ and $z\in B_0$, then $\left(x_n,z\right)\in E(G)$. Otherwise, using the Fatou property one has

$$dist_{\rho}\left(A,B\right)\leq\rho\left(x_n-z\right)\leq\liminf_{p}\rho\left(x_n-z_p\right)=dist_{\rho}\left(A,B\right),$$

for all $n\geq 0$. Thus, $\rho\left(x_n-z\right)=dist_{\rho}\left(A,B\right)$, for all $n\geq 0$. Therefore, C_n is ρ-closed for all $n\geq 0$. Using the property (R), we obtain $\bigcap\limits_{n}C_n$ is nonempty ρ-closed and convex. Hence, there exists $w\in B_0$ such that $\left(x_n,w\right)\in E(G)$ and $\rho\left(x_n-w\right)=dist_{\rho}\left(A,B\right)$ for all $n\geq 0$. By Fatou property, we have $\rho\left(x-w\right)\leq\liminf\limits_{n}\rho\left(x_n-w\right)$. Thus, $\rho\left(x-w\right)=dist_{\rho}\left(A,B\right)$. Otherwise, since T is noncyclic relatively (ρG)-nonexpansive one has

$$\begin{aligned}\rho\left(x_{n+1}-Tw\right)&\leq\alpha_n\rho\left(y_n-Tw\right)+(1-\alpha_n)\rho\left(z_n-Tw\right)\\&\leq\alpha_n\beta_n\rho\left(z_n-w\right)+\alpha_n\left(1-\beta_n\right)\rho\left(z_n-Tw\right)+\left(1-\alpha_n\right)\rho\left(z_n-Tw\right)\end{aligned}$$

$$\leq \alpha_n\beta_n\delta_n\rho\left(Tx_n-w\right)+\alpha_n\beta_n\left(1-\delta_n\right)\rho\left(x_n-w\right)+\delta_n\left(1-\alpha_n\beta_n\right)\rho\left(x_n-w\right)$$
$$+\left(1-\delta_n\right)\left(1-\alpha_n\beta_n\right)\rho\left(x_n-Tw\right)$$
$$\leq \alpha_n\beta_n\delta_n\rho\left(Tx_n-w\right)+\left(\delta_n+\alpha_n\beta_n-2\alpha_n\beta_n\right)\rho\left(x_n-w\right)$$
$$+\left(1-\delta_n\right)\left(1-\alpha_n\beta_n\right)\rho\left(x_n-Tw\right)$$

for all $n\geq 0$. Using the $\liminf$ as n goes to infinity and conditions (C_1) and (C_2), we obtain

$$\liminf_n \rho\left(x_n-Tw\right)\leq \theta\liminf_n \rho\left(x_n-w\right)+(1-\theta)\liminf_n \rho\left(x_n-Tw\right).$$

Hence,

$$\liminf_n \rho\left(x_n-Tw\right)\leq \liminf_n \rho\left(x_n-w\right)=dist_\rho\left(A,B\right).$$

Therefore, by Fatou property we get $\rho\left(x-Tw\right)\leq \liminf_n \rho\left(x_n-Tw\right)$. Thus, $\rho\left(x-Tw\right)= dist_\rho\left(A,B\right)$. Moreover,

$$dist_\rho\left(A,B\right)\leq \rho\left(\frac{(x-Tw)+(x-w)}{2}\right)\leq \frac{\rho(x-Tw)+\rho(x-w)}{2}=dist_\rho\left(A,B\right).$$

Hence, $\rho\left(\frac{(x-Tw)+(x-w)}{2}\right)=dist_\rho\left(A,B\right)$. Since ρ is (UUC1), then ρ is strictly convex. Therefore, $x-Tw=x-w$. Thus, $w=Tw$. Since T is noncyclic relatively (ρG)-nonexpansive mapping then

$$\rho\left(Tx-w\right)=\rho\left(Tx-Tw\right)\leq \rho\left(x-w\right)=dist_\rho\left(A,B\right).$$

The strict convexity of ρ implies that $x=Tx$, that is, x is a fixed point of T on A_0. Otherwise, since T is noncyclic relatively (ρG)-nonexpansive, then

$$\rho\left(x-Tx'\right)=\rho\left(Tx-Tx'\right)\leq \rho\left(x-x'\right).$$

Hence, $\rho\left(x-Tx'\right)=dist_\rho\left(A,B\right)$. By the strict convexity of ρ, one has $x-Tx'=x-x'$. Thus, $x'=Tx'$, that is, x' is a fixed point of T in B_0. In order to complete the proof, we show that $(x'_n)_n$ ρ-converges to x'. By contradiction, we assume that there exists a subsequence $\left(x'_{\psi(n)}\right)_n$ which converge to $x''\neq x'$. Since (A_0,B_0) is proximal ρ-compactness, then there exists a subsequence $\left(x_{\psi(n)}\right)_n \subset A_0$ which ρ-converges to x. We have $\rho\left(x-x'\right) = dist_\rho\left(A,B\right)$ and $\rho\left(x-x''\right) = dist_\rho\left(A,B\right)$. Moreover,

$$\begin{aligned} dist_\rho\left(A,B\right) \le \rho\left(x-\frac{x'+x''}{2}\right) &= \rho\left(\frac{(x-x')+(x-x'')}{2}\right) \\ &\le \frac{1}{2}\rho\left(x-x'\right)+\frac{1}{2}\rho\left(x-x''\right) = dist_\rho\left(A,B\right). \end{aligned}$$

Hence, the strict convexity (SC) leads to $x' = x''$, contradiction. Therefore, the sequence $(x'_n)_n$ ρ-converges to x'. Thus, (x_n, x'_n) ρ-converges to a best proximity pair (x,x') of T. $\square$

If we assume that $\alpha_n = \beta_n = 0$, then the sequence (3) will be Mann iteration defined by

$$x_{n+1} = \delta_n T x_n + (1-\delta_n)x_n \quad \text{and} \quad x'_{n+1} = \delta_n T x'_n + (1-\delta_n)x'_n, \tag{5}$$

where $(\delta_n)_n$ is a sequence in $(0,1)$ such that $0 < a \le \delta_n \le b < 1$ and $\lim\limits_n \delta_n = \theta$.

Corollary 3. *Under the same assumptions of Theorem 2, the sequence* (5) *ρ-converges to a best proximity pair of T.*

Example 1. *Let $X = \mathbb{R}^2$, we define the modular $\rho : X \longrightarrow [0,+\infty[$ by $\rho(x) = |x_1|^2 + |x_2|^2$, for all $x = (x_1,x_2) \in \mathbb{R}^2$. The modular ρ is convex satisfying the Fatou property and (UUC1), and X_ρ is a ρ-complete modular space.*
Let

$$A = \left\{(x,y) \in \mathbb{R}^2 \,:\, -2 \le x \le -1 \;\textit{and}\; -\frac{1}{2} \le y \le \frac{1}{2}\right\}$$

and

$$B = \left\{(x,y) \in \mathbb{R}^2 \,:\, 1 \le x \le 2 \;\textit{and}\; -\frac{1}{2} \le y \le \frac{1}{2}\right\}.$$

The pair (A,B) is convex and ρ-bounded in X_ρ with $d_\rho(A,B) = 4$. Moreover,

$$A_0 = \left\{(-1,y) \,:\, -\frac{1}{2} \le y \le \frac{1}{2}\right\} \;\textit{and}\; B_0 = \left\{(1,y) \,:\, -\frac{1}{2} \le y \le \frac{1}{2}\right\}$$

are nonempty and proximal ρ-compactness. Consider the convex graph $G = (V(G),E(G))$ such that $V(G) = A \cup B$ and for $x = (x_1,y_1)$ and $y = (x_2,y_2)$, we have $(x,y) \in E(G)$ if and only if

$x_1 \leq x_2$. *Moreover, for all* $x \in A_0$ *and* $y \in B_0$, *one has* $(x,y) \in E(G)$. *Consider the mapping* $T : A \cup B \longrightarrow A \cup B$ *defined by*

$$T(x,y) = \begin{cases} \left(\frac{x-1}{2}, -\frac{y}{2}\right) & \text{if } (x,y) \in A, \\ \\ \left(\frac{x+1}{2}, -\frac{y}{2}\right) & \text{if } (x,y) \in B. \end{cases}$$

It easy to see that T *is noncyclic relatively* (ρG)*-nonexpansive.*

Consider the following parameter sequences $(\alpha_n)_n$, $(\beta_n)_n$ *and* $(\delta_n)_n$ *given by*

$$\alpha_n = \frac{2(n+1)}{3(n^2+2)} \ , \ \beta_n = \frac{n+2}{2n+3} \ \text{ and } \ \delta_n = \frac{n^2+1}{3n^2+6},$$

for all $n \geq 0$. *We have*

$$\lim_n \alpha_n = 0 \ , \ \lim_n \delta_n = \frac{1}{6} \ \text{ and } \ 0 < \frac{1}{9} \leq \delta_n \leq \frac{1}{6} < 1.$$

Now, let us compute the pair of the sequences $((x_n)_n ; (x'_n)_n)$ *generated by the iterative process* (3). *Let* $x_0 = (u_0, v_0) = \left(-1, -\frac{1}{2}\right)$ *and* $x'_0 = (u'_0, v'_0) = \left(1, \frac{1}{2}\right)$. *We have*

$$\begin{aligned} z_0 &= \delta_0 \left(-1, \frac{-v_0}{2}\right) + (1-\delta_0)(-1, v_0) \\ &= \left(-1, \frac{-\delta_0}{2} v_0 + (1-\delta_0) v_0\right) \\ &= \left(-1, \left(1 - \frac{3}{2}\delta_0\right) v_0\right). \end{aligned}$$

We have $Tz_0 = \left(-1, -\frac{1}{2}\left(1 - \frac{3}{2}\delta_0\right) v_0\right)$. *Thus,*

$$\begin{aligned} y_0 &= \beta_0 \left(-1, -\frac{1}{2}\left(1 - \frac{3}{2}\delta_0\right) v_0\right) + (1-\beta_0)\left(-1, \left(1 - \frac{3}{2}\delta_0\right) v_0\right) \\ &= \left(-1, \left(1 - \frac{3}{2}\beta_0\right)\left(1 - \frac{3}{2}\delta_0\right) v_0\right). \end{aligned}$$

Therefore, one has

$$\begin{aligned} x_1 &= \alpha_0 \left(-1, \left(1 - \frac{3}{2}\beta_0\right)\left(1 - \frac{3}{2}\delta_0\right) v_0\right) + (1-\alpha_0)\left(-1, \left(1 - \frac{3}{2}\delta_0\right) v_0\right) \\ &= \left(-1, \left(1 - \frac{3}{2}\alpha_0\beta_0\right)\left(1 - \frac{3}{2}\delta_0\right) v_0\right). \end{aligned}$$

Moreover, $Tx_1 = \left(-1, -\frac{1}{2}\left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right)$. *As the same way , we compute* x_2 *as follows:*

$$z_1 = \delta_1\left(-1, -\frac{1}{2}\left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right) + (1-\delta_1)\left(-1, \left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right)$$
$$= \left(-1, \left(1-\frac{3}{2}\delta_1\right)\left(1-\frac{3}{2}\delta_0\right)\left(1-\frac{3}{2}\alpha_0\beta_0\right)v_0\right).$$

and $Tz_1 = \left(-1, -\frac{1}{2}\left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_1\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right)$. *Hence,*

$$\begin{aligned} y_1 &= \beta_1\left(-1, -\frac{3}{2}\left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_1\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right) \\ &\quad + (1-\beta_1)\left(-1, \left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_1\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right) \\ &= \left(-1, \left(1-\frac{3}{2}\beta_1\right)\left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_1\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right). \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= \alpha_1\left(-1, \left(1-\frac{3}{2}\beta_1\right)\left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_1\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right) \\ &\quad + (1-\alpha_1)\left(-1, \left(1-\frac{3}{2}\alpha_0\beta_0\right)\left(1-\frac{3}{2}\delta_1\right)\left(1-\frac{3}{2}\delta_0\right)v_0\right) \end{aligned}$$

Thus,

$$x_n = \left(-1, \prod_{k=0}^{n-1}\left(1-\frac{3}{2}\delta_k\right)\prod_{j=0}^{n-1}\left(1-\frac{3}{2}\alpha_j\beta_j\right)v_0\right).$$

By induction, we prove that

$$x_{n+1} = \left(-1, \prod_{k=0}^{n}\left(1-\frac{3}{2}\delta_k\right)\prod_{j=0}^{n}\left(1-\frac{3}{2}\alpha_j\beta_j\right)v_0\right).$$

One has,

$$\begin{aligned} z_n &= \delta_n\left(-1, \frac{-1}{2}\prod_{k=0}^{n-1}\left(1-\frac{3}{2}\delta_k\right)\prod_{j=0}^{n-1}\left(1-\frac{3}{2}\alpha_j\beta_j\right)v_0\right) \\ &\quad + (1-\delta_n)\left(-1, \prod_{k=0}^{n-1}\left(1-\frac{3}{2}\delta_k\right)\prod_{j=0}^{n-1}\left(1-\frac{3}{2}\alpha_j\beta_j\right)v_0\right) \\ &= \left(-1, \prod_{k=0}^{n}\left(1-\frac{3}{2}\delta_k\right)\prod_{j=0}^{n-1}\left(1-\frac{3}{2}\alpha_j\beta_j\right)v_0\right). \end{aligned}$$

We have

$$\begin{aligned}
y_n &= \beta_n \left(-1, \prod_{k=0}^{n} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n-1} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v_0 \right) \\
&\quad + (1-\beta_n) \left(-1, \prod_{k=0}^{n-1} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n-1} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v_0 \right) \\
&= \left(-1, \left(1 - \frac{3}{2}\beta_n\right) \prod_{k=0}^{n} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n-1} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v_0 \right)
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
x_{n+1} &= \alpha_n \left(-1, \prod_{k=0}^{n} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n-1} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v_0 \right) + \left(-1, \prod_{k=0}^{n} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n-1} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v_0 \right) \\
&= \left(-1, \prod_{k=0}^{n} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v_0 \right).
\end{aligned}$$

Thus, for all $n \geq 0$

$$x_n = \left(-1, \prod_{k=0}^{n-1} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n-1} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v_0 \right).$$

As the same way, we compute the sequence $(x'_n)_n$ as follows

$$x'_n = \left(-1, \prod_{k=0}^{n-1} \left(1 - \frac{3}{2}\delta_k\right) \prod_{j=0}^{n-1} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) v'_0 \right),$$

for all $n \geq 0$. Moreover, $\prod_{k=0}^{\infty} \left(1 - \frac{3}{2}\delta_k\right) = 0$ and $\prod_{j=0}^{\infty} \left(1 - \frac{3}{2}\alpha_j\beta_j\right) = 0$. Therefore, the pair of the sequences $((x_n)_n ; (x'_n)_n)$ ρ-converges to $((-1,0);(1,0)) \in A_0 \times B_0$ best proximity pair of the mapping T.

Next, we will show via the Table 1 and figures 1, 2, 3 and 3 that the new iterative process (3) *converges faster than Mann's iteration by using MatLab R2020a software. By taking the initial couple $(x_0, x'_0) = \left(\left(-1, \frac{1}{2}\right), \left(1, \frac{1}{2}\right) \right)$ and the parameters sequences $\alpha_n = \frac{2(n+1)}{3(n^2+2)}$, $\beta_n = \frac{n+2}{2n+3}$ and $\delta_n = \frac{n^2+1}{3n^2+6}$, for all $n \geq 0$, we get Table 1 and figures 1, 2, 3 and 4 which illustrate clearly the fastness of the proposed algorithm to converges to a best proximity pair of T. The numerical results of Table 1 confirm that the new iteration is more advantageous than*

Mann's iteration and its requires less numbers of iteration than Mann's iteration. In fact, it can be easily seen from the 1 that the proposed scheme requires 361 iterations to achieve the best proximity pair of T against 1076 iterations for Mann's scheme. The figures 3 and 4 illustrate the behaviour of the sequences $(x_n)_n$ *and* $(x'_n)_n$ *defined by* (4) *and Mann's iteration* (3) *on the subsets A and B, respectively.*

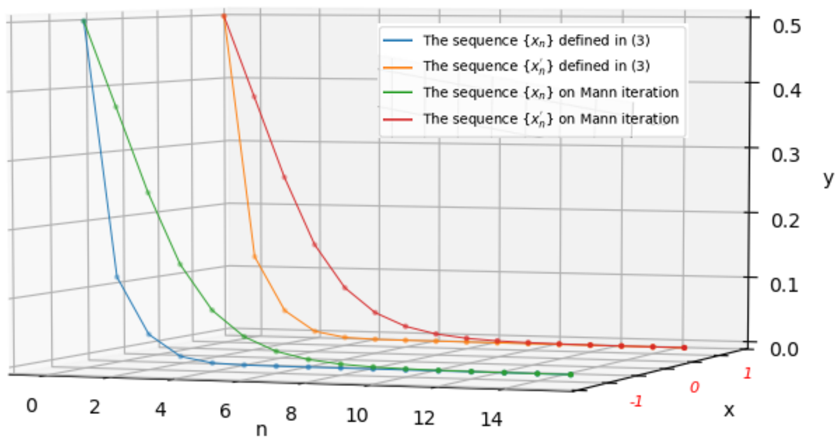

FIGURE 1. Graphic simulation of the convergence for $(x_0, x'_0) = \left(\left(-1, \frac{1}{2}\right), \left(1, \frac{1}{2}\right)\right)$.

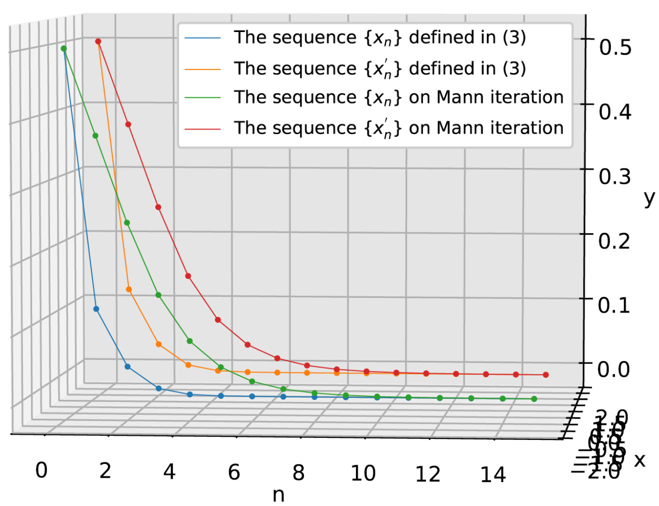

FIGURE 2. Side view of the graphic simulation of the convergence for $(x_0, x'_0) = \left(\left(-1, \frac{1}{2}\right), \left(1, \frac{1}{2}\right)\right)$.

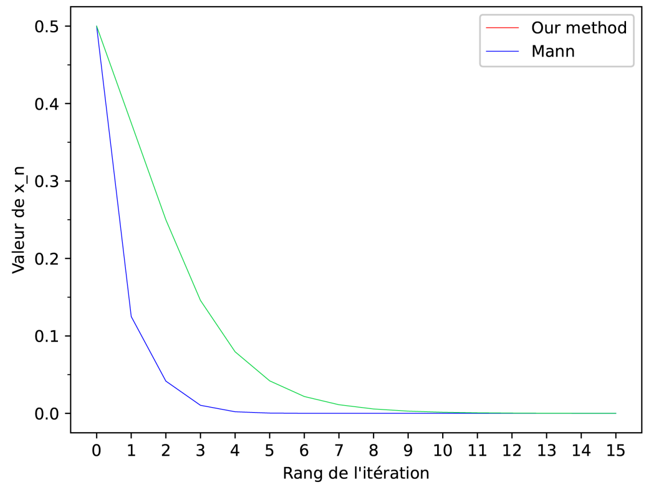

FIGURE 3. Comparisons of the sequence $(x_n)_n$ of the new iteration and Mann's iteration on A.

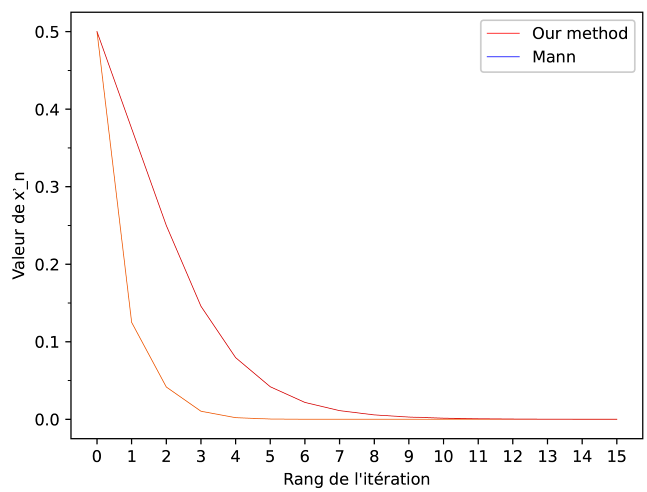

FIGURE 4. Comparisons of the sequence $(x'_n)_n$ of the new iteration and Mann's iteration on B.

	Mann iteration (5)				*New iteration* (3)			
Steps	x_n		x'_n		x_n		x'_n	
	a_n	b_n	a'_n	b'_n	a_n	b_n	a'_n	b'_n
0	-1	0.5	1	0.5	-1	0.5	1	0.5
1	-1	0.375000000000000	1	0.375000000000000	-1	0.125000000000000	1	0.125000000000000
2	-1	0.250000000000000	1	0.250000000000000	-1	0.0416666666666667	1	0.0416666666666667
3	-1	0.145833333333333	1	0.145833333333333	-1	0.0104166666666667	1	0.0104166666666667
4	-1	0.0795454545454545	1	0.0795454545454545	-1	0.00209481175390266	1	0.00209481175390266
5	-1	0.0419823232323232	1	0.0419823232323232	-1	0.000368531697445839	1	0.000368531697445839
6	-1	0.0217686120463898	1	0.0217686120463898	-1	$5,96137060445786.10^{-5}$	1	$5,96137060445786.10^{-5}$
7	-1	0.0111707351290685	1	0.0111707351290685	-1	$9,12370431845697.10^{-6}$	1	$9,12370431845697.10^{-6}$
8	-1	0.00569488457560353	1	0.00569488457560353	-1	$1,34388778583147.10^{-6}$	1	$1,34388778583147.10^{-6}$
9	-1	0.00289058535276846	1	0.00289058535276846	-1	$1,92561563628817.10^{-7}$	1	$1,92561563628817.10^{-7}$
10	-1	0.00146270584115994	1	0.00146270584115994	-1	$2,70296170749117.10^{-8}$	1	$2,70296170749117.10^{-8}$
11	-1	0.000738523047252324	1	0.000738523047252324	-1	$3,73468494286397.10^{-9}$	1	$3,73468494286397.10^{-9}$
12	-1	0.000372263649834505	1	0.000372263649834505	-1	$5,09658968407492.10^{-10}$	1	$5,09658968407492.10^{-10}$
$\vdots$								
359	-1	$1,41168347350775.10^{-108}$	1	$1,41168347350775.10^{-108}$	-1	$1,930.10^{-322}$	1	$1,930.10^{-322}$
360	-1	$7,05847213362402.10^{-109}$	1	$7,05847213362402.10^{-109}$	-1	$2,0.10^{-323}$	1	$2,0.10^{-323}$
361	-1	$3,52926329815155.10^{-109}$	1	$3,52926329815155.10^{-109}$	-1	0	1	0
362	-1	$1,76464518952258.10^{-109}$	1	$1,76464518952258.10^{-109}$	-1	0	1	0
$\vdots$								
1074	-1	$1,0.10^{-323}$	1	$1,0.10^{-323}$	-1	0	1	0
1075	-1	$4,9.10^{-324}$	1	$4,9.10^{-324}$	-1	0	1	0
1076	-1	0	1	0	-1	0	1	0
1076	-1	0	1	0	-1	0	1	0

TABLE 1. Comparison of rate converges

CONCLUSION

A best proximity pair existence theorem for noncyclic relatively ρ-nonexpansive mappings has been proven in the context of modular spaces endowed with a graph. Furthermore, a new iterative method has been introduced in order to approximate a best proximity pair of such mapping in the case of modular spaces endowed with a graph. The result has been validated with a numerical example.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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