COMMON COUPLED FIXED AND COINCIDENCE POINTS RESULTS FOR RATIONAL TYPE CONTRACTION MAPPINGS IN COMPLEX VALUED $S_b$-METRIC SPACES

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Abstract. In this paper, we introduce a new rational type contraction mapping in complex valued $S_b$-metric space and find some common coupled fixed and coincidence points. Some results are also given as corollaries.

Keywords: complex valued $S_b$-metric space; coupled common fixed point; coupled coincidence point; rational type contraction.

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1. INTRODUCTION AND PRELIMINARIES

Azam et al. [1] introduced the concept of complex valued metric space and proved some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. Moreover, Shin Min Kang et al. [2] introduced the notion of complex valued $G$-metric space and proved contraction principle in this space. In 2014, Nabil M. Mlaiki [3] introduced the

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complex valued $S$-metric space and proved the existence and the uniqueness of a common fixed point of two self mappings in this space. Recently, Priyobarta et al. [4] introduced the concept of complex valued $S_b$-metric space and some topological properties. They also proved some fixed point theorems. Some more results on complex valued can be seen in [5-6].

The concept of rational type contraction is one of the interest for researchers and these can be found in [7-15]. On the other hand, there are various forms of generalization of metric space in the literature. Some of them can be found in [16-24]. The concept of coupled fixed point was introduced by Guo and Lakshmikantham [25]. The concept is further used by various authors in [26-27].

In this paper, we prove some common coupled and coincidence points theorems for rational type contractive mappings in complex valued $S_b$-metric space.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } \Re(z_1) \leq \Re(z_2) \text{ and } \Im(z_1) \leq \Im(z_2).$$

If follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

(C1): $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$,

(C2): $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$,

(C3): $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$,

(C4): $\Re(z_1) < \Re(z_2)$ and $\Im(z_1) < \Im(z_2)$.

Particularly, we write $z_1 \not\preceq z_2$ if $z_1 \neq z_2$ and one (C2), (C3) and (C4) is satisfied and we write $z_1 \prec z_2$ if only (C4) is satisfied. The following statements hold:

(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \prec bz$ for all $z \in \mathbb{C}$.

(2) If $0 \preceq z_1 \not\preceq z_2$, then $|z_1| < |z_2|$.

(3) If $z_1 \preceq z_2$ and $z_2 \not\preceq z_3$, then $z_1 \prec z_3$.

**Definition 1.1.** [1] Let $X$ be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies the following conditions:

($d_1$): $0 \not\preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

($d_2$): $d(x, y) = d(y, x)$ for all $x, y \in X$;

($d_3$): $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$. 

Then \(d\) is called a complex valued metric on \(X\), and \((X, d)\) is called a complex valued metric space.

**Example 1.** [5] Let \(X = \mathbb{C}\) be a set of complex number. Define \(d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\), by

\[
d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|
\]

where \(z_1 = x_1 + iy_1\) and \(z_2 = x_2 + iy_2\). Then \((X, d)\) is a complex valued metric space.

**Example 2.** [6] Let \(X = \mathbb{C}\) be a set of complex number. Define \(d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\), by

\[
d(z_1, z_2) = e^{ik}|z_1 - z_2|
\]

where \(0 \leq k \leq \frac{\pi}{2}\), \(z_1 = x_1 + iy_1\) and \(z_2 = x_2 + iy_2\). Then \((X, d)\) is a complex valued metric space.

**Definition 1.2.** [4] Let \(X\) be a nonempty set and \(s \geq 1\) be a given real number. Suppose that a mapping \(S : X^3 \to \mathbb{C}\) satisfies:

\[
(CS_b1): 0 < S(x, y, z) \text{ for all } x, y, z \in X \text{ with } x \neq y \neq z \neq x,
\]

\[
(CS_b2): S(x, y, z) = 0 \iff x = y = z,
\]

\[
(CS_b3): S(x, x, y) = S(y, y, x), \text{ for all } x, y \in X,
\]

\[
(CS_b4): S(x, y, z) \preceq s(S(x, x, a) + S(y, y, a) + S(z, z, a)) \text{ for all } x, y, z, a \in X.
\]

Then, \(S\) is called a complex valued \(S_b\)-metric and \((X, S)\) is called a complex valued \(S_b\)-metric space.

**Definition 1.3.** [4] A complex valued \(S_b\)-metric space \((X, S)\) is said to be symmetric if

\[
S(x, x, y) = S(y, y, x).
\]

**Definition 1.4.** [4] Let \((X, S)\) be a complex valued \(S_b\)-metric space, let \(\{x_n\}\) be a sequence in \(X\).

(i): \(\{x_n\}\) is a complex valued \(S_b\)-convergent to \(x\) if for every \(a \in \mathbb{C}\) with \(0 < a\), there exists \(k \in \mathbb{N}\) such that \(S(x_n, x_n, x) < a\) or \(S(x, x, x_n) < a\) for all \(n \geq k\) and denoted by

\[
\lim_{n \to \infty} x_n = x.
\]

(ii): A sequence \(\{x_n\}\) is called complex valued \(S_b\)-Cauchy if for every \(a \in \mathbb{C}\) with \(0 < a\), there exists \(k \in \mathbb{N}\) such that \(S(x_n, x_n, x_m) < a\) for each \(m, n \geq k\).
(iii): If every complex valued $S_b$-Cauchy sequence is complex valued $S_b$-convergent in $(X, S)$, then $(X, S)$ is said to be complex valued $S_b$-complete.

**Proposition 1.1.** [4] Let $(X, S)$ be a complex valued $S_b$-metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is complex valued $S_b$-convergent to $x$ if and only if $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$ or $|S(x, x, x_n)| \to 0$ as $n \to \infty$.

**Theorem 1.2.** [4] Let $(X, S)$ be a complex valued $S_b$-metric space, then for a sequence $\{x_n\}$ in $X$ and point $x \in X$, the following are equivalent

(1): $\{x_n\}$ is a complex valued $S_b$-convergent to $x$.

(2): $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$.

**Theorem 1.3.** [4] Let $(X, S)$ be a complex valued $S_b$-metric space and $\{x_n\}$ be a sequence in $X$. Then, $\{x_n\}$ is complex valued $S_b$-Cauchy sequence if and only if $|S(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.

**Definition 1.5.** [25] An element $(x, y) \in X \times X$ is called a

(1) coupled fixed point of a mapping $A : X \times X \to X$ if $x = A(x, y)$ and $y = A(y, x)$;

(2) coupled common fixed point of two mappings $A, B : X \times X \to X$ if $x = A(x, y) = B(x, y)$ and $y = A(y, x) = B(y, x)$.

**Definition 1.6.** [25, 26] Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called

i) a coupled fixed point of the mapping $f : X \times X \to X$ if $x = f(x, y)$ and $y = f(y, x)$.

ii) a coupled coincidence point of mappings $f : X \times X \to X$ and $T : X \to X$ if $T(x) = f(x, y)$ and $T(y) = f(y, x)$.

iii) a common coupled fixed point of mappings $f : X \times X \to X$ and $T : X \to X$ if $x = T(x) = f(x, y)$ and $y = T(y) = f(y, x)$.

2. **Main Results**

Now we prove the following theorems
\textbf{Theorem 2.1.} Let \((X, S)\) be a complete complex valued symmetric \(S_b\)-metric space with parameter \(s \geq 1\) and let the mappings \(f, g : X^2 \to X\) satisfying

\[
S(f(x, y), f(x, y), g(u, v)) \lesssim a_1 \frac{S(x, x, u) + S(y, y, v)}{2}
\]

\[
+ a_2 \frac{S(f(x, y), f(x, y), g(u, v))S(x, x, u)}{1 + S(x, x, u) + S(y, y, v)}
\]

\[
+ a_3 \frac{S(f(x, y), f(x, y), g(u, v))S(y, y, v)}{1 + S(x, x, u) + S(y, y, v)}
\]

\[
+ a_4 \frac{S(x, x, f(x, y))S(x, x, u)}{1 + S(x, x, u) + S(y, y, v)}
\]

\[
+ a_5 \frac{S(x, x, f(x, y))S(y, y, v)}{1 + S(x, x, u) + S(y, y, v)}
\]

\[
+ a_6 \frac{S(u, u, g(u, v))S(x, x, u)}{1 + S(x, x, u) + S(y, y, v)}
\]

\[
+ a_7 \frac{S(u, u, g(u, v))S(y, y, v)}{1 + S(x, x, u) + S(y, y, v)}
\]

(1)

for all \(x, y, u, v \in X\) and \(a_i \geq 0\) with \(\sum_{i=1}^{7} a_i < 1\), \(i = 1, 2, \ldots, 7\) and \(s < \frac{1 - a_2 - a_1 - a_6 - a_7}{a_1 + a_4 + a_5}\). Then \(f\) and \(g\) have a unique common coupled fixed point in \(X\).

\textit{Proof.} Let \(x_0, y_0 \in X\) be arbitrary points.

Define

\[
x_{2k+1} = f(x_{2k}, y_{2k}) \quad , \quad y_{2k+1} = f(y_{2k}, x_{2k})
\]

\[
x_{2k+2} = g(x_{2k+1}, y_{2k+1}) \quad , \quad y_{2k+2} = g(y_{2k+1}, x_{2k+1})
\]

for \(k = 0, 1, 2, \ldots\). Then

\[
S(x_{2k+1}, x_{2k+1}, x_{2k+2}) = S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))
\]

\[
\lesssim a_1 \frac{S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}{2}
\]

\[
+ a_2 \frac{S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))S(x_{2k}, x_{2k}, x_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}
\]

\[
+ a_3 \frac{S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))S(y_{2k}, y_{2k}, y_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}
\]

\[
= \frac{S(x_{2k+1}, x_{2k+1}, x_{2k+2})}{2}
\]

\[
+ a_2 \frac{S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))S(x_{2k}, x_{2k}, x_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}
\]

\[
+ a_3 \frac{S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))S(y_{2k}, y_{2k}, y_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}
\]

\[
= \frac{S(x_{2k+1}, x_{2k+1}, x_{2k+2})}{2}
\]

\[
+ a_2 \frac{S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))S(x_{2k}, x_{2k}, x_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}
\]

\[
+ a_3 \frac{S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))S(y_{2k}, y_{2k}, y_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})}
\]
\[
+ a_4 \frac{S(x_{2k}, x_{2k}, f(x_{2k}, y_{2k}))S(x_{2k}, x_{2k}, x_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})} \\
+ a_5 \frac{S(x_{2k}, x_{2k}, f(x_{2k}, y_{2k}))S(y_{2k}, y_{2k}, y_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})} \\
+ a_6 \frac{S(x_{2k+1}, x_{2k+1}, g(x_{2k+1}, y_{2k+1}))S(x_{2k}, x_{2k}, x_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})} \\
+ a_7 \frac{S(x_{2k+1}, x_{2k+1}, g(x_{2k+1}, y_{2k+1}))S(y_{2k}, y_{2k}, y_{2k+1})}{1 + S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})} \\
= \frac{a_1}{2} S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1}) \\
+ (a_2 + a_3)S(x_{2k+1}, x_{2k+1}, x_{2k+2}) + (a_4 + a_5)S(x_{2k}, x_{2k}, x_{2k+1}) \\
+ (a_6 + a_7)S(x_{2k+1}, x_{2k+1}, x_{2k+2})
\]

\[\Rightarrow (1 - a_2 - a_3 - a_6 - a_7)S(x_{2k+1}, x_{2k+1}, x_{2k+2}) \approx \frac{a_1}{2} + a_4 + a_5)S(x_{2k}, x_{2k}, x_{2k+1}) \\
+ \frac{a_1}{2} S(y_{2k}, y_{2k}, y_{2k+1}) \\
\Rightarrow S(x_{2k+1}, x_{2k+1}, x_{2k+2}) \approx \frac{a_1}{2} + a_4 + a_5 \frac{1}{1 - a_2 - a_3 - a_6 - a_7} S(x_{2k}, x_{2k}, x_{2k+1}) \\
+ \frac{a_1}{1 - a_2 - a_3 - a_6 - a_7} S(y_{2k}, y_{2k}, y_{2k+1})
\]

(2)
Proceeding similarly one can prove that

\[ S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \lesssim \frac{a_1 + a_4 + a_5}{1 - a_2 - a_3 - a_6 - a_7} \cdot S(y_{2k}, y_{2k}, y_{2k+1}) \]

Adding (2) and (3) we have

\[ S(x_{2k+1}, x_{2k+1}, x_{2k+2}) + S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \lesssim \frac{a_1 + a_4 + a_5}{1 - a_2 - a_3 - a_6 - a_7} [S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})] \]

Therefore

\[ S(x_{2k+1}, x_{2k+1}, x_{2k+2}) + S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \lesssim h[S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})] \]

where \( h = \frac{a_1 + a_4 + a_5}{1 - a_2 - a_3 - a_6 - a_7} < 1 \).

Also, we can show that

\[ S(x_{2k+2}, x_{2k+2}, x_{2k+3}) + S(y_{2k+2}, y_{2k+2}, y_{2k+3}) \lesssim h[S(x_{2k+1}, x_{2k+1}, x_{2k+2}) + S(y_{2k+1}, y_{2k+1}, y_{2k+2})] \]

\[ \lesssim h^2[S(x_{2k}, x_{2k}, x_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1})] \]

Continuing this way, we have

\[ S(x_{n}, x_{n}, x_{n+1}) + S(y_{n}, y_{n}, y_{n+1}) \lesssim h[S(x_{n-1}, x_{n-1}, x_{n}) + S(y_{n-1}, y_{n-1}, y_{n})] \]

\[ \lesssim h^2[S(x_{n-2}, x_{n-2}, x_{n-1}) + S(y_{n-2}, y_{n-2}, y_{n-1})] \]

\[ \lesssim \cdots \lesssim h^n[S(x_0, x_0, x_1) + S(y_0, y_0, y_1)] \]

If \( S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = S_n \), then

\[ S_n \lesssim hS_{n-1} \lesssim h^2S_{n-2} \lesssim \cdots \lesssim h^nS_0 \]
So for $m > n$,

\[
S(x_n, x_m) + S(y_n, y_m) \lesssim s[2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m) \\
+ 2S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_m)]
\]

\[
= 2s[2S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) \\
+ sS(x_{n+1}, x_{n+1}, x_m) + S(y_{n+1}, y_{n+1}, y_m)]
\]

\[
\lesssim 2s[2S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) \\
+ s^2[2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(x_{n+2}, x_{n+2}, x_m) \\
+ 2S(y_{n+1}, y_{n+1}, y_{n+2}) + S(y_{n+2}, y_{n+2}, y_m)]
\]

\[
\lesssim 2s[2S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) \\
+ 2s^2[2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(y_{n+1}, y_{n+1}, y_{n+2}) \\
+ \ldots + 2s^{m-n-1}[S(x_{m-2}, x_{m-2}, x_{m-1}) + S(y_{m-2}, y_{m-2}, y_{m-1})] \\
+ s^{m-n}[S(x_{m-1}, x_{m-1}, x_m) + S(y_{m-1}, y_{m-1}, y_m)]
\]

\[
\lesssim 2s[2S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) \\
+ 2s^2[2S(x_{n+1}, x_{n+1}, x_{n+2}) + S(y_{n+1}, y_{n+1}, y_{n+2}) \\
+ 2s^3[2S(x_{n+2}, x_{n+2}, x_{n+3}) + S(y_{n+2}, y_{n+2}, y_{n+3})] \\
+ \ldots + 2s^{m-n-1}[S(x_{m-2}, x_{m-2}, x_{m-1}) + S(y_{m-2}, y_{m-2}, y_{m-1})] \\
+ 2s^{m-n}[S(x_{m-1}, x_{m-1}, x_m) + S(y_{m-1}, y_{m-1}, y_m)]
\]

\[
\lesssim 2\left\{sh^n + s^2h^{n+1} + s^3h^{n+2} + \ldots + s^{m-n}h^{m-1}\right\}S_0
\]

\[
< 2sh^n[1 + sh + (sh)^2 + \ldots]S_0
\]

\[
= \frac{2sh^n}{1 - sh}S_0 \to 0 \text{ as } n \to \infty
\]
Hence, which is a contradiction, so since 

\[
S(x, x, f(x, y)) = l_1 > 0 \quad \text{and} \quad S(y, y, f(y, x)) = l_2 > 0.
\]

Using inequality (1)

\[
l_1 = S(x, x, f(x, y)) \leq s[2S(x, x, x_{n+1}) + S(x_{n+1}, x_{n+1}, f(x, y))] = s[2S(x, x, x_{n+1}) + S(f(x_n, y_n), f(x_n, y_n), f(x, y))]
\]

\[
\leq 2sS(x, x, x_{n+1}) + s \left[ a_1 \frac{S(x_n, x_n, x) + S(y_n, y_n, y)}{2} + a_2 \frac{S(f(x_n, y_n), f(x_n, y_n), f(x, y))S(x_n, x_n, x)}{1 + S(x_n, x_n) + S(y_n, y_n)} + a_3 \frac{S(f(x_n, y_n), f(x_n, y_n), f(x, y))S(y_n, y_n, y)}{1 + S(x_n, x_n) + S(y_n, y_n)} + a_4 \frac{S(x_n, x_n, f(x_n, y_n))S(x_n, x_n, x)}{1 + S(x_n, x_n) + S(y_n, y_n)} + a_5 \frac{S(x_n, x_n, f(x_n, y_n))S(y_n, y_n, y)}{1 + S(x_n, x_n) + S(y_n, y_n)} + a_6 \frac{S(x, x, f(x, y))S(x_n, x_n, x)}{1 + S(x_n, x_n) + S(y_n, y_n)} + a_7 \frac{S(x, x, f(x, y))S(y_n, y_n, y)}{1 + S(x_n, x_n) + S(y_n, y_n)} \right].
\]

Since \( \{x_n\} \) and \( \{y_n\} \) are convergent to \( x \) and \( y \), therefore by taking limit as \( n \to \infty \) we get \( l_1 \leq 0 \), which is a contradiction, so \( |S(x, x, f(x, y))| = 0 \) which gives \( x = f(x, y) \).

Similarly, we can prove that \( y = f(y, x) \). Also, we can prove that \( x = g(x, y) \) and \( y = g(y, x) \).

Hence \( (x, y) \) is a common coupled fixed point of \( f \) and \( g \).

In order to prove the uniqueness of the coupled fixed point, if possible let \( (p, q) \) be the second common coupled fixed point of \( f \) and \( g \).

Then by using inequality (1), we have

\[
S(x, x, p) = S(f(x, y), f(x, y), g(p, q)) \leq \frac{a_1}{2} \{S(x, x, p) + S(y, y, q)\}
\]

\[
+ a_2 \frac{S(f(x, y), f(x, y), g(p, q))S(x, x, p)}{1 + S(x, x, p) + S(y, y, q)} + a_3 \frac{S(f(x, y), f(x, y), g(p, q))S(y, y, q)}{1 + S(x, x, p) + S(y, y, q)}
\]

\[
+ a_4 \frac{S(x, x, f(x, y))S(x, x, p)}{1 + S(x, x, p) + S(y, y, q)} + a_5 \frac{S(x, x, f(x, y))S(y, y, q)}{1 + S(x, x, p) + S(y, y, q)}
\]

\[
+ a_6 \frac{S(x, x, f(x, y))S(x, x, p)}{1 + S(x, x, p) + S(y, y, q)} + a_7 \frac{S(x, x, f(x, y))S(y, y, q)}{1 + S(x, x, p) + S(y, y, q)}.\]
Similarly, \( (4) \) and \( (5) \) we have

\[
S(x, x, p) + S(y, y, q) \lesssim \frac{a_1}{2 - a_1 - 2a_2 - 2a_3} [S(x, x, p) + S(y, y, q)]
\]

\[
\Rightarrow [1 - \frac{a_1}{2 - a_1 - 2a_2 - 2a_3}] [S(x, x, p) + S(y, y, q)] \lesssim 0
\]

\[
\Rightarrow \frac{2(1 - a_1 - a_2 - a_3)}{2 - a_1 - 2a_2 - a_3} [S(x, x, p) + S(y, y, q)] \lesssim 0
\]

Since \( a_1 + a_2 + a_3 < 1 \), \( \frac{2(1 - a_1 - a_2 - a_3)}{2 - a_1 - 2a_2 - 2a_3} > 0 \).

Hence \( |S(x, x, p) + S(y, y, q)| = 0 \),

which implies that \( x = p \) and \( y = q \) \( \Rightarrow (x, y) = (p, q) \).

Thus \( f \) and \( g \) have unique coupled common fixed point. This completes the proof.
Corollary 2.1. Let \((X, S)\) be a complete complex valued symmetric \(S_b\)-metric space with parameter \(s \geq 1\) and let the mapping \(f : X^2 \rightarrow X\) satisfy

\[
S(f(x, y), f(x, y), f(u, v)) \preceq a_1 \frac{S(x, x, u) + S(y, y, v)}{2} + a_2 \frac{S(f(x, y), f(x, y), f(u, v))S(x, x, u)}{1 + S(x, x, u) + S(y, y, v)} + a_3 \frac{S(f(x, y), f(x, y), f(u, v))S(y, y, v)}{1 + S(x, x, u) + S(y, y, v)} + a_4 \frac{S(x, x, f(x, y))S(x, x, u)}{1 + S(x, x, u) + S(y, y, v)} + a_5 \frac{S(x, x, f(x, y))S(y, y, v)}{1 + S(x, x, u) + S(y, y, v)} + a_6 \frac{S(u, u, f(u, v))S(x, x, u)}{1 + S(x, x, u) + S(y, y, v)} + a_7 \frac{S(u, u, f(u, v))S(y, y, v)}{1 + S(x, x, u) + S(y, y, v)}
\]

for all \(x, y, u, v \in X\) and \(a_i \geq 0\) with \(\sum_{i=1}^{7} a_i < 1\), \(i = 1, 2, \ldots, 7\). Then \(f\) has a unique coupled fixed point in \(X\).

Theorem 2.2. Let \((X, S)\) be a complete complex valued symmetric \(S_b\)-metric space with parameter \(s \geq 1\) and let the mappings \(f, g : X^2 \rightarrow X\) satisfy

\[
S(f(x, y), f(x, y), g(u, v)) \preceq \beta_1 \frac{S(x, x, u) + S(y, y, v)}{2} + \beta_2 \frac{S(x, x, f(x, y))S(u, u, g(u, v))}{1 + s[S(x, x, g(x, y) + S(u, u, f(u, v)) + S(x, x, u) + S(y, y, v))]
\]

(6)

for all \(x, y, u, v \in X\) and \(\beta_1, \beta_2\) are non-negative real numbers with \(\beta_1 + \beta_2 < 1\) and \(s < \frac{1-\beta_2}{\beta_1}\). Then \(f\) and \(g\) have unique common coupled fixed point.

Proof. Let \(x_0, y_0\) be arbitrary points. Define

\[
x_{2k+1} = f(x_{2k}, x_{2k}) \quad y_{2k+1} = f(y_{2k}, x_{2k})
\]

\[
x_{2k+2} = g(x_{2k+1}, y_{2k+1}) \quad y_{2k+2} = g(y_{2k+1}, x_{2k+1})
\]

for \(k = 0, 1, 2, \ldots\). Then

\[
S(x_{2k+1}, x_{2k+1}, x_{2k+2}) = S(f(x_{2k}, y_{2k}), f(x_{2k}, y_{2k}), g(x_{2k+1}, y_{2k+1}))
\]
\[ \beta_1 \frac{S(x_{2k}, y_{2k}, y_{2k+1})}{2} + \beta_2 \frac{S(x_{2k}, y_{2k}, y_{2k+1}) S(x_{2k+1}, y_{2k+1})}{1 + S(x_{2k}, y_{2k+1}) + S(x_{2k+1}, y_{2k+1})} \]

\[ = \beta_1 S(x_{2k}, y_{2k}, y_{2k+1}) + S(x_{2k+1}, y_{2k+1}) \]

\[ + \beta_2 \frac{S(x_{2k}, y_{2k}, y_{2k+1}) S(x_{2k+1}, y_{2k+1})}{1 + S(x_{2k}, y_{2k+1}) + S(x_{2k+1}, y_{2k+1})} \]

\[ \geq \frac{\beta_1}{2} \{ S(x_{2k}, y_{2k}, y_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1}) \} + \beta_2 S(x_{2k+1}, y_{2k+1}, y_{2k+2}) \]

\[ \Rightarrow (1 - \beta_2) S(x_{2k+1}, y_{2k+1}, y_{2k+2}) \geq \frac{\beta_1}{2} \{ S(x_{2k}, y_{2k}, y_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1}) \} \]

\[ \Rightarrow S(x_{2k+1}, y_{2k+1}, y_{2k+2}) \geq \frac{\beta_1}{2(1 - \beta_2)} \{ S(x_{2k}, y_{2k}, y_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1}) \} \]

Similarly we can show that

\[ S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \geq \frac{\beta_1}{2(1 - \beta_2)} \{ S(x_{2k}, y_{2k}, y_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1}) \} \]

Adding (7) and (8) we have

\[ S(x_{2k+1}, y_{2k+1}, y_{2k+3}) \geq \frac{\beta_1}{2(1 - \beta_2)} \{ S(x_{2k}, y_{2k}, y_{2k+1}) + S(y_{2k}, y_{2k}, y_{2k+1}) \} \]

where \( k = \frac{\beta_1}{1 - \beta_2} \).

Similarly, we can show that

\[ S(x_{2k+2}, y_{2k+2}, y_{2k+3}) \geq k \{ S(x_{2k+1}, y_{2k+1}, y_{2k+3}) + S(y_{2k+1}, y_{2k+1}, y_{2k+3}) \} \]

Now, if \( S(x_n, x_{n+1}) + S(y_n, y_{n+1}) = S_n \) then

\[ S_n \geq k S_{n-1} \geq k^2 S_{n-2} \geq \cdots \geq k^n S_0 \]

So, for \( m > n \) we have
Therefore \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences in \( X \). Since \( X \) is complete \( S_b \)-metric space, there exist \( x, y \in X \) such that \( x_n \to x \) and \( y_n \to y \) as \( n \to \infty \).
Now, we will show that \( x = f(x, y) \) and \( y = f(y, x) \). Suppose on contrary that \( x \neq f(x, y) \) and \( y \neq f(y, x) \), so that \( S(x, x, f(x, y)) = l_1 > 0 \) and \( S(y, y, f(y, x)) = l_2 > 0 \). Consider the following and using inequality \((6)\), we get

\[
I_1 = S(x, x, f(x, y))
\]

\[
\preceq s[2S(x, x, x_{n+1}) + S(x_{n+1}, x_{n+1}, f(x, y))]
\]

\[
= sS(x, x, x_{n+1}) + sS(f(x, y), f(x, y), f(x, y))
\]

\[
\preceq sS(x, x, x_{n+1}) + s\left[ \frac{\beta_1}{2} S(x_n, x_n, x) + S(y_n, y_n, y) \right]
\]

\[
+ \frac{\beta_2}{1 + s[S(x_n, x_n, f(x, y)) + S(x, x, x_{n+1}) + S(x_n, x_n, x) + S(y_n, y_n, y)]}
\]

Taking the limit as \( n \to \infty \) we get

\[
S(x, x, f(x, y)) \preceq 0
\]

Therefore

\[
S(x, x, f(x, y)) = 0
\]

which implies that \( x = f(x, y) \). Similarly, we can prove that \( y = f(y, x) \). Also, we can prove that \( x = g(x, y) \) and \( y = g(y, x) \). Hence, \((x, y)\) is a common coupled fixed point of \( f \) and \( g \).

In order to prove the uniqueness of the common coupled fixed point of \( f \) and \( g \), if possible let \((p, q)\) be the second common coupled fixed point of \( f \) and \( g \).

Then by using inequality \((6)\), we have

\[
S(x, x, p) = S(f(x, y), f(x, y), g(p, q))
\]

\[
\preceq \frac{\beta_1}{2} \left[ S(x, x, p) + S(y, y, q) \right]
\]

\[
+ \frac{\beta_2}{1 + s[S(x, x, g(p, q)) + S(p, p, f(x, y)) + S(x, x, p) + S(y, y, q)]}
\]
\[ \Rightarrow S(x, x, p) \preceq \frac{\beta_1}{2} \{S(x, x, p) + S(y, y, q)\} \]

\[ \Rightarrow (1 - \frac{\beta_1}{2})S(x, x, p) \preceq \frac{\beta_1}{2}S(y, y, q) \]

(9)

\[ \Rightarrow S(x, x, p) \preceq \frac{\beta_1}{2 - \beta_1}S(y, y, q) \]

Similarly

(10)

\[ S(y, y, q) \preceq \frac{\beta_1}{2 - \beta_1}S(x, x, p) \]

Adding (9) and (10) we have

\[ S(x, x, p) + S(y, y, q) \preceq \frac{\beta_1}{2 - \beta_1}[S(x, x, p) + S(y, y, q)] \]

\[ \Rightarrow (1 - \frac{\beta_1}{2 - \beta_1})|S(x, x, p) + S(y, y, q)| \leq 0 \]

\[ \Rightarrow 2(1 - \frac{\beta_1}{2 - \beta_1})|S(x, x, p) + S(y, y, q)| \leq 0. \]

But \(\frac{2(1 - \beta_1)}{2 - \beta_1} > 0\). Therefore \(|S(x, x, p) + S(y, y, q)| = 0\). Which implies that \(x = p\) and \(y = q\) \(\Rightarrow (x, y) = (p, q)\). Thus \(f\) and \(g\) have a unique common coupled fixed point.

\[ \square \]

**Corollary 2.2.** Let \((X, S)\) be a complete complex valued symmetric \(S_b\)-metric space with parameter \(s \geq 1\) and let the mapping \(f : X^2 \rightarrow X\) satisfying

\[ S(f(x, y), f(x, y), f(u, v)) \preceq \beta_1 \frac{S(x, x, u) + S(y, y, v)}{2} + \beta_2 \frac{S(x, x, f(x, y))S(u, u, f(u, v))}{1 + s[S(x, x, f(u, v) + S(u, u, f(x, y)) + S(x, x, u) + S(y, y, v)]} \]

for all \(x, y, u, v \in X\) and \(\beta_1, \beta_2\) are non-negative real numbers with \(\beta_1 + \beta_2 < 1\). Then \(f\) has a unique coupled fixed point.

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.
REFERENCES


