ON WEAKLY $\alpha$-SHIFTING RING

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Abstract. For a ring endomorphism $\alpha$, we introduce weakly $\alpha$-shifting ring which is an extension of reduced as well as $\alpha$-shifting ring. The notion of weakly $\alpha$-shifting ring is a generalization of weak $\alpha$-compatible ring. We investigate various properties of this ring including some kinds of examples in the process of development of this new concept.

Keywords: weak $\alpha$-symmetric ring; weak $\alpha$-reversible ring; weak $\alpha$-compatible ring; weak $\alpha$-rigid ring; semi-commutative ring.

2010 AMS Subject Classification: 08A35, 16S50.

1. INTRODUCTION

Throughout this article, all rings are associative with identity 1 and $\alpha : R \rightarrow R$ is an ring endomorphism of a ring $R$. An element $x$ of a ring $R$ is nilpotent whenever $x^m = 0$ for some positive integer $m$. We denote $Nil(R)$, the set of nilpotent elements of $R$. We recall that a ring is said to be reduced whenever it has no non zero nilpotent elements. Again a ring is defined as symmetric in [1] whenever $xyz = 0 \Rightarrow xzy = 0$ for any $x, y, z \in R$. In 1999, Cohn [2] defined that
a ring is said to be reversible if $xy = 0$ implies $yx = 0$ for any $x, y \in R$. Again a ring is called semicommutative if for any $x, y \in R$, $xy = 0$ implies $xRy = 0$, this ring is also called ZI ring in [14]. If a ring is commutative, then it is always reversible, symmetric and semicommutative. It is mentioned that reduced rings are symmetric [3, Theorem I.3]. We can see that symmetric rings are reversible and reversible rings are always semicommutative by using their definitions. D.D. Anderson and V. Camillo provided the examples of non reduced symmetric ring [3, Example II.5] and a non symmetric reversible ring [3, Example I.5]. Moreover Example 1.5 of [15] is given to establish that a semicommutative ring may not be reversible.

In 1996, J. Krempa furnished a new concept rigid endomorphism of a ring in [5]. An endomorphism $\alpha$ of $R$ is stated as rigid if $x\alpha(x) = 0$ implies $x = 0$ for any $x \in R$. A ring is said to be $\alpha$-rigid if there exists a some rigid endomorphism $\alpha$. Motivated by that new term, L. Ouyang defined weak $\alpha$-rigid ring [6] in the context of $\text{Nil}(R)$ in 2008. A ring is weak $\alpha$-rigid if $x\alpha(x) \in \text{Nil}(R) \Leftrightarrow x \in \text{Nil}(R)$ for any $x \in R$. Another term $\alpha$-reversible ring [7] was introduced in 2009. A ring $R$ is right (left) $\alpha$-reversible whenever $xy = 0$ implies $y\alpha(x) = 0$ ($\alpha(y)x = 0$) for any $x, y \in R$. A ring is said to be $\alpha$-reversible if it satisfies the both conditions of right and left $\alpha$-reversible. In 2014, A. Bahlekeh introduced weak $\alpha$-reversible ring [8] by extending $\alpha$-reversible ring with the help of the set $\text{Nil}(R)$. Whenever $xy \in \text{Nil}(R)$ for any $x, y \in R$ implies $y\alpha(x) \in \text{Nil}(R)$, then $R$ is said to be weak $\alpha$-reversible. On the other hand, T.K. Kwak extended the concept of symmetric ring to $\alpha$-symmetric [9] by using ring endomorphism $\alpha$ in 2007. A ring $R$ is right(left) $\alpha$-symmetric if $xyz = 0 \Rightarrow xz\alpha(y) = 0$ ($\alpha(y)xz = 0$) for any $x, y, z \in R$. Motivated by this above definition, L. Ouyang and H. Chen introduced weak $\alpha$-symmetric ring [10] in 2010. A ring $R$ is weak $\alpha$-symmetric ring if $xyz \in \text{Nil}(R)$ implies $xz\alpha(y) \in \text{Nil}(R)$ for any $x, y, z \in R$. A ring $R$ is $\alpha$-compatible [11] if $xy = 0 \Leftrightarrow x\alpha(y) = 0$ for any $x, y \in R$. Again in 2011, weak $\alpha$-compatible ring [12] was introduced by using the weak condition to $\alpha$-compatible ring. A ring $R$ is weak $\alpha$-compatible if $xy \in \text{Nil}(R) \Leftrightarrow x\alpha(y) \in \text{Nil}(R)$ for any $x, y \in R$. In 2010, the concept of reversible ring extend to $\alpha$-shifting ring [13] by using ring endomorphism $\alpha$. They defined $R$ is right(left) $\alpha$-shifting whenever $x\alpha(y) = 0$ ($\alpha(x)y = 0$) implies $y\alpha(x) = 0$ ($\alpha(y)x = 0$) for any $x, y \in R$. The ring is $\alpha$-shifting whenever it satisfies both the conditions of right and left $\alpha$-shifting.
Motivated by all of the above definitions, we have introduced the concept of weakly $\alpha$-shifting ring which is an extension of reduced as well as $\alpha$-shifting ring. The notion of weakly $\alpha$-shifting ring is a generalization of weak $\alpha$-compatible ring.

2. **Weakly $\alpha$-Shifting Ring**

In this section we introduce and study a class of rings, called weakly $\alpha$-shifting ring which is an extension of $\alpha$-shifting rings. We prove that weakly $\alpha$-shifting ring is a generalization of weak $\alpha$-compatible ring. We investigate the connections of weakly $\alpha$-shifting ring to weak $\alpha$-reversible ring, weak $\alpha$-rigid ring and weak $\alpha$-symmetric rings. Moreover some results of $\alpha$-shifting rings can be extended to weakly $\alpha$-shifting ring. We now start with the following definition:

**Definition 2.1.** A ring $R$ is called weakly $\alpha$-shifting if $x\alpha(y) \in \text{Nil}(R) \Rightarrow y\alpha(x) \in \text{Nil}(R)$ for any $x, y \in R$.

It is very easy to check that

**Lemma 2.1.** If $xy \in \text{Nil}(R)$ for any $x, y \in R$ then $yx \in \text{Nil}(R)$.

We get the following remark from the above Lemma and the definition of weakly $\alpha$-shifting ring.

**Remark 2.1.** All rings are always weakly Id-shifting rings where Id is the identity ring endomorphism.

It is shown that the concept of reduced ring and $\alpha$-shifting ring do not depend on each other by the Example 1.1(2) and Example 2.3 of [13]. Now the next proposition shows the connection between $\alpha$-shifting and weakly $\alpha$-shifting ring.

**Proposition 2.1.** If $R$ is reduced and $\alpha$-shifting ring then $R$ is weakly $\alpha$-shifting ring.

**Proof.** Let $x\alpha(y) \in \text{Nil}(R)$ for any $x, y \in R$. It implies $x\alpha(y) = 0$ as $R$ is reduced ring. Since $R$ is $\alpha$-shifting, so $x\alpha(y) = 0$ implies $y\alpha(x) = 0$. As $R$ is reduced, $y\alpha(x) \in \text{Nil}(R)$. Thus $R$ is weakly $\alpha$-shifting ring.
Let $T_n(R)$ denote $n \times n$ upper triangular matrix ring over $R$. Then the map $\bar{\alpha} : T_n(R) \longrightarrow T_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ for all $(a_{ij}) \in T_n(R)$ is a ring endomorphism of $T_n(R)$.

**Proposition 2.2.** $R$ is weakly $\alpha$-shifting ring if and only if $T_n(R)$ is weakly $\bar{\alpha}$-shifting ring for any $n \in \mathbb{N}$.

**Proof.** Let $R$ be a weakly $\alpha$-shifting ring. Let us consider $A\bar{\alpha}(B) \in \text{Nil}(T_n(R))$ for

$$
\begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{1n} \\
    0 & a_{22} & \ldots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & a_{nn}
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
    b_{11} & b_{12} & \ldots & b_{1n} \\
    0 & b_{22} & \ldots & b_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & b_{nn}
\end{bmatrix}
$$

in $T_n(R)$. Therefore $(A\bar{\alpha}(B))^k = 0$ for some positive integer $k$. It implies $(a_{ii}\alpha(b_{ii}))^k = 0$ for $i = 1, 2, \ldots, n$. Then $a_{ii}\alpha(b_{ii}) \in \text{Nil}(R)$. Consequently $b_{ii}\alpha(a_{ii}) \in \text{Nil}(R)$ as $R$ is weakly $\alpha$-shifting ring. So $(b_{ii}\alpha(a_{ii}))^{k_i} = 0$ for some positive integer $k_i$. Now $(B\bar{\alpha}(A))^\bar{k} \in \text{Nil}(T_n(R))$ where $\bar{k} = \max\{k_1, k_2, \ldots, k_n\}$. Thus $B\bar{\alpha}(A) \in \text{Nil}(T_n(R))$ and so $T_n(R)$ is weakly $\alpha$-shifting ring.

Conversely let $T_n(R)$ be weakly $\bar{\alpha}$-shifting ring. Now let us consider $x\alpha(y) \in \text{Nil}(R)$ for $x, y \in R$. It implies $(x\alpha(y))^m = 0$ for some positive integer $m$. It leads to

$$
\begin{bmatrix}
    x & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & a
\end{bmatrix} \bar{\alpha} \begin{bmatrix}
    y & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0
\end{bmatrix}^m = 0.
$$

Now by using the definition of weakly $\bar{\alpha}$-shifting of $T_n(R)$,

$$
\begin{bmatrix}
    y & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0
\end{bmatrix} \alpha(x) \begin{bmatrix}
    0 & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0
\end{bmatrix} \in \text{Nil}(T_n(R)).
$$

Now it is very easy to check that $y\alpha(x) \in \text{Nil}(R)$. Thus we have $R$ is weakly $\alpha$-shifting ring.

The next example shows that there exists a weakly $\alpha$-shifting ring which is not $\alpha$-shifting.
Example 2.1. We prove that the ring $R \oplus R$ is weakly $\alpha$-shifting ring as shown in Example[2.15]. Then $T_2(R \oplus R)$ is weakly $\alpha$-shifting by immediate consequence of above Proposition 2.5. Let us consider $A = \begin{pmatrix} (0,0) & (0,0) \\ (0,0) & (1,0) \end{pmatrix}$ and $B = \begin{pmatrix} (0,0) & (0,1) \\ (0,0) & (0,0) \end{pmatrix}$ in $T_2(R \oplus R)$. Therefore we have $A\alpha(B) = 0$ but $B\alpha(A) = \begin{pmatrix} (0,0) & (0,1) \\ (0,0) & (0,0) \end{pmatrix} \neq 0$. Thus $T_2(R \oplus R)$ is not $\alpha$-shifting.

Lemma 2.2. Let $\alpha : R \rightarrow S$ be any ring endomorphism. If $x \in \text{Nil}(R)$ for any $x \in R$, then $\alpha(x) \in \text{Nil}(S)$.

Remark 2.2. The converse of the Lemma 2.2 holds whenever $\alpha$ is a monomorphism.

Proposition 2.3. Let $R$ be a weak $\alpha$-compatible ring. Then we have the following:

(i) $R$ is weak $\alpha$-reversible.

(ii) $R$ is weakly $\alpha$-shifting.

Proof. (i) Let $xy \in \text{Nil}(R)$ for any $x, y \in R$. Then $yx \in \text{Nil}(R) \Rightarrow y\alpha(x) \in \text{Nil}(R)$ by using Lemma 2.2 and the condition that $R$ is weak $\alpha$-compatible ring. Thus $R$ is a weak $\alpha$-reversible ring.

(ii) Let $x\alpha(y) \in \text{Nil}(R)$ for any $x, y \in R$. It implies $xy \in \text{Nil}(R)$ as $R$ is weak $\alpha$-compatible ring. Again $xy \in \text{Nil}(R)$ implies $y\alpha(x) \in \text{Nil}(R)$ by using Proposition 2.9(i). Thus $R$ is weakly $\alpha$-shifting ring.

Proposition 2.4. Let $R$ be a weak $\alpha$-reversible ring for a monomorphism $\alpha$. Then we have the following:

(i) $R$ is weak $\alpha$-compatible.

(ii) $R$ is weakly $\alpha$-shifting.

Proof. (i) Let us consider $xy \in \text{Nil}(R)$ for any $x, y \in R$. Now $xy \in \text{Nil}(R) \Rightarrow yx \in \text{Nil}(R) \Rightarrow x\alpha(y) \in \text{Nil}(R)$ by using Lemma 2.2 and $R$ is weak $\alpha$-reversible ring.

Conversely, let $x\alpha(y) \in \text{Nil}(R)$ for any $x, y \in R$. Then $\alpha(y)\alpha(x) \in \text{Nil}(R)$ by using the definition of weak $\alpha$-reversible ring of $R$. It implies $\alpha(yx) \in \text{Nil}(R) \Rightarrow yx \in \text{Nil}(R)$ by using Remark 2.8.

Now we have $xy \in \text{Nil}(R)$ by using Lemma 2.2. So for any $x, y \in R$, $xy \in \text{Nil}(R) \Leftrightarrow x\alpha(y) \in \text{Nil}(R)$. Thus $R$ is weak $\alpha$-compatible.
(ii) From Proposition 2.10(i), we have $R$ is weak $\alpha$-compatible. Now $R$ is weakly $\alpha$-shifting by using Proposition 2.9(ii).

**Proposition 2.5.** Weak $\alpha$-symmetric rings are always weak $\alpha$-reversible.

*Proof.* Let $R$ be a weak $\alpha$-symmetric ring. Let $xy \in Nil(R)$ for any $x, y \in R$. Since $R$ is weak $\alpha$-symmetric ring, so $xy = 1.x.y \in Nil(R)$ implies $y\alpha(x) = 1.y.\alpha(x) \in Nil(R)$. Thus $R$ is weak $\alpha$-reversible.

The next corollary is a direct deduction of Proposition 2.11 and Proposition 2.10(ii).

**Corollary 2.1.** Weak $\alpha$-symmetric rings are weakly $\alpha$-shifting whenever $\alpha$ is monomorphism.

The next example provides a weak $\alpha$-symmetric which is not weak $\alpha$-compatible.

**Example 2.2.** Let us consider that $F$ be any field and $R = F[x]$. Let $\alpha : R \longrightarrow R$ such that $\alpha(f(x)) = f(0)$ for all $f(x) \in F[x]$. Clearly $\alpha$ is a ring endomorphism of $F[x]$. We know that $R$ is a domain. We can easily show that for any ring endomorphism $\alpha$, domains are weak $\alpha$-symmetric ring. Thus $F[x]$ is weak $\alpha$-symmetric. Now let $f(x) = x \neq 0$ and $g(x) = a \neq 0$. So clearly $g(x)\alpha(f(x)) = 0 \in Nil(R)$. But $g(x)f(x) \neq 0 \not\in Nil(R)$ where $Nil(R) = 0$ as $R$ is domain. Thus we can see that $R$ is not weak $\alpha$-compatible.

**Remark 2.3.** Since the ring $R = F[x]$ given in Example 2.13 is also a weak $\alpha$-reversible ring by Proposition 2.11. Therefore the above example also provides a weak $\alpha$-reversible ring which is not weak $\alpha$-compatible.

In the next example, we give a weakly $\alpha$-shifting ring which is not weak $\alpha$-reversible.

**Example 2.3.** Let $R$ be a commutative ring. Let $\alpha : R \oplus R \longrightarrow R \oplus R$ such that $\alpha((a, b)) = (b, a)$ for all $(a, b) \in R \oplus R$. Then $\alpha$ is a ring endomorphism of $R \oplus R$. Now our first motive is to show that $R \oplus R$ is weakly $\alpha$-shifting ring. Therefore let us consider $(a, b)\alpha((c, d)) \in Nil(R \oplus R)$ for any $(a, b), (c, d) \in R \oplus R$. It implies $(ad, bc) \in Nil(R \oplus R)$. So there exists a positive integer $m$ such that $((ad, bc))^m = 0$. So we have $((ad))^m = ((bc))^m = 0$. Since $R$ is commutative, so $((da))^m = ((cb))^m = 0$. Now $((c, d)\alpha(a, b))^m = ((cb, da))^m = 0 \Rightarrow (c, d)\alpha((a, b)) \in Nil(R \oplus R)$. 
Therefore $R \oplus R$ is weakly $\alpha$-shifting ring.

Now we see that $(1, 0)(0, 1) = 0 \in \text{Nil}(R \oplus R)$. But $(0, 1)\alpha(1, 0) = (0, 1)$ is not nilpotent element of $R \oplus R$. So $R \oplus R$ is not weak $\alpha$-reversible.

**Remark 2.4.** We can see that $R \oplus R$, the weakly $\alpha$-shifting ring given in Example 2.15 is neither weak $\alpha$-compatible nor weak $\alpha$-symmetric by using Proposition 2.9(i) and Proposition 2.11 respectively.

**Proposition 2.6.** If $R$ is weak $\alpha$-rigid ring and $\text{Nil}(R)$ forms an ideal, then $R$ is weakly $\alpha$-shifting.

**Proof.** Let us consider $R$ is weak $\alpha$-rigid ring. Let $x\alpha(y) \in \text{Nil}(R)$ for any $x, y \in R$. It implies $yx\alpha(y)\alpha(x) = yx\alpha(yx) \in \text{Nil}(R)$ as $\text{Nil}(R)$ forms an ideal. Since $R$ is weak $\alpha$-rigid ring, we have $yx \in \text{Nil}(R)$. Now $yx \in \text{Nil}(R) \Rightarrow \alpha(yx) \in \text{Nil}(R)$ by using Lemma 2.7. Since $\text{Nil}(R)$ forms an ideal, then we have $\alpha^2(x)\alpha(y)\alpha(x)y \in \text{Nil}(R)$. It implies $\alpha(\alpha(x)y)\alpha(x)y \in \text{Nil}(R)$. Now by using the definition of weak $\alpha$-rigid ring, we have $\alpha(x)y \in \text{Nil}(R)$. Now $\alpha(x)y \in \text{Nil}(R) \Rightarrow y\alpha(x) \in \text{Nil}(R)$ by using Lemma 2.2. Thus $R$ is weakly $\alpha$-shifting ring.

**Example 2.4.** From the example of weakly $\alpha$-shifting ring given in Example 2.15, we can see that $(1, 0)\alpha(1, 0) = (1, 0)(0, 1) = 0 \in \text{Nil}(R \oplus R)$ but $(1, 0)$ is not nilpotent element of $R \oplus R$. Thus $R \oplus R$ is not weak $\alpha$-rigid ring.

**Lemma 2.3.** [16] $R$ is semicommutative $\Rightarrow \text{Nil}(R)$ forms an ideal.

We have the following corollary from the Proposition 2.17 and Lemma 2.19.

**Corollary 2.2.** If $R$ is weak $\alpha$-rigid ring and semicommutative then $R$ is weakly $\alpha$-shifting.

**Proposition 2.7.** Let $R$ be weakly $\alpha$-shifting ring. Then we have the following:

(i) If $x\alpha^k(y) \in \text{Nil}(R)$, then $y\alpha^k(x) \in \text{Nil}(R)$ for any positive integer $k$.

(ii) If $xy \in \text{Nil}(R)$, then $x\alpha^k(y), y\alpha^k(x) \in \text{Nil}(R)$ for any positive integer $k = 2m$.

**Proof.** (i) For $k = 1$, $x\alpha(y) \in \text{Nil}(R)$ implies $y\alpha(x) \in \text{Nil}(R)$ by the definition of weakly $\alpha$-shifting ring. Let us consider $x\alpha^m(y) \in \text{Nil}(R)$ implies $y\alpha^m(x) \in \text{Nil}(R)$ for some $m > 1$.
Now let \( x\alpha^{m+1}(y) \in \text{Nil}(R) \). It implies \( x\alpha(\alpha^m(y)) \in \text{Nil}(R) \Rightarrow \alpha^m(y)\alpha(x) \in \text{Nil}(R) \) as \( R \) is weakly \( \alpha \)-shifting ring. By using the Lemma 2.2, we have \( \alpha(x)\alpha^m(y) \in \text{Nil}(R) \). Again by using our assumption \( y\alpha^{m+1}(x) = y\alpha^m(\alpha(x)) \in \text{Nil}(R) \). Thus \( x\alpha^k(y) \in \text{Nil}(R) \) implies \( y\alpha^k(x) \in \text{Nil}(R) \) for any positive integer \( k \) by using principle of mathematical induction.

(ii) Let \( xy \in \text{Nil}(R) \). By using Lemma 2.7, we have \( \alpha(xy) = \alpha(x)\alpha(y) \in \text{Nil}(R) \). Since \( R \) is weakly \( \alpha \)-shifting, then \( y\alpha^2(x) = y\alpha(\alpha(x)) \in \text{Nil}(R) \). Again by using Lemma 2.7, we have \( \alpha(y\alpha^2(x)) \in \text{Nil}(R) \). It implies \( \alpha(y)\alpha^3(x) \in \text{Nil}(R) \). Now by using Lemma 2.2, we have \( \alpha^3(x)\alpha(y) \in \text{Nil}(R) \). Since \( R \) is weakly \( \alpha \)-shifting ring, therefore \( y\alpha^4(x) = y\alpha(\alpha^3(x)) \in \text{Nil}(R) \). Continuing the same process, we get \( y\alpha^k(x) \in \text{Nil}(R) \) for any positive integer \( k = 2m \). On the other hand, if \( xy \in \text{Nil}(R) \), then \( xy \in \text{Nil}(R) \) by using Lemma 2.2. Using the above method for \( yx \) in lieu \( xy \), we get \( x\alpha^k(y) \in \text{Nil}(R) \) for any positive integer \( k = 2m \).

**Proposition 2.8.** Let \( R \) be weakly \( \alpha \)-shifting ring for monomorphism \( \alpha \). Then the following are equivalent:

(i) \( xy \in \text{Nil}(R) \) for any \( x, y \in R \).

(ii) \( x\alpha^k(y) \in \text{Nil}(R) \) for any positive integer \( k = 2m \).

**Proof.** (i)\( \Rightarrow \) (ii) is obvious by Proposition 2.21 (ii).

(ii)\( \Rightarrow \) (i). If \( x\alpha^k(y) \in \text{Nil}(R) \) for any positive integer \( k = 2m \), then \( x\alpha(\alpha^{k-1}(y)) \in \text{Nil}(R) \). Since \( R \) is weakly \( \alpha \)-shifting ring, we get \( \alpha^{k-1}(y)\alpha(x) \in \text{Nil}(R) \). It implies \( \alpha(\alpha^{k-2}(y)x) \in \text{Nil}(R) \). By using Remark 2.8, we have \( \alpha^{k-2}(y)x \in \text{Nil}(R) \). Again by using Lemma 2.2, we get \( x\alpha^{k-2}(y) \in \text{Nil}(R) \). It implies \( \alpha(\alpha^{k-3}(y)) \in \text{Nil}(R) \). Since \( R \) is weakly \( \alpha \)-shifting ring, we get \( \alpha^{k-3}(y)\alpha(x) \in \text{Nil}(R) \). It implies \( \alpha(\alpha^{k-4}(y)x) \in \text{Nil}(R) \). Again by using Remark 2.8 and Lemma 2.2, we get \( x\alpha^{k-4}(y) \in \text{Nil}(R) \). Now continuing this procedure, we obtain \( xy \in \text{Nil}(R) \).

**Lemma 2.4.** [8] If \( R \) is semicommutative and \( f(x) = r_0 + r_1x + r_2x^2 + \ldots + r_nx^n \in R[x] \). Then \( f(x) \in \text{Nil}(R[x]) \iff r_0, r_1, \ldots, r_n \in \text{Nil}(R) \).

Let us define \( \bar{\alpha} : R[x] \rightarrow R[x] \) such that \( \bar{\alpha}(r_0 + r_1x + r_2x^2 + \ldots + r_nx^n) = \alpha(r_0) + \alpha(r_1)x + \ldots + \alpha(r_n)x^n \) for all \( r(x) = r_0 + r_1x + r_2x^2 + \ldots + r_nx^n \in R[x] \). Then \( \bar{\alpha} \) is a ring endomorphism of \( R[x] \).
Proposition 2.9. Let \( R \) be semicommutative, then \( R \) is weakly \( \alpha \)-shifting iff \( R[x] \) is weakly \( \bar{\alpha} \)-shifting whereas \( \bar{\alpha}(r_0 + r_1x + r_2x^2 + \ldots + r_nx^n) = \alpha(r_0) + \alpha(r_1)x + \ldots + \alpha(r_n)x^n \).

Proof. Let us consider that \( R \) be weakly \( \alpha \)-shifting ring. Now let \( r(x) = r_0 + r_1x + r_2x^2 + \ldots + r_nx^n \) and \( s(x) = s_0 + s_1x + s_2x^2 + \ldots + s_mx^m \) in \( R[x] \) so that \( r(x)\bar{\alpha}(s(x)) \in Nil(R[x]) \). We know that

\[
(2.1) \quad r(x)\bar{\alpha}(s(x)) = \Sigma_{k=0}^{m+n} (\Sigma_{i+j=k} r_i \alpha(s_j)) x^k
\]

Now by using Lemma 2.23, we have

\[
(2.2) \quad \Sigma_{i+j=k} r_i \alpha(s_j) \in Nil(R)
\]

For \( k = 0 \), (2) implies \( r_0 \alpha(s_0) \in Nil(R) \) and it implies \( \alpha(s_0)r_0 \in Nil(R) \) by using Lemma 2.2.

Now for \( k = 1 \), \( r_0 \alpha(s_1) + r_1 \alpha(s_0) \in Nil(R) \) from Eq(2). Again it implies \( (r_0 \alpha(s_1) + r_1 \alpha(s_0))r_0 \in Nil(R) \) by using Lemma 2.19. By using the same Lemma 2.19, we have \( r_0 \alpha(s_1) \in Nil(R) \). Similarly we can show that \( r_0 \alpha(s_1) + r_1 \alpha(s_0) \) implies \( r_1 \alpha(s_0) \in Nil(R) \).

So \( r_i \alpha(s_j) \in Nil(R) \) for \( k = i + j = 0, 1 \).

Now let us assume that there exists some positive integer \( p > 1 \) such that \( r_i \alpha(s_j) \in Nil(R) \) where \( i + j \leq p \). Therefore \( r_0 \alpha(s_p), r_1 \alpha(s_{p-1}), \ldots, r_p \alpha(s_0) \in Nil(R) \). Then we have

\[
\alpha(s_p)r_0, \alpha(s_{p-1})r_1, \ldots, \alpha(s_0)r_p \in Nil(R) \text{ by using Lemma 2.2.}
\]

Now we will show that \( r_i \alpha(s_j) \in Nil(R) \) for \( i + j = p + 1 \). From Eq. (2) for \( k = p + 1 \), we have

\[
(2.3) \quad r_0 \alpha(s_{p+1}) + r_1 \alpha(s_p) + \ldots + r_{p+1} \alpha(s_0) \in Nil(R)
\]

Now multiplying Eq. (3) by \( r_0 \) from the right hand side, we have

\[
(2.4) \quad (r_0 \alpha(s_{p+1}) + r_1 \alpha(s_p) + \ldots + r_{p+1} \alpha(s_0))r_0 \in Nil(R).
\]
Again by using our assumption that $r_i\alpha(s_j) \in \text{Nil}(R)$ for $i + j \leq p$ and Lemma 2.19, we have $r_0\alpha(s_{p+1})r_0 \in \text{Nil}(R)$ and it leads to $r_0\alpha(s_{p+1}) \in \text{Nil}(R)$.

Again multiplying Eq. (3) by $r_1$ from the right hand side and continuing with the same procedure as above, we can show that $r_1\alpha(s_p) \in \text{Nil}(R)$. Similarly we can get $r_2\alpha(s_{p-1}), \ldots, r_{p+1}\alpha(s_0) \in \text{Nil}(R)$. Thus $r_i\alpha(s_j) \in \text{Nil}(R)$ for $i + j = p + 1$. Now by induction hypothesis we can conclude that $r_i\alpha(s_j) \in \text{Nil}(R)$ for any $k = i + j$ where $k = 0, 1, \ldots, m + n$. Again $r_i\alpha(s_j) \in \text{Nil}(R) \Rightarrow s_j\alpha(r_i) \in \text{Nil}(R)$ as $R$ is weakly $\alpha$-shifting ring.

Therefore it can be easily shown that

$$s(x)\bar{\alpha}(r(x)) = \sum_{k=0}^{m+n} \left( \sum_{i+j=k} r_i \alpha(r_i) \right) x^k \in \text{Nil}(R[x])$$

by using Lemma 2.23 and hence $R[x]$ is weakly $\bar{\alpha}$-shifting. Converse part is trivial.

Let $I$ be an ideal and $\alpha$ be a ring endomorphism of a ring $R$. Then the map $\bar{\alpha}: R/I \rightarrow R/I$ defined by $\bar{\alpha}(x + I) = \alpha(x) + I$ for all $x + I \in R/I$ is a ring endomorphism of quotient ring $R/I$.

**Proposition 2.10.** If $I \subseteq \text{Nil}(R)$. Then $R$ is weakly $\alpha$-shifting $\Leftrightarrow R/I$ is weakly $\bar{\alpha}$-shifting.

**Proof.** Let $R$ be weakly $\alpha$-shifting ring. Now let us consider that $(x + I)\bar{\alpha}(y + I) \in \text{Nil}(R/I)$ for any $x + I, y + I \in R/I$. It implies clearly that $(x\alpha(y) + I)^m = I$ for some positive integer $m$. It implies $(x\alpha(y))^m + I = I$. So we have $(x\alpha(y))^m \in \text{Nil}(R)$ by using the condition that $I \subseteq \text{Nil}(R)$. So there exists some positive integer $k$ such that $(x\alpha(y))^m = 0$. Clearly, $x\alpha(y) \in \text{Nil}(R)$. Since $R$ is weakly $\alpha$-shifting, therefore $x\alpha(y) \in \text{Nil}(R) \Rightarrow y\alpha(x) \in \text{Nil}(R)$. Thus $(y\alpha(x))^n = 0$ for some positive integer $n$. It implies $(y\alpha(x))^n + I = I \Rightarrow ((y + I)\bar{\alpha}(x + I))^n = I \Rightarrow (y + I)\bar{\alpha}(x + I) \in \text{Nil}(R/I)$. Thus $R/I$ is weakly $\bar{\alpha}$-shifting.

Conversely let us consider $R/I$ is weakly $\bar{\alpha}$-shifting ring. Now we have to prove that $R$ is weakly $\alpha$-shifting ring. Let $x\alpha(y) \in \text{Nil}(R)$ for any $x, y \in R$. So we have $(x\alpha(y))^t = 0$ for some $t \in \mathbb{N}$. Then $(x\alpha(y))^t + I = I$. Therefore $((x + I)\bar{\alpha}(y + I))^t = I$. It implies $(x + I)\bar{\alpha}(y + I) \in \text{Nil}(R/I)$. Since $R/I$ is weakly $\bar{\alpha}$-shifting ring, so $(y + I)\bar{\alpha}(x + I) \in \text{Nil}(R/I)$. It implies $(y\alpha(x) + I)^r = I$ for some $r \in \mathbb{N}$. Thus $(y\alpha(x))^r \in I \subseteq \text{Nil}(R)$. Now it leads to $y\alpha(x) \in \text{Nil}(R)$. Therefore $R$ is weakly $\alpha$-shifting.

**Proposition 2.11.** If $R$ is weakly $\alpha$-shifting for a monomorphism $\alpha$, then $\alpha(1) = 1$. 
Proof. Let $R$ be weakly $\alpha$-shifting ring for a monomorphism $\alpha$. Here $\sigma = (1 - \alpha(1))\alpha(1) = 0 \in Nil(R)$ as $\alpha(1)$ is an idempotent element of $R$. Now by using the definition of weakly $\alpha$-shifting of $R$, $\alpha(1 - \alpha(1)) = 1.\alpha(1 - \alpha(1)) \in Nil(R)$. It implies $1 - \alpha(1) \in Nil(R)$ by using Remark 2.7. Therefore $(1 - \alpha(1))^m = 0$ for some integer $m$. It implies $1 - \alpha(1) = 0$ as $1 - \alpha(1)$ is an idempotent element. Thus $\alpha(1) = 1$.

**Proposition 2.12.** Let $\sigma : R \rightarrow S$ be a ring isomorphism. Then $R$ is a weakly $\alpha$-shifting ring if and only if $S$ is weakly $\sigma\alpha\sigma^{-1}$-shifting ring.

**Proof.** Let $R$ be a weakly $\alpha$-shifting ring. Let $\bar{x}, \bar{y} \in S$ so that $\bar{x}(\sigma\alpha\sigma^{-1})(\bar{y}) \in Nil(S)$. Since $\sigma$ is onto, therefore there exist $x$ and $y$ in $R$ such that $\sigma(x) = \bar{x}$ and $\sigma(y) = \bar{y}$. It implies $\sigma(x)(\sigma\alpha\sigma^{-1})(\sigma(y)) \in Nil(S)$. It leads to $\sigma(x\alpha(y)) \in Nil(S)$. Now by using the Remark 2.8, we have $\sigma(y\alpha(x)) \in Nil(S)$. It leads to $\sigma(y)(\sigma\alpha\sigma^{-1})(\sigma(x)) \in Nil(S)$.

Conversely let $S$ be a weakly $\sigma\alpha\sigma^{-1}$-shifting ring. Let $r\alpha(s) \in Nil(R)$ for any $r, s \in R$. Then $\sigma(r\alpha(s)) \in Nil(S)$ by Lemma 2.7. It implies $\sigma(r)(\sigma\alpha\sigma^{-1})(\sigma(s)) \in Nil(S) \Rightarrow \sigma^r(\sigma\alpha\sigma^{-1})(\bar{s}) \in Nil(S)$ where $\sigma(r) = \bar{r}$ and $\sigma(s) = \bar{s}$. Since $S$ is weakly $\sigma\alpha\sigma^{-1}$-shifting, so $\bar{s}(\sigma\alpha\sigma^{-1})(\bar{r}) \in Nil(S)$. It implies $\sigma(s\alpha(r)) \in Nil(S)$. Now using Remark 2.8, we get $s\alpha(r) \in Nil(R)$. Thus $R$ is weakly $\alpha$-shifting.

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.

**References**


