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WELL-POSEDNESS AND EXPONENTIAL STABILITY OF SWELLING POROUS ELASTIC SOILS WITH A SECOND SOUND AND DISTRIBUTED DELAY TERM

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Abstract. In this paper we consider a one-dimensional swelling porous-elastic system with second sound and distributed delay term. We prove that the combination of the frictional damping with the heat flux effect is strong enough to provoke an exponential decay of the energy even if the delay is a source of destabilization.

Keywords: Swelling porous system; semi-group; second sound; distributed delay time; exponential stability; Lyapunov functional.

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1. INTRODUCTION

Using the second law of thermodynamics, Eringen [9] developed general and linear constitutive equations of mixtures of viscous liquids, elastic solids and gas. Then established a relation between the continuum theory of swelling porous elastic soils and the classical diffusion theories. For more discussion on continuum theories that have been developed to model mixtures we

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refer the reader to [3]. As discussed deeply in [14], expansive soils cause minor to major structural damages to buildings, that includes floor slab on grade cracking, buckling of pavements and cracking of buried pipes. Thus to deal with this problematic soil, it is essential to evaluate its swelling potential, and then propose several techniques to prevent structural damages, such as reducing the swelling, using sufficiently strong structures and isolating the structure from the swelling soil (see [22]). For more practical applications of the theory in architecture and civil engineering, we mention for example Handy [11], Hung [12], see also ([10], [13]).

The linear theory of swelling porous elastic soils as considered in [23] and [24] is given by the system

$$(1.1) \quad \begin{aligned} \rho_1 \varphi_{tt} &= P_{1x} - G_1 + H_1 \\ \rho_2 \psi_{tt} &= P_{2x} + G_2 + H_2 \end{aligned}$$

where φ represent the displacement of the fluid with density ρ_1 and ψ is the elastic solid material with density ρ_2 . The functions (P_1, P_2) represent the partial tension, (G_1, G_2) the internal body forces, and (H_1, H_2) the external forces, acting on the displacement and on the elastic solid, respectively. Moreover, the partial tensions (P_1, P_2) are given by

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = A \begin{pmatrix} \varphi_x \\ \psi_x \end{pmatrix},$$

where A is the positive definite matrix $\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$, with a_1, a_3 are positive constants and $a_2 \neq 0$ is a real number.

Wang and Guo [26] considered (1.1) by taking

$$\begin{aligned} G_1 &= G_2 = 0, \\ H_1 &= -\rho_1 \gamma(x) \varphi_t, \\ H_2 &= 0, \end{aligned}$$

where $\gamma(x)$ is an internal viscous damping function with positive mean. By using the spectral method they established an exponential stability result. More recently, in [7], the authors studied

(1.1) with different conditions

$$\begin{aligned} G_1 &= G_2 = H_1 = 0, \\ H_2 &= - \int_0^t g(t-s) \phi_{xx}(x,s) ds - \beta_1 \phi_t \\ &\quad - \int_{\tau_1}^{\tau_2} |\beta_2(\sigma)| \phi_t(x,t-\sigma) d\sigma, \end{aligned}$$

they used the multiplier method to establish a general decay result.

In the literature, many interesting results on the swelling porous system with different conditions on G_1, G_2, H_1 and H_2 are considered (see [1], [2], [4], [16], [17], [18]), where the stability results obtained by using either multiplier or spectral methods.

A system which is asymptotically stable may be destabilized under the effects of time delay, that make it a property of practical and theoretical importance for many physical systems. As mentioned in [21], by a change of variable, distributed delay can be regarded as a memory acting only on the time interval $(t - \tau_2, t - \tau_1)$,

$$\int_{\tau_1}^{\tau_2} \mu(s) u_t(x, t-s) ds = \int_{t-\tau_2}^{t-\tau_1} \mu(t-s) u_t(x, s) ds,$$

for more discussions (see [8], [27]).

Models governed by the Fourier's law of heat conduction leads to an infinite speed of heat propagation, which means that any thermal disturbance at one point has an instantaneous effect somewhere else. By replacing Fourier's law $\beta q + \theta_x = 0$ with a wave propagation process described by Cattaneo's law $\tau q_t + \beta q + \theta_x = 0$, the problem of the infinite speed of heat propagation is eliminated (see [6], [25]).

In the present work, we consider (1.1) with distributed delay term on the elastic solid is in the form of distributed delay term, that is:

$$\begin{aligned} G_1 &= G_2 = 0, \\ H_1 &= \mu_1 \phi_t + \int_{\tau_1}^{\tau_2} \mu_2(\sigma) \phi_t(x, t-\sigma) d\sigma, \\ H_2 &= 0. \end{aligned}$$

Thus, we are concerned with the following thermoelastic system of swelling porous elastic soils with a linear frictional damping and an internal distributed delay acting on the transverse

displacement, where the heat flux is given by Cattaneo's law:

$$(1.2) \quad \left\{ \begin{array}{l} \rho_1 \varphi_{tt} - a_1 \varphi_{xx} - a_2 \psi_{xx} + \mu_1 \varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s) \varphi_t(x, t-s) ds = 0 \text{ in } (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - a_3 \psi_{xx} + a_2 \varphi_{xx} + \delta \theta_x = 0, \text{ in } (0, 1) \times (0, \infty), \\ \rho_3 \theta_t - q_x + \delta \psi_{tx} = 0, \text{ in } (0, 1) \times (0, \infty), \\ \tau q_t + \beta q + \theta_x = 0, \text{ in } (0, 1) \times (0, \infty), \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \theta(x, 0) = \theta_0(x) \text{ in } (0, 1), \\ \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), q(x, 0) = q_0(x) \text{ in } (0, 1), \\ \varphi(0, t) = \varphi(1, t) = \psi_x(0, t) = \psi_x(1, t) = \theta(0, t) = \theta(1, t) = 0 \text{ in } (0, \infty), \end{array} \right.$$

where the functions $(\varphi, \psi, \theta, q)$ are the transverse displacement of the beam, the rotation angle, the difference temperature, the heat flux, respectively. The coefficients, $\rho_1, \rho_2, \rho_3, a_1, a_2, a_3, \beta, \delta, \mu_1, \tau$ are positive constants. τ_1 and τ_2 are two real numbers with $0 \leq \tau_1 < \tau_2$, $\mu_1 > 0$ is a positive constant, $\mu_2 : [\tau_1, \tau_2] \rightarrow \mathbb{R}$ is L^∞ function, $\mu_2 \geq 0$ almost everywhere, such that

$$(1.3) \quad \mu_1 \geq \int_{\tau_1}^{\tau_2} \mu_2(s) ds.$$

Finally, $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0$ are the initial data and f_0 is history function, belong to an appropriate functional spaces.

The purpose of this paper is to study the well-posedness and the asymptotic behavior of the solution of (1.2) regardless of the speeds of wave propagation.

2. PRELIMINARIES

As in [19], we introduce the new variable

$$(2.1) \quad z(x, \rho, s, t) = \varphi_t(x, t - \rho s) \text{ in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

It is straight forward to check that z satisfies

$$sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0 \text{ in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Consequently, problem (1.2) is equivalent to

$$(2.2) \quad \left\{ \begin{array}{l} \rho_1 \varphi_{tt} - a_1 \varphi_{xx} - a_2 \psi_{xx} + \mu_1 \varphi_t + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds = 0, \text{ in } (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - a_3 \psi_{xx} + a_2 \varphi_{xx} + \delta \theta_x = 0, \text{ in } (0, 1) \times (0, \infty), \\ \rho_3 \theta_t - q_x + \delta \psi_{tx} = 0, \text{ in } (0, 1) \times (0, \infty), \\ \tau q_t + \beta q + \theta_x = 0, \text{ in } (0, 1) \times (0, \infty), \\ sz_t(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0, \text{ in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty), \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \theta(x, 0) = \theta_0(x) \text{ in } (0, 1), \\ \psi(x, 0) = \psi_0(x), \psi_t(x, 0) = \psi_1(x), q(x, 0) = q_0(x) \text{ in } (0, 1), \\ \varphi(0, t) = \varphi(1, t) = \psi_x(0, t) = \psi_x(1, t) = \theta(0, t) = \theta(1, t) = 0 \text{ in } (0, \infty), \\ z(x, 0, t, s) = \varphi_t(x, t) \text{ in } (0, 1) \times (0, \infty) \times (\tau_1, \tau_2), \\ z(x, \rho, 0, s) = f_0(x, \rho, s) \text{ in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2). \end{array} \right.$$

For any regular solution of (2.2), we define the energy by

$$(2.3) \quad \begin{aligned} E(t) &= \frac{1}{2} \int_0^1 \left\{ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_3 \theta_t^2 + \theta q^2 + \left(a_1 - \frac{a_2^2}{a_3} \right) \varphi_x^2 \right. \\ &\quad \left. + \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 \right\} dx \\ &\quad + \frac{1}{2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

We assume (1.3) holds and establish the well-posedness as well as the exponential stability results of the energy.

3. WELL-POSEDNESS OF THE PROBLEM

In this section, we prove the existence and uniqueness of solutions for (2.2) using semigroup theory. Introducing the vector function

$$\phi = (\varphi, u, \psi, v, \theta, q, z)^T,$$

and the two new dependent variables

$$u = \varphi_t \text{ and } v = \psi_t,$$

then the system (2.2) can be written as

$$(3.1) \quad \begin{cases} \phi'(t) + \mathcal{A}\phi(t) = 0, & t > 0, \\ \phi(0) = \phi_0 = (\varphi_0, u_0, \psi_0, v_0, \theta_0, q_0, z_0)^T. \end{cases}$$

Where the operator \mathcal{A} is defined by

$$\mathcal{A}\phi = \begin{pmatrix} -u \\ \frac{1}{\rho_1} \left(-a_1 \varphi_{xx} - a_2 \psi_{xx} + \mu_1 u + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) \\ -v \\ \frac{1}{\rho_2} (-a_3 \psi_{xx} + a_2 \varphi_{xx} + \delta \theta_x) \\ \frac{1}{\rho_3} (-q_x + \delta v_x) \\ \frac{1}{\tau} (\beta q + \theta_x) \\ \frac{1}{s} z_\rho(x, \rho, s, t) \end{pmatrix},$$

so

$$\mathcal{A} = \begin{pmatrix} 0 & -I & 0 & 0 & 0 & 0 & 0 \\ \frac{-a_1}{\rho_1} \partial_{xx} & \frac{\mu_1}{\rho_1} & \frac{-a_2}{\rho_1} \partial_{xx} & 0 & 0 & 0 & \frac{1}{\rho_1} \int_{\tau_1}^{\tau_2} \mu_2(s) ds \\ 0 & 0 & 0 & -I & 0 & 0 & 0 \\ \frac{-a_2}{\rho_2} \partial_{xx} & 0 & \frac{-a_3}{\rho_2} \partial_{xx} & 0 & \frac{\delta}{\rho_2} \partial_x & 0 & 0 \\ 0 & 0 & 0 & \frac{\delta}{\rho_3} \partial_x & 0 & \frac{1}{\rho_3} \partial_x & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\tau} \partial_x & \frac{\beta}{\tau} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} \partial_\rho \end{pmatrix}.$$

We have reserved the following spaces

$$L_*^2(0, 1) = \left\{ \psi \in L^2(0, 1) : \int_0^1 \psi(x) dx = 0 \right\},$$

$$H_*^1(0, 1) = H^1(0, 1) \cap L_*^2(0, 1),$$

$$H_*^2(0, 1) = \{ \psi \in H^2(0, 1) : \psi_x(0) = \psi_x(1) = 0 \},$$

and \mathcal{H} is the energy space given by

$$\begin{aligned} \mathcal{H} &= H_0^1(0, 1) \times L^2(0, 1) \times H_*^1(0, 1) \times L_*^2(0, 1) \times L^2(0, 1) \\ &\quad \times L_*^2(0, 1) \times L^2((0, 1) \times (0, 1) \times (\tau_1, \tau_2)). \end{aligned}$$

We will show that \mathcal{A} generates a C_0 Semi-group on \mathcal{H} .

Let $\phi = (\varphi, u, \psi, v, \theta, q, z)^T$, $\bar{\phi} = (\bar{\varphi}, \bar{u}, \bar{\psi}, \bar{v}, \bar{\theta}, \bar{q}, \bar{z})^T$, we equip the Hilbert space \mathcal{H} with the inner product through

$$(3.2) \quad \begin{aligned} \langle \phi, \bar{\phi} \rangle_{\mathcal{H}} = & \rho_1 \int_0^1 u \bar{u} dx + \rho_2 \int_0^1 v \bar{v} dx + \rho_3 \int_0^1 \theta \bar{\theta} + \tau \int_0^1 q \bar{q} + \left(a_1 - \frac{a_2}{q_3} \right) \\ & \int_0^1 \bar{\varphi}_x dx + \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right) \left(\frac{a_2}{\sqrt{a_3}} \bar{\varphi}_x + \sqrt{a_3} \bar{\psi}_x \right) dx \\ & + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z(x, \rho, s, t) \bar{z}(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

\mathcal{H} is the Hilbert space. In this box, the inner product above is equivalent to the natural inner product set to \mathcal{H} .

The domain of \mathcal{A} is

$$D(\mathcal{A}) = \left\{ \begin{array}{l} \phi \in \mathcal{H} : \varphi \in H^2(0, 1) \cap H_0^1(0, 1), \psi \in H_*^2(0, 1) \cap H_*^1(0, 1), \\ v, q \in H_*^1(0, 1), u, \theta \in H_0^1(0, 1), \\ z, z_\rho \in L_\omega^2((0, L) \times (0, 1) \times (\tau_1, \tau_2)), u(x) = (x, 0, s) \text{ in } (0, L). \end{array} \right\}.$$

Clearly, $D(\mathcal{A})$ is dense in \mathcal{H} . Now we can give the following existence and uniqueness result.

Theorem 1. *Suppose that hypothesis (1.3) holds. Then, for any initial data $\phi_0 \in \mathcal{H}$ there exists a unique solution $\phi \in C([0, \infty), \mathcal{H})$ of problem (3.1). Moreover, if $\phi_0 \in D(\mathcal{A})$, then $\phi \in C([0, \infty), D(\mathcal{A})) \cap C^1([0, \infty), \mathcal{H})$.*

Proof. To obtain the above result, we will prove that $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$ is a maximal monotone operator. For this purpose, we need the following two steps.

Step 1: In this step, we prove that the operator \mathcal{A} is monotone. Let $\phi \in D(\mathcal{A})$,

$$(3.3) \quad \begin{aligned} \langle A\phi, \phi \rangle_{\mathcal{H}} = & \beta \int_0^1 q^2 + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s) ds dx \\ & + \left(\mu_1 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_0^1 u_2 dx + \int_0^1 u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s) ds dx. \end{aligned}$$

Using Young's inequality, the last term in (3.3), we have

$$(3.4) \quad \begin{aligned} & - \int_0^1 u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s) ds dx \\ & \leq \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 u^2 dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s) ds dx, \end{aligned}$$

substitute (3.4) in (3.3) yield,

$$\langle A\phi, \phi \rangle_{\mathcal{H}} \geq \beta \int_0^1 q^2 dx + \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_0^1 u^2 dx.$$

By (1.3), $\langle A\phi, \phi \rangle_{\mathcal{H}} \geq 0$, then we conclude that \mathcal{A} is monotone

Step 2: We prove that operator $\mathcal{A} + I$ is surjective. That is to say, for everything $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7) \in \mathcal{H}$, we are looking for $\phi \in D(A)$ satisfying

$$(\mathcal{A} + I)\phi = G,$$

so we get

$$(3.5) \quad \left\{ \begin{array}{l} \varphi - u = g_1, \\ \rho_1 u - a_1 \varphi_{xx} - a_2 \psi_{xx} + \mu_1 u + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds = \rho_1 g_2, \\ \psi - v = g_3, \\ \rho_2 v - a_3 \psi_{xx} + a_2 \varphi_{xx} + \delta \theta_x = \rho_2 g_4, \\ \rho_3 \theta - q_x + \delta v_x = \rho_3 g_5, \\ (\tau + \beta) q + \theta_x = \tau g_6, \\ z + \frac{1}{s} z_\rho(x, \rho, t) = g_7. \end{array} \right.$$

Suppose we have found u and v . Therefore, the first and the third equation in (3.5) given

$$(3.6) \quad \left\{ \begin{array}{l} u = \varphi - g_1, \\ v = \psi - g_3. \end{array} \right.$$

It is clear that $u \in H_0^1(0, 1)$, $v \in H_0^1(0, 1)$. And we can find

$$(3.7) \quad z(x, 0, s) = u(x), \text{ if } x \in (0, 1), s \in (\tau_1, \tau_2).$$

Following the same approach as in [20], we obtain by using equations for z in (3.6),

$$(3.8) \quad z(x, \rho, s) + s^{-1} z_\rho(x, \rho, s) = g_6(x, \rho, s), \text{ if } x \in (0, 1), s \in (\tau_1, \tau_2).$$

Then by (3.7) and (3.8),

$$(3.9) \quad z(x, \rho, s) = e^{-\rho s} u(x) + s e^{-\rho s} \int_0^\rho g_7(x, \tau, s) e^{\tau s} d\tau.$$

From (3.5)₆ we have

$$\theta_x = \tau g_6 - (\beta + \tau)q,$$

so

$$(3.10) \quad \theta = \tau \int_0^x g_6 dx - (\beta + \tau) \int_0^x q dx,$$

then $\theta(0) = \theta(1) = 0$. Using (3.6) and (3.10) in (3.5), we get

$$(3.11) \quad \begin{cases} -a_1 \varphi_{xx} - a_2 \psi_{xx} + \mathcal{M} \varphi = h_1 \in L^2(0, 1), \\ -a_3 \psi_{xx} - a_2 \varphi_{xx} + \rho_2 \psi - (\beta + \tau) \delta q = h_2 \in L(0, 1), \\ -\delta g_{3x} - \rho_3 (g_5 - \tau) \int_0^1 g_6(y) dy = h_3 \in L^2(0, 1), \end{cases}$$

where

$$(3.12) \quad \begin{aligned} \mathcal{M} &= \mu_1 + \rho_1 + \int_{\tau_1}^{\tau_2} \mu_2(s) e^{-s} ds, \\ h_1 &= \mu g_1 + \rho_1 g_2 - \int_{\tau_1}^{\tau_2} s \mu_2(s) e^{-s} \int_0^1 e^{st} g_7(x, \tau, s) d\tau ds, \\ h_2 &= \rho_2 (g_3 + g_4) - \tau \delta g_6, \\ h_3 &= -\delta g_{3x} - \rho (g_5 - \tau \int_0^x g_6(y) dy). \end{aligned}$$

The variational formulation associated with (3.11) takes the form

$$(3.13) \quad B((\varphi, \psi, q), (\tilde{\varphi}, \tilde{\psi}, \tilde{q})) = F(\tilde{\varphi}, \tilde{\psi}, \tilde{q}),$$

where $B: [H_0^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1)]^2 \rightarrow \mathbb{R}$ is the bilinear form

$$\begin{aligned} B((\varphi, \psi, q), (\tilde{\varphi}, \tilde{\psi}, \tilde{q})) &= a_2 \int_0^1 (\psi_x \tilde{\varphi}_x + \varphi_x \tilde{\psi}_x) dx + \mathcal{M} \int_0^1 \varphi \tilde{\varphi} dx + \rho_2 \int_0^1 \psi \tilde{\psi} dx \\ &+ a_1 \int_0^1 \varphi_x \tilde{\varphi}_x dx - \delta (\gamma + \beta) \int_0^1 q \tilde{\psi} dx \\ &+ (\tau + \beta) \int_0^1 q \tilde{q} dx + \delta (\tau + \beta) \int_0^1 \psi \tilde{q} dx \\ &+ \rho_3 (\tau + \beta)^2 \int_0^1 \left(\int_0^x q(y) dy \int_0^x \tilde{q}(y) dy \right) dx, \end{aligned}$$

and $F: [H_*^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1)] \rightarrow \mathbb{R}$ is the linear form

$$F(\tilde{\varphi}, \tilde{\psi}, \tilde{q}) = \int_0^1 h_1 \tilde{\varphi} dx + \int_0^1 h_2 \tilde{\psi} dx + \int_0^1 h_3 \int_0^x \tilde{q}(y) dy dx,$$

where the space $V = H_*^1(0, 1) \times H_0^1(0, 1) \times L_*^2(0, 1)$ quipped with the norm

$$\|(\varphi, \psi, q)\|_V^2 = a_1 \int_0^1 \varphi_x^2 dx + a_3 \int_0^1 \psi_x^2 dx + (\beta + \tau) \int_0^1 q^2 dx,$$

which is equivalent to

$$\|(\varphi, \psi, q)\|_V^2 := \|\varphi_x\|_2^2 + \|\psi_x\|_2^2 + \|\varphi\|_2^2 + \|\psi\|_2^2 + \|q\|_2^2.$$

To solve the problem (3.11), it suffices to show that B is continuous and coercive, and that F is continuous, we can therefore easily see that B and F are bounded, and moreover, we have for a $c > 0$:

$$\begin{aligned} B((\varphi, \psi, q), (\tilde{\varphi}, \tilde{\psi}, \tilde{q}))_V &= 2a_2 \int_0^1 \psi_x \varphi_x dx + (\beta + \tau) \int_0^1 q^2 dx + a_1 \int_0^1 \varphi_x^2 dx \\ &\quad + \rho_2 \int_0^1 \psi^2 dx + \mathcal{M} \int_0^1 \varphi^2 dx + \rho_3 (\beta + \tau)^2 \int_0^1 \left(\int_0^1 q(y) dy \right)^2 dx \\ &\geq C \|(\varphi, \psi, q)\|_V^2, \end{aligned}$$

therefore, according to the Lax-Milgram theorem, the system (3.11) admits a unique solution

$$\varphi \in H_*^1(0, 1), \psi \in H_0^1(0, 1) \text{ and } q \in L_*^2(0, 1).$$

By replacing φ in (3.6)₁, ψ in (3.6)₂ and q in (3.10) we get

$$\mathcal{M} \in H_0^1(0, 1), V \in H_*^1(0, 1), \theta \in H_0^1(0, 1),$$

similarly by inserting u in (3.9) we obtain $z, z_\rho \in L^2((0, 1) \times (0, 1) \times (\tau_1, \tau_2))$ if $(\tilde{\varphi}, \tilde{q}) = (0, 0) \in H_0^1(0, 1) \times L_*^2(0, 1)$ then (3.13) reduces to

$$\begin{aligned} &a_2 \int_0^1 \varphi_x \tilde{\psi}_x dx + \rho_2 \int_0^1 \psi \tilde{\psi} dx - \delta (\beta + \tau) \int_0^1 q \tilde{\psi} dx \\ (3.14) \quad &= \int_0^1 h_2 \tilde{\psi} dx \quad \forall \tilde{\psi} \in H_*^1(0, 1), \end{aligned}$$

which implies

$$(3.15) \quad \rho_2 \psi = a_2 \varphi_{xx} + (B + \tau) \delta q + h_2 \in L^2(0, 1),$$

by the theory of regularity for linear equations, it follows that

$$\psi \in H^2(0, 1) \cap H_*^1(0, 1).$$

Moreover (3.14) is also true for all $\phi \in C^1([0, 1]) \subset H_*^1(0, 1)$ so we have

$$\rho_2 \int_0^1 \psi \phi \, dx + \int_0^1 (-a_2 \phi_{xx} - \delta(B + \tau)q - h_2) \phi \, dx = 0,$$

for all $\phi \in C^1([0, 1])$, thus using integration by parts and bearing in the mind (3.15),

$$\psi_x(1)\phi(1) - \psi_x(0)\phi(0) = 0 \quad \forall \phi \in C^1([0, 1]).$$

So, $\psi_x(0) = \psi_x(1) = 0$. Therefore we get

$$\begin{aligned} -a_1 \phi_{xx} &= a_2 \psi_{xx} - \mathcal{M} \phi + h_1 \in L^2(0, 1), \\ -q_x &= \delta \psi_x - (B + \tau) \rho_3 \int_0^x q(y) \, dy + h_3 \in L^2(0, 1), \end{aligned}$$

thus we have

$$\phi \in H^2(0, 1) \cap H_0^1(0, 1), \quad q \in H_*^1(0, 1).$$

Finally, we ensure the existence of unique $\phi \in D(\mathcal{A})$ such that (3.5) is satisfied, by the application of the regularity theory of linear elliptic equations. Consequently, \mathcal{A} is maximal operator. Hence, from Lumer-Phillips theorem the result of Theorem 1 follows. \square

4. EXPONENTIAL STABILITY

In this section, we state and prove our stability result for the energy of the solution of system (2.2), using the multiplier technique. To achieve our goal, we need the following lemmas.

Lemma 1. *Let $(\phi, \psi, \theta, q, z)$ be a solution (2.2) and assume (1.3) holds. Then the energy functional defined by (2.3) satisfied*

$$(4.1) \quad \frac{d}{dt}(E(t)) \leq -\beta \int_0^1 q^2 \, dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \int_0^1 \phi_t^2 \, dx, \quad t \geq 0.$$

Proof. Multiplying the equations (2.2)₁, (2.2)₂, (2.2)₃, and (2.2)₄ by ϕ_t , ψ_t , θ and q respectively, and we integrate along $[0, 1]$ using the boundary conditions, and by addition, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \{ \rho_1 \phi_t^2 + a_1 \phi_x^2 + \rho_2 \psi^2 + a_3 \psi_x^2 + \rho_3 \theta^2 + \tau q^2 + 2a_2 \psi_x \phi_x \} \, dx \\ (4.2) \quad & = -\beta \int_0^1 q^2 \, dx - \mu_1 \int_0^1 \phi_t^2 \, dx - \int_0^1 \phi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) \, ds \, dx. \end{aligned}$$

We multiply (2.2)₄ by $|\mu_2(s)|z$, and integrates over $(0, 1) \times (0, 1) \times (\tau_1, \tau_2)$, we have

$$\begin{aligned} \frac{1}{2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(s)| z^2(x, \rho, s, t) d\rho ds dx \\ &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 0, s, t) ds dx, \end{aligned}$$

and recall that $z(x, 0, s, t) = \varphi_t$,

$$(4.3) \quad \begin{aligned} \frac{1}{2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx &= -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \varphi_t^2(x, t) ds dx, \end{aligned}$$

combining (4.2) and (4.3), we get

$$(4.4) \quad \begin{aligned} \frac{d}{dt}(E(t)) &= -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 \varphi_t^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds dx \\ &\quad - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \varphi_t^2(x, t) ds dx. \end{aligned}$$

From Young's inequality

$$(4.5) \quad \begin{aligned} - \int_0^1 \varphi_t \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx &\leq \frac{1}{2} \int_0^1 \varphi_t^2 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx. \end{aligned}$$

Inserting (4.5) in (4.4), we get

$$(4.6) \quad \begin{aligned} \frac{d}{dt}(E(t)) &\leq - \left(\mu_1 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \int_0^1 \varphi_t^2 dx - \beta \int_0^1 q^2 dx \\ &\quad + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \varphi_t^2(x, t) ds dx, \quad t \geq 0. \end{aligned}$$

Therefore, we get (4.1). □

Lemma 2. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2). Then the functional*

$$F_1(t) := \rho_1 \int_0^1 \varphi_t \varphi dx - \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi dx,$$

satisfies the estimate

$$(4.7) \quad \begin{aligned} F'_1(t) \leq & -\frac{1}{2} \left(a_1 - \frac{a_2^2}{a_3} \right) \int_0^1 \varphi_x^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx + \varepsilon_0 \int_0^1 \psi_t^2 dx + C_{\varepsilon_0} \int_0^1 \varphi_t^2 dx \\ & + C_0 \int_0^1 \theta^2 dx + C_0 \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx. \end{aligned}$$

Proof. Differentiating $F_1(t)$ using (2.2)₁ and (2.2)₂, gives

$$\begin{aligned} F'_1(t) &= \rho_1 \int_0^1 \varphi_t^2 dx + \int_0^1 \varphi \left[a_1 \varphi_{xx} + a_2 \psi_{xx} - \mu_1 \varphi_t - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right] dx \\ &\quad - \frac{a_2}{a_3} \int_0^1 \varphi [a_3 \psi_{xx} + a_2 \varphi_{xx} - \delta \theta_x] dx - \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx \\ &= \rho_1 \int_0^1 \varphi_t^2 dx - a_1 \int_0^1 \varphi_x^2 dx - \mu_1 \int_0^1 \varphi \varphi_t dx \\ &\quad - \int_0^1 \varphi \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx - \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx \\ &\quad + \frac{a_2^2}{a_3} \int_0^1 \varphi_x^2 dx - \frac{\delta a_2}{a_3} \int_0^1 \varphi_x \theta dx. \end{aligned}$$

Then,

$$(4.8) \quad \begin{aligned} F'_1(t) &= - \left(a_1 - \frac{a_2^2}{a_3} \right) \int_0^1 \varphi_x^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx - \int_0^1 \varphi \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx \\ &\quad - \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx - \frac{\delta a_2}{a_3} \int_0^1 \varphi_x \theta dx. \end{aligned}$$

Now, we estimate the terms in the right hand side of (4.8). Using Young's inequality, we have

$$(4.9) \quad -\frac{\delta a_2}{a_3} \int_0^1 \varphi_x \theta dx \leq \frac{1}{4} \left(a_1 - \frac{a_2^2}{a_3} \right) \int_0^1 \varphi_x^2 dx + C_0 \int_0^1 \theta^2 dx,$$

$$(4.10) \quad -\frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi_t dx \leq \varepsilon_0 \int_0^1 \psi_t^2 dx + C_{\varepsilon_0} \int_0^1 \varphi_t^2 dx.$$

By using the Cauchy-Schwarz's, Young's and Poincaré inequalities, we obtain

$$\begin{aligned}
-\int_0^1 \varphi \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx &\leq C_0 \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\
(4.11) \qquad \qquad \qquad &+ \frac{1}{4} \left(a_1 - \frac{a_2^2}{a_3} \right) \int_0^1 \varphi_x^2 dx.
\end{aligned}$$

Finally, by substituting (4.9), (4.10) and (4.11) in (4.8), we obtain estimate (4.7). \square

Lemma 3. *Let be the solution of (2.2). Then the functional satisfies the estimate*

$$\begin{aligned}
F_2'(t) &\leq -\frac{a_2^2 \rho_2}{4\sqrt{a_3}} \int_0^1 \psi_t^2 dx + C_{\varepsilon_1} \int_0^1 \varphi_x^2 dx + C_1 \int_0^1 \varphi_t^2 dx \\
&+ C_{\varepsilon_1} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds \right) dx \\
(4.12) \qquad \qquad \qquad &+ 2\varepsilon_1 \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 dx.
\end{aligned}$$

Proof. Taking the derivative of $F_2(t)$, using (2.2) and integration by parts, we obtain

$$\begin{aligned}
F_2'(t) &= a_2 \int_0^1 [a_1 \varphi_{xx} + a_2 \psi_{xx} - \mu_1 \varphi_t - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds] \left(\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right) dx \\
&+ \rho_1 a_2 \int_0^1 \varphi_t \left(\frac{a_2}{\sqrt{a_3}} \varphi_t + \sqrt{a_3} \psi_t \right) dx \\
&- \frac{a_2^2}{a_3} \int_0^1 [a_3 \psi_{xx} + a_2 \varphi_{xx} - \delta \theta_x] \left(\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right) dx \\
&- \frac{a_2^2}{a_3} \rho_2 \int_0^1 \psi_t \left(\frac{a_2}{\sqrt{a_3}} \varphi_t + \sqrt{a_3} \psi_t \right) dx + \frac{\mu_1 a_2^2}{\sqrt{a_3}} \int_0^1 \varphi \varphi_t dx \\
&+ \mu_1 a_2 \sqrt{a_3} \left(\int_0^1 \psi_t \varphi dx + \int_0^1 \psi \varphi_t dx \right),
\end{aligned}$$

next, we write

$$\begin{aligned}
F'_2(t) &= -a_1 a_2 \int_0^1 \varphi_x \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right) dx - a_2^2 \int_0^1 \psi_x \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right) dx \\
&\quad - \frac{\mu_1 a_2^2}{\sqrt{a_3}} \int_0^1 \varphi_t \varphi dx - \mu_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t \psi dx \\
&\quad - a_2 \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right) \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx \\
&\quad + \rho_1 \frac{a_2^2}{\sqrt{a_3}} \int_0^1 \varphi_t^2 dx + \rho_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t \psi_t dx + a_2^2 \int_0^1 \psi_x \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right) dx \\
&\quad + \frac{a_2^3}{\sqrt{a_3}} \int_0^1 \varphi_x \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right) dx - \frac{a_2^3 \rho_2}{a_3 \sqrt{a_3}} \int_0^1 \psi_t \varphi_t dx \\
&\quad - \frac{a_2^2 \rho_2}{\sqrt{a_3}} \int_0^1 \psi_t^2 dx + \mu_1 a_2 \sqrt{a_3} \int_0^1 \psi_t \varphi dx.
\end{aligned}$$

Then,

$$\begin{aligned}
F'_2(t) &= -a_2 \left(a_1 - \frac{a_2^2}{\sqrt{a_3}} \right) \int_0^1 \varphi_x \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right) dx \\
&\quad - a_2 \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right) \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx \\
&\quad + \rho_1 \frac{a_2^2}{\sqrt{a_3}} \int_0^1 \varphi_t^2 dx + \rho_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t \psi_t dx + \\
(4.13) \quad &\quad - \frac{a_2^3 \rho_2}{a_3 \sqrt{a_3}} \int_0^1 \psi_t \varphi_t dx - \frac{a_2^2 \rho_2}{\sqrt{a_3}} \int_0^1 \psi_t^2 dx + \mu_1 a_2 \sqrt{a_3} \int_0^1 \psi_t \varphi dx.
\end{aligned}$$

Using Young's inequality in (4.13), we get

$$\begin{aligned}
&\quad -a_2 \left(a_1 - \frac{a_2^2}{\sqrt{a_3}} \right) \int_0^1 \varphi_x \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right) dx \\
(4.14) \quad &\leq \varepsilon_1 \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 dx + C_{\varepsilon_1} \int_0^1 \varphi_x^2 dx,
\end{aligned}$$

$$(4.15) \quad -\frac{a_2^3 \rho_2}{a_3 \sqrt{a_3}} \int_0^1 \psi_t \varphi_t dx \leq \frac{a_2^2 \rho_2}{4 \sqrt{a_3}} \int_0^1 \psi_t^2 dx + C_1 \int_0^1 \varphi_t^2 dx,$$

$$(4.16) \quad \rho_1 a_2 \sqrt{a_3} \int_0^1 \varphi_t \psi_t dx \leq \frac{a_2^2 \rho_2}{4 \sqrt{a_3}} \int_0^1 \psi_t^2 dx + C_1 \int_0^1 \varphi_t^2 dx,$$

and

$$(4.17) \quad \mu_1 a_2 \sqrt{a_3} \int_0^1 \psi_t \varphi dx \leq \frac{a_2^2 \rho_2}{4 \sqrt{a_3}} \int_0^1 \psi_t^2 dx + C_1 \int_0^1 \varphi_x^2 dx.$$

By using the Cauchy-Schwarz inequality and those of Young and Poincaré in (4.13), we obtain:

$$(4.18) \quad \begin{aligned} & -a_2 \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right) \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx \\ & \leq \varepsilon_1 \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 dx \\ & \quad + C_{\varepsilon_1} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_0^1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds \right) dx. \end{aligned}$$

Finally, substituting (4.14), (4.15), (4.16), (4.17) and (4.18), we obtain the estimate (4.12). \square

Lemma 4. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2). Then the functional*

$$F_3(t) = \rho_2 \int_0^1 \psi \psi_t dx + \rho_1 \int_0^1 \varphi \varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx,$$

satisfies

$$(4.19) \quad \begin{aligned} F_3'(t) & \leq -\frac{1}{2} \left(a_1 - \frac{a_2^2}{a_3} \right) \int_0^1 \varphi_x^2 dx - \frac{1}{2} \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 dx \\ & \quad + C_2 \int_0^1 \theta^2 dx + |\mu_1| \int_0^1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds \right) dx \\ & \quad + \rho_1 \int_0^1 \varphi_t^2 dx + \rho_2 \int_0^1 \psi_t^2 dx. \end{aligned}$$

Proof. By differentiating $F_3(t)$, taking in account the second and the third equations in (2.2), and integrating by parts, we obtain

$$\begin{aligned}
F'_3(t) &= \rho_2 \int_0^1 \psi_t^2 dx + \int_0^1 (\alpha_3 \psi_{xx} + \alpha_2 \varphi_{xx} - \delta \theta_x) \psi dx + \rho_1 \int_0^1 \varphi_t^2 dx \\
&\quad + \int_0^1 \varphi \left[\alpha_1 \varphi_{xx} + \alpha_2 \psi_{xx} - \mu_1 \varphi_t - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right] dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx \\
&= \rho_2 \int_0^1 \psi_t^2 dx - \alpha_3 \int_0^1 \psi_x^2 dx - \alpha_2 \int_0^1 \varphi_x \psi_x dx + \delta \int_0^1 \theta \psi_x dx + \rho_1 \int_0^1 \varphi_t^2 dx \\
&\quad + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx - \alpha_1 \int_0^1 \varphi_x dx - \alpha_2 \int_0^1 \varphi_x \psi_x dx - \frac{\mu_1}{2} \int_0^1 \varphi^2 dx \\
&\quad - \int_0^1 \varphi \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s, t) ds \right) dx.
\end{aligned}$$

Finally, the estimate (4.19) is established, using Young's and Poincaré inequalities. \square

Lemma 5. Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2). Then the functional $F_4(t) = \tau \rho_3 \int_0^1 \theta \left(\int_0^x q(y) dy \right) dx$, satisfies the estimate

$$(4.20) \quad F'_4(t) \leq -\frac{\rho_3}{2} \int_0^1 \theta^2 dx + \varepsilon_2 \int_0^1 \psi_t^2 dx + c \left(1 + \frac{1}{\varepsilon_2} \right) \int_0^1 q^2 dx.$$

Proof. Differentiating $F_4(t)$, and using the equations in (2.2), we obtain

$$\begin{aligned}
F'_4(t) &= \tau \rho_3 \int_0^1 \theta_t \left(\int_0^x q(y) dy \right) dx + \tau \rho_3 \int_0^1 \theta \frac{d}{dt} \left(\int_0^x q(y) dy \right) dx \\
&= \tau \int_0^1 (-q_x - \delta \psi_{tx}) \left(\int_0^x q(y) dy \right) dx + \tau \rho_3 \int_0^1 \theta_t dy dx \\
&= \tau \int_0^1 q^2 dx + \delta \tau \int_0^1 \psi_t q dx + \rho_3 \int_0^1 \theta \int_0^x (-\beta q - \theta_x) dy dx \\
&= \tau \int_0^1 q^2 dx + \delta \tau \int_0^1 \psi_t q dx - \beta \rho_3 \int_0^1 \theta \int_0^x q dy dx - \rho_3 \int_0^1 \theta \int_0^x \theta_x dy dx.
\end{aligned}$$

Now we use Cauchy-Schwarz and Young's inequalities

$$\begin{aligned} F'_4(t) &\leq \tau \int_0^1 q^2 dx + \frac{\delta\tau}{4\varepsilon} \int_0^1 \psi_t^2 dx + \delta\tau\varepsilon_3 \int_0^1 q^2 dx - \rho_3 \int_0^1 \theta^2 dx \\ &\quad + \beta\rho_3\varepsilon \int_0^1 \theta^2 dx + \frac{\beta\rho_3}{4\varepsilon} \int_0^1 \left(\int_0^x q dy \right)^2 dx, \end{aligned}$$

then

$$F'_4(t) \leq \left(\tau + \delta\tau\varepsilon + \frac{\beta\rho_3}{4\varepsilon} \right) \int_0^1 q^2 dx + (\beta\rho_3\varepsilon - \rho_3) \int_0^1 \theta^2 dx + \frac{\tau\delta}{4\varepsilon} \int_0^1 \psi_t^2 dx.$$

Thus we get the estimate (4.20). □

Lemma 6. *Let $(\varphi, \psi, \theta, q, z)$ be the solution of (2.2). Then the functional*

$$F_5(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx,$$

satisfies, for some positive constant m_1 , the following estimate

$$\begin{aligned} (4.21) \quad F'_5(t) &\leq -m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \\ &\quad - m_1 \int_0^1 \int_0^1 |\mu_2(s)| z^2(x, \rho, s, t) ds dx + \mu_1 \int_0^1 \varphi_t^2 dx. \end{aligned}$$

Proof. Differentiating $F_5(t)$, and using (2.2)₅, we obtain

$$\begin{aligned} F'_5(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z(x, \rho, s, t) z_\rho(x, \rho, s, t) ds d\rho dx \\ &= -\frac{d}{d\rho} \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx \\ &= -\int_0^1 \int_{\tau_1}^{\tau_2} [|\mu_2(s)| e^{-s} z^2(x, \rho, s, t) ds d\rho dx - z^2(x, 0, s, t)] \end{aligned}$$

$$- \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx.$$

Since $z(x, 0, s, t) = \varphi_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $\rho \in [0, 1]$, we get

$$\begin{aligned} F'_5(t) &\leq - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t) ds dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int \varphi_t^2 dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

Using the fact that $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$.

Finally, setting $m_1 = e^{-\tau_2}$ and by (1.3), we obtain (4.21). \square

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional $E(t)$.

Lemma 7. *For N sufficiently large, the functional defined by*

$$(4.22) \quad L(t) := NE(t) + N_1 F_1 + N_2 F_2 + 2F_3 + N_3 F_4 + N_4 F_5,$$

where N_1, N_2, N_3 and N_4 are positive real numbers to be chosen appropriately later, satisfies

$$(4.23) \quad c_1 E(t) \leq L(t) \leq c_2 E(t), \quad \forall t \geq 0,$$

for two positive constants c_1 and c_2 .

Proof. Let $L(t) := NE(t) + N_1 F_1 + N_2 F_2 + 2F_3 + N_3 F_4 + N_4 F_5$,

$$\begin{aligned} |L(t) - NE(t)| &\leq N_1 \rho_1 \int_0^1 \varphi_t \varphi dx - N_1 \frac{a_2}{a_3} \rho_2 \int_0^1 \psi_t \varphi dx \\ &\quad + N_2 \rho_1 a_2 \int_0^1 \varphi_t \left(\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right) dx \\ &\quad - N_2 \frac{a_2^2}{a_3} \rho_2 \int_0^1 \psi_t \left(\frac{a_2}{\sqrt{a_3}} \varphi + \sqrt{a_3} \psi \right) dx \end{aligned}$$

$$\begin{aligned}
& + N_2 \frac{\mu_1 a_2^2}{2\sqrt{a_3}} \int_0^1 \varphi^2 dx + N_2 \mu_1 a_2 \sqrt{a_3} \int_0^1 \psi \varphi dx \\
& + 2\rho_2 \int_0^1 \psi \psi_t dx + 2\rho_1 \int_0^1 \varphi \varphi_t dx + \mu_1 \int_0^1 \varphi^2 dx \\
& + N_3 \gamma \rho_3 \int_0^1 \theta \left(\int_0^x q(y) dy \right) dx \\
& + N_4 \tau \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| e^{-s\rho} z^2(x, \rho, s, t) ds d\rho dx.
\end{aligned}$$

Exploiting Cauchy-Schwarz's and Young's and Poincaré inequalities, gives

$$\begin{aligned}
|L(t) - NE(t)| & \leq C \int_0^1 \left(\varphi_t^2 + \psi_t^2 + \theta^2 + q^2 + \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 + \varphi_x^2 \right) \\
& + C \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| e^{-s\rho} z^2(x, \rho, s, t) ds d\rho dx. \\
& \leq CE(t).
\end{aligned}$$

Which yields

$$(N - C)E(t) \leq L(t) \leq (N + C)E(t).$$

Consequently, By choosing N large enough. We obtain estimate (4.23). \square

Now, we are ready to state and prove the main result of this section.

Theorem 2. *Let $(\varphi, \psi, \theta, q, z)$ be the solution (2.2). Then there are two positive constants α and γ such that*

$$(4.24) \quad E(t) \leq \alpha E(0) e^{-\gamma t}, \quad t \geq 0.$$

Proof. by differentiating (4.22) and recalling (4.1), (4.7), (4.12), (4.19), (4.20), (4.21) we obtain

$$\begin{aligned}
L'(t) & \leq - \left[N(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds) - N_1(\rho_1 + C_{\varepsilon_0}) - C_1 N_2 - 2\rho_1 - N_4 \mu_1 \right] \int_0^1 \varphi_t^2 dx \\
& - \left[\frac{1}{2} \left(a_1 - \frac{a_2^2}{a_3} \right) N_1 - N_2 C_{\varepsilon_1} + \left(a_1 - \frac{a_2^2}{a_3} \right) \right] \int_0^1 \varphi_x^2 dx \\
& - [1 - 2\varepsilon_1 N_2] \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 dx - [\beta N - N_3 c (1 + \frac{1}{\varepsilon_2})] \int_0^1 q^2 dx
\end{aligned}$$

$$\begin{aligned}
& -\left[\frac{\rho_3}{2}N_3 - C_0N_1 - 2C_2\right] \int_0^1 \theta^2 dx - \left[\frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2 - \varepsilon_0N_1 - 2\rho_2 - \varepsilon_2N_3\right] \int_0^1 \psi_t^2 dx \\
& - [N_1C_0\mu_1 - C_{\varepsilon_1}N_2\mu_1 - 2\mu_1 + \tau N_4] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\
& - \tau N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx.
\end{aligned}$$

By setting

$$\varepsilon_0 = \frac{1}{N_1}, \quad \varepsilon_2 = \frac{1}{N_3}, \quad \varepsilon_1 = \frac{1}{4N_2},$$

we obtain

$$\begin{aligned}
L'(t) & \leq - \left[N(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds) - N_1(\rho_1 + C_{\varepsilon_0}) - C_1N_2 - 2\rho_1 - N_4\mu_1 \right] \int_0^1 \varphi_t^2 dx \\
& - \left[\frac{1}{2} \left(a_1 - \frac{a_2^2}{a_3} \right) N_1 - N_2C_{\varepsilon_1} + \left(a_1 - \frac{a_2^2}{a_3} \right) \right] \int_0^1 \varphi_x^2 dx \\
& - \frac{1}{2} \int_0^1 \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 dx - [\beta N - N_3c(1 + N_3)] \int_0^1 q^2 dx \\
& - \left[\frac{\rho_3}{2}N_3 - C_0N_1 - 2C_2 \right] \int_0^1 \theta^2 dx - \left[\frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2 - 2\rho_2 - 2 \right] \int_0^1 \psi_t^2 dx \\
& - [N_1C_0\mu_1 - C_{\varepsilon_1}N_2\mu_1 - 2\mu_1 + \tau N_4] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\
& - \tau N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx.
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose N_2 large enough such that

$$\alpha_1 = \frac{a_2^2\rho_2}{4\sqrt{a_3}}N_2 - 2\rho_2 - 2 > 0,$$

then we choose N_1 large enough such that

$$\alpha_2 = \frac{N_1}{2} \left(a_1 - \frac{a_2^2}{a_3} \right) - N_2 C_{\varepsilon_1} + \left(a_1 - \frac{a_2^2}{a_3} \right) > 0.$$

Once N_2 is fixed, then we choose N_3 large enough such that

$$\alpha_3 = \frac{\rho_3}{2} N_3 - C_0 N_1 - 2C_2 > 0.$$

For any N_1, N_2 and N_3 , choosing N_4 large enough that

$$N_1 C_0 \mu_1 - C_{\varepsilon_1} N_2 \mu_1 - 2\mu_1 + \tau N_4.$$

Finally, we choose N large enough (even larger so that (4.23) remains valid) so that

$$N \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| \right) - N_1 (\rho_1 + C_{\varepsilon_0}) - C_1 N_2 - 2\rho_1 - N_4 \mu_1 > 0,$$

and

$$\alpha_5 = \beta N - c(1 + N_3) N_3 > 0.$$

Take, $\alpha_0 = \tau N_4$, we obtain

$$\begin{aligned} L'(t) \leq & - \int_0^1 \left(\alpha_4 \varphi_t^2 + \alpha_1 \psi_t^2 + \alpha_3 \theta^2 + \alpha_5 q^2 + \frac{1}{2} \left(\frac{a_2}{\sqrt{a_3}} \varphi_x + \sqrt{a_3} \psi_x \right)^2 + \alpha_2 \varphi_x^2 \right) \\ & - \alpha_0 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

By (2.3), we get

$$(4.25) \quad L'(t) \leq -\sigma_0 E(t), \quad \forall t \geq 0,$$

for some $\sigma_0 > 0$. A combination of (4.23) and (4.25) gives

$$(4.26) \quad L'(t) \leq -\gamma L(t), \quad \forall t \geq 0,$$

Where $\gamma = \alpha_0/C_2$. A simple integration of (4.26) over $(0, t)$ yields

$$(4.27) \quad L(t) \leq L(0) e^{-\gamma t}, \quad \forall t \geq 0.$$

Finally, by combining (4.23) and (4.27) we obtain (4.24) with $\alpha = \frac{C_2 E(0)}{C_1}$, which completes the proof. \square

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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