ON THE NONLINEAR CIRCLE PLUS OPERATOR RELATED TO THE LAPLACIAN

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Abstract. In this paper, we study the solution of nonlinear equation

\[ \oplus^k u(x) = f(x, \triangle^{k-1} \square^k L^k u(x)), \]

where the operator \( \oplus^k \) is defined by

\[ \oplus^k = \left[ \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^4 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k, \]

or the operator \( \oplus^k \) can be express by \( \oplus^k = \triangle^k \square^k L^k \). The operator \( \triangle^k \) is Laplacian operator, \( \square^k \) is ultrahyperbolic operator and \( L^k \) is operator defined by

\[ L^k = \left[ \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 + \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k, \]

\( p + q = n \) is the dimension of the \( n \)-dimension Euclidean space \( \mathbb{R}^n \), \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( k \) is a positive integer, \( u(x) \) is an unknown and \( f \) is a given function. It is found that the existence of the solution \( u(x) \) of such equation depending on the condition of \( f \) and \( \triangle^{k-1} \square^k L^k u(x) \) and moreover such solution \( u(x) \) related to the Laplacian depending on the conditions of \( p, q \) and \( k \).

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1. Introduction

The operator $\oplus^k$ has been studied first by Kananthai, Suantai and Longani [5] and is defined by

$$\oplus^k = \left[ \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \times \left[ \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \times \left[ \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k,$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$, $i = \sqrt{-1}$ and $k$ is a nonnegative integer. The diamond operator is denoted by

$$\Diamond^k = \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2.$$

The operator $L_1$ and $L_2$ are defined by

$$L_1 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$$

and

$$L_2 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}.$$

Thus equation (1) can be written as

$$\oplus^k = \Diamond^k L_1^k L_2^k.$$

Otherwise, the operator $\Diamond$ can also be expressed in the form $\Diamond = \Box \Delta = \Delta \Box$, where $\Box$ is the ultra-hyperbolic operator defined by

$$\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2},$$

where $p + q = n$ and $\Delta$ is the Laplacian defined by

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$
The linear equation $\hat{\triangle}^k u(x) = f(x)$, see [6], has been already studied and the convolution $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ has been obtained as a solution of such an equation where $K_{2k,2k}(x) = R_{2k}^H(x) * R_{2k}^e(x)$. The function $R_{2k}^H(x)$ and $R_{2k}^e(x)$ are defined by (9) and (11), respectively, with $\alpha = \beta = 2k$.

Kananthai, Suantai and Longani, see[4], has been studied the operator $\oplus^k$. They obtained

$$K(x) = [R_{2k}^H(u) * (-1)^k R_{2k}^e] * (-1)^k (-i)^{q/2} S_{2k}(w) * (-1)^k (i)^{q/2} T_{2k}(z)$$

is the elementary solution of such operator.

In this work, we study the nonlinear equation of the form

$$(7) \quad \oplus^k u(x) = f(x, \Delta^{k-1} \square^k L^k u(x)).$$

with $f$ defined and continuous for all $x \in \Omega \cup \partial \Omega$ where $\Omega$ is an open subset of $\mathbb{R}^n$ and $\partial \Omega$ denotes the boundary of $\Omega$. We can find the solution $u(x)$ of (7) which is unique under the condition $|f(x, \Delta^{k-1} \square^k L^k u(x))| \leq N$ where $N$ is a constant for all $x \in \Omega$ and the boundary condition $\Delta^{k-1} \square^k L^k u(x) = 0$ for $x \in \partial \Omega$.

2. Preliminaries

**Definition 2.1.** Let $x = (x_1, x_2, ..., x_n)$ be a point in the space $\mathbb{R}^n$ of the $n$-dimensional Euclidean space and write

$$(8) \quad v = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2,$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$.

Denote by $\Gamma_+ = \{ x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0 \}$ the set of an interior of the forward cone and $\overline{\Gamma}_+$ denotes it closure and $\mathbb{R}^n$ is the $n$-dimensional Euclidean space.

For any complex number $\alpha$, define

$$(9) \quad R_{\alpha}^H(v) = \begin{cases} \frac{\alpha - n}{\bar{K}_n(\alpha)}, & \text{for } x \in \Gamma_+ \\ 0, & \text{for } x \notin \Gamma_+ \end{cases}$$
where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}.$$  

The function $R^H_\alpha(v)$ is called the hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [4, p72]. It is well known that $R^H_\alpha(v)$ is an ordinary function if $\text{Re}(\alpha) \geq n$ and is a distribution of $\alpha$ if $\text{Re}(\alpha) < n$. Let $\text{supp } R^H_\alpha(v)$ denote the support of $R^H_\alpha(v)$ and suppose $\text{supp } R^H_\alpha(v) \subset \bar{\Gamma}_+$, that is $\text{supp } R^H_\alpha(v)$ is compact.

**Definition 2.2.** Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and write

$$|x| = x_1^2 + x_2^2 + ... + x_n^2.$$  

For any complex number $\beta$, define

$$R^e_\beta(x) = 2^{-\beta} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\beta}{2}\right) \frac{|x|^{\frac{\beta-n}{2}}}{\Gamma\left(\frac{\beta}{2}\right)}.$$  

The function $R^e_\beta(x)$ is called the elliptic kernel of Marcel Riesz and is ordinary function for $\text{Re}(\beta) \geq n$ and is a distribution of $\beta$ for $\text{Re}(\beta) < n$.

**Definition 2.3.** Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and write

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right)$$  

and

$$w = x_1^2 + x_2^2 + ... + x_p^2 - i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right),$$  

For any complex number $\gamma$ and $\nu$, we define

$$T_\nu(z) = 2^{-\nu} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\nu}{2}\right) z^{\frac{\nu-n}{2}} \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$$  

and

$$S_\gamma(w) = 2^{-\gamma} \pi^{-\frac{n}{2}} \Gamma\left(\frac{n-\gamma}{2}\right) w^{\frac{\gamma-n}{2}} \frac{\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}.$$  

The function $S_\gamma(w)$ and $T_\nu(z)$ is an ordinary function if $\text{Re}(\gamma) \geq n$ and $\text{Re}(\nu) \geq n$, is a distribution of $\gamma$ for $\text{Re}(\gamma) < n$ and $\nu$ for $\text{Re}(\nu) < n$.

**Lemma 2.1.** Given the equation

$$\Delta^k u(x) = 0,$$  

where the constant $K_n(\alpha)$ is given by the formula
where \( \triangle^k \) is the Laplacian operator iterated \( k \)-times defined by equation (6) we obtain
\[ u(x) = ((-1)^{k-1} R^e_{2(k-1)}(x))^{(m)} \]
as a solutions of (16) where \( m = (n - 4)/2, n \geq 4 \) is non-negative integer and \( n \) is even and \( R^e_{2(k-1)}(x) \) defined by equation (11) with \( m \) derivatives and \( \beta = 2(k - 1) \).

**Proof.** see [6, Lemma 2.2].

**Lemma 2.2.** Given the equation

\[ (17) \]
\[ \Box^k u(x) = 0, \]

where \( \Box^k \) is the Ultra-hyperbolic operator iterated \( k \)-times defined by equation (5) we obtain
\[ u(x) = (R^H_{2(k-1)}(v))^{(m)} \]
as a solutions of (17) where \( m = (n - 4)/2, n \geq 4 \) is non-negative integer and \( n \) is even and \( R^H_{2(k-1)}(v) \) defined by equation (9) with \( m \) derivatives and \( \alpha = 2(k - 1) \).

**Proof.** see [6, Lemma 2.3].

**Lemma 2.3.** The function \( T_{2k}(z) * S_{2k}(w) \) is an elementary solutions of the operator
\[ L^k = L^k_1 L^k_2 = L^k_2 L^k_1, \]
denoted by

\[ (18) \]
\[ L^k = \left[ \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k, \]

where \( T_{2k}(z) \) and \( S_{2k}(w) \) are defined by equation (14) and (15), respectively, with \( \gamma = \nu = 2k \). The operator \( L^k_1 \) and \( L^k_2 \) are defined by equation (3) and (4), respectively.

**Proof.** We need to show that
\[ L^k_1[(-1)^k(i)^{\frac{3}{2}} T_{2k}(z)] = \delta \]
and
\[ L^k_2[(-1)^k(-i)^{\frac{3}{2}} S_{2k}(w)] = \delta. \]
At first we have to show that

\[ (19) \]
\[ L^k_1 T_{\nu}(z) = (-1)^k T_{\nu-2k}(z), \quad L^k_2 S_{\gamma}(w) = (-1)^k S_{\gamma-2k}(w) \]

and also

\[ (20) \]
\[ T_{-2k}(z) = (-1)^k(-i)^{\frac{3}{2}} L^k_1 \delta, \quad S_{-2k}(w) = (-1)^k(i)^{\frac{3}{2}} L^k_2 \delta. \]
Now for $k = 1$,

$$L_1 T_\nu(z) = \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) T_\nu(z)$$

$$= 2^{-\nu} \pi^{-n} \frac{\Gamma\left(\frac{n-\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} (\nu - n)(\nu - 2) z^{\nu - n - 2}$$

$$= (-1)^{\nu - 2} \pi^{-n} \frac{\Gamma\left(\frac{n-\nu-2}{2}\right)}{\Gamma\left(\frac{\nu-2}{2}\right)} z^{\nu - n - 2}$$

$$= -T_{\nu-2}(z).$$

By repeating $k$-times in operating $L_1$ to $T_\nu(z)$, we obtain $L_1^k T_\nu(z) = (-1)^k T_{\nu-2k}(z)$.

Similarly, $L_2^k S_\gamma(w) = (-1)^k S_{\gamma-2k}(w)$.

Now consider

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i \left( x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2 \right), p + q = n$$

by changing the variable

$$x_1 = y_1, x_2 = y_2, ..., x_p = y_p,$$

$$x_{p+1} = \frac{y_{p+1}}{\sqrt{1}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{1}}, ..., x_{p+q} = \frac{y_{p+q}}{\sqrt{1}}.$$

Thus we have $z = y_1^2 + y_2^2 + ... + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + ... + y_{p+q}^2$.

Denote $z = r^2 = y_1^2 + y_2^2 + ... + y_n^2$ and consider the generalized function $z^\lambda = r^{2\lambda}$ where $\lambda$ is any complex number. Now $\langle z^\lambda, \varphi \rangle = \int_{R^n} z^\lambda \varphi(x) dx$, where $\varphi \in \mathcal{D}$ the space of infinitely differentiable functions with compact supports. Thus

$$\langle z^\lambda, \varphi \rangle = \int_{R^n} r^{2\lambda} \varphi \frac{\partial(x_1, x_2, ..., x_n)}{\partial(y_1, y_2, ..., y_n)} dy_1 dy_2 \cdots dy_n$$

$$= \frac{1}{(i)^{q/2}} \int_{R^n} r^{2\lambda} \varphi dy$$

$$= \frac{1}{(i)^{q/2}} \langle r^{2\lambda}, \varphi \rangle.$$

By Gelfand and Shilov [3, p.271], the function $r^{2\lambda}$ have simple poles at $\lambda = (-n/2) - k, k$ is nonnegative and for $k = 0$ we can find the residue of $r^{2\lambda}$ at $\lambda = -n/2$ and by [3, p.73], we obtain

$$\text{res}_{\lambda=-\frac{n}{2}} (r^{2\lambda}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \delta(x).$$
Thus

(21) \[ \text{res}_{\lambda=-n/2} (z^\lambda) = (-i)^{\frac{n}{2}} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \delta(x). \]

We next find the residues of \( z^\lambda \) at \( \lambda = (-n/2) - k \). Now, by computing directly we have

\[
L_1 z^{\lambda+1} = 2(\lambda + 1)(2\lambda + n) z^\lambda.
\]

By \( k \)-fold iteration, we obtain

\[
L_k z^{\lambda+k} = 4^k (\lambda + 1)(\lambda + 2) \cdots (\lambda + k) (\lambda + \frac{n}{2}) \times
\]

\[
(\lambda + \frac{n}{2} + 1) \cdots (\lambda + \frac{n}{2} + k - 1) z^\lambda
\]

or

\[
z^\lambda = \frac{1}{4^k(\lambda + 1)(\lambda + 2) \cdots (\lambda + k)} \times
\]

\[
\frac{1}{(\lambda + \frac{n}{2})(\lambda + \frac{n}{2} + 1) \cdots (\lambda + \frac{n}{2} + k - 1)} L_k z^{\lambda+k}.
\]

Thus

\[
\text{res}_{\lambda=-n/2-k} (z^\lambda) = \frac{1}{4^k k (\frac{n}{2} + k - 1) (\frac{n}{2} + k - 2) \cdots (\frac{n}{2})} \text{res}_{\lambda=-n/2} L_k z^{\lambda+k}.
\]

By (21) and the properties of Gamma functions, we obtain

(22) \[ \text{res}_{\lambda=-n/2-k} (z^\lambda) = (-i)^{\frac{n}{2}} \frac{2\pi^{\frac{n}{2}}}{4^k \Gamma(\frac{n}{2} + k)} L_k \delta(x). \]

Now we consider \( T_{-2k}(z) \) we have

\[
T_{-2k}(z) = \lim_{\nu \to -2k} T(z)
\]

\[
= \pi^{-\frac{n}{2}} \lim_{\nu \to -2k} \frac{\Gamma(\nu + n/2)}{\Gamma(\nu/2)} \lim_{\nu \to -2k} 2^{-\nu} \Gamma \left( \frac{n - \nu}{2} \right)
\]

\[
= \pi^{-\frac{n}{2}} \lim_{\nu \to -2k} \frac{\Gamma(\nu + 2k)(\nu/2)}{\Gamma(\nu + 2k)(\nu/2)} \lim_{\nu \to -2k} 2^{2k} \Gamma \left( \frac{n + 2k}{2} \right)
\]

\[
= 4^k \pi^{-\frac{n}{2}} \lim_{\nu \to -2k} \text{res}_{\nu=-n/2} z^{(\nu-n)/2} \Gamma \left( \frac{n + 2k}{2} \right),
\]
Since \( \text{res}_{\lambda=-\frac{k}{2}} z^\lambda = \text{res}_{\nu=-2k} z^{(\nu-n)/2} \) and \( \text{res}_{\nu=-2k} \Gamma\left(\frac{\nu}{2}\right) = \frac{2(-1)^k}{k!} \), by (22) and the properties of Gamma function we obtain

\[
T_{-2k}(z) = (-1)^k (-i)^{\frac{\nu}{2}} L_1^k \delta(x).
\]

Similarly

\[
S_{-2k}(w) = (-1)^k (i)^{\frac{\nu}{2}} L_2^k \delta(x).
\]

Thus we have

\[
T_0(z) = (-i)^{\frac{\nu}{2}} \delta(x), \quad S_0(w) = (i)^{\frac{\nu}{2}} \delta(x).
\]

Now, from (19) \( L_1^k T_{2k}(z) = (-1)^k T_0(z) \) for \( \nu = 2k \). Thus by (23) we obtain \( L_1^k (-1)^k (i)^{\frac{\nu}{2}} T_{2k}(z) = \delta(x) \). It follows that \( (-1)^k (i)^{\frac{\nu}{2}} T_{2k}(z) \) is an elementary solution of the operator \( L_1^k \). Similarly \( (-1)^k (-i)^{\frac{\nu}{2}} S_{2k}(w) \) is also an elementary solution of the operator \( L_2^k \). Thus we have

\[
L^k (T_{2k}(z) * S_{2k}(w)) = L_2^k (-1)^k (i)^{\frac{\nu}{2}} T_{2k}(z) * L_1^k (-1)^k (-i)^{\frac{\nu}{2}} S_{2k}(w) = \delta.
\]

**Lemma 2.4.** Given the equation

\[
\Delta u(x) = f(x, u(x)),
\]

where \( f \) is defined and has continuous first derivatives for all \( x \in \Omega \cup \partial \Omega \), \( \Omega \) is an open subset of \( \mathbb{R}^n \) and \( \partial \Omega \) denotes the boundary of \( \Omega \). Assume \( f \) is a bounded, that is \( |f(x, u)| \leq N \) and the boundary condition \( u(x) = 0 \) for \( x \in \partial \Omega \). Then we obtain \( u(x) \) as a unique solution of (24).

**Proof.** We can prove this lemma by the method of iterations and the Schauder’s estimates, see [1, pp. 369-372].

**3. Main results**

**Theorem 3.1.** *Given the nonlinear equation*

\[
\oplus^k u(x) = f(x, \Delta^{k-1} \square^k L^k u(x)),
\]
where $\oplus^k$ is the operator iterated $k$ times, defined by (1), $\Delta^{k-1}$ is the Laplacian iterated $k-1$ times defined by (6) and $\Box^k$ is the ultrahyperbolic operator iterated $k$ times defined by (5). Let $f$ be defined and have continuous first derivatives for all $x \in \Omega \cup \partial \Omega$, $\Omega$ is an open subset of $\mathbb{R}^n$ and $\partial \Omega$ denotes the boundary of $\Omega$ and $n$ is even with $n \geq 4$. Let $f$ be a bounded function, that is

$$|f(x, \Delta^{k-1} \Box^k L^k u(x))| \leq N$$

and the boundary condition

$$\Delta^{k-1} \Box^k L^k u(x) = 0, \text{ for } x \in \partial \Omega,$$

then we obtain

$$u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * R_{2k}^H(v) * S_{2k}(w) * T_{2k}(z) * W(x)$$

as a solution of (25) with the boundary condition

$$u(x) = S_{2k}(w) * T_{2k}(z) * R_{2k}^H(v) * (-1)^{k-2} (R_{2(k-2)}^e(x))^{(m)}$$

for $x \in \partial \Omega, m = (n-4)/2$, $k = 2, 3, 4, \ldots$ and $v$ is given by (8), $W(x)$ is a continuous function for $x \in \Omega \cup \partial \Omega$, $R_{2(k-2)}^e(x)$ and $R_{2k}^H(v)$ are given by (11) and (9), respectively, with $\beta = 2(k-2)$ and $\alpha = 2k$. Moreover, for $q = 0$ then (25) becomes

$$\Delta_p^{4k} u(x) = f(x, \Delta^{4k-1} u(x)),$$

with boundary condition

$$\Delta^{4k-1} u(x) = 0, \text{ for } x \in \partial \Omega,$$

where $\Delta_p^{4k}$ is the Laplacian of $p$-dimension iterated $4k$-times. we have

$$u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * R_{6k}^e(x) * W(x)$$

as a solution of (29) where $|x| = x_1^2 + x_2^2 + \ldots + x_p^2$.

**Proof.** From equation (25), we have

$$\oplus^k u(x) = \Delta(\Delta^{k-1} \Box^k L^k u(x)) = f(x, \Delta^{k-1} \Box^k L^k u(x)).$$
Since \( u(x) \) has continuous derivatives up to order \( 4k \) for \( k = 1,2,3,\ldots \) we can assume

\[
\triangle^{k-1} \Box^k L^k u(x) = W(x), \text{ for } x \in \partial \Omega.
\]

Thus, (32) can be written in the form

\[
\oplus^k u(x) = \Delta W(x) = f(x,W(x)),
\]

by (26)

\[
|f(x,W(x))| \leq N,
\]

and by (27), \( W(x) = 0 \) or

\[
\triangle^{k-1} \Box^k L^k u(x) = 0, \text{ for } x \in \partial \Omega.
\]

Thus by Lemma 2.4 there exist a unique solution \( W(x) \) of (34) which satisfies (35). Now consider (33), we have \( \triangle^{k-1}(-1)^{k-1} R^e_{2(k-1)}(x) = \delta \) and \( \Box^k R^H_{2k}(v) = \delta \) where \( \delta \) is the Dirac-delta distribution, that is \( R^H_{2k}(v) \) and \( (-1)^{k-1} R^e_{2(k-1)}(x) \) are the elementary solutions of the operators \( \Box^k \) and \( \triangle^{k-1} \), respectively, see\[8, \text{p.11}\] and see\[2, \text{p.118}\]. The functions \( R^H_{2k}(v) \) and \( R^e_{2(k-1)}(x) \) are defined by (9) and (11), respectively, with \( \beta = 2(k-1), \alpha = 2k \). And by Lemma 2.3, the function \( T_{2k}(z) * S_{2k}(w) \) is an elementary solutions of the operator \( L^k \), are defined by equation (14) and (15), respectively, with \( \gamma = \nu = 2k \). Thus, convolving both sides of (33) by

\[
(-1)^{k-1} R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w),
\]

we obtain

\[
[(-1)^{k-1} R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w)] * \triangle^{k-1} \Box^k L^k u(x)
\]

\[
= [(-1)^{k-1} R^e_{2(k-1)}(x) * R^H_{2k}(v) * T_{2k}(z) * S_{2k}(w)] * W(x).
\]
By properties of convolution, we obtain

$$\Delta^{k-1} (1)^{k-1} R_{2(k-1)}^e (x) \ast [\square^k R_{2k}^H (v)] \ast [L^k T_{2k} (z) \ast S_{2k} (w)] \ast u(x) =$$

$$\delta \ast \delta \ast u(x) =$$

$$[(1)^{k-1} R_{2(k-1)}^e (x) \ast R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w)] \ast W(x).$$

Thus

$$u(x) = (1)^{k-1} R_{2(k-1)}^e (x) \ast R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w) \ast W(x)$$

as required. Consider $\Delta^{k-1} \square^k L^k u(x) = 0$, for $x \in \partial \Omega$. By Lemma 2.1, we have

$$\square^k L^k u(x) = (1)^{k-2} (R_{2(k-2)}^e (x))^{(m)}.$$

Convolving both sides of the above equation by $R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w)$, we obtain

$$R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w) \ast \square^k L^k u(x)$$

$$= R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w) \ast (1)^{k-2} (R_{2(k-2)}^e (x))^{(m)},$$

$$\square^k R_{2k}^H (v) \ast [L^k \ast T_{2k} (z) \ast S_{2k} (w)] \ast u(x)$$

$$= R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w) \ast (1)^{k-2} (R_{2(k-2)}^e (x))^{(m)},$$

$$\delta \ast \delta \ast u(x)$$

$$= R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w) \ast (1)^{k-2} (R_{2(k-2)}^e (x))^{(m)},$$

$$u(x) = R_{2k}^H (v) \ast T_{2k} (z) \ast S_{2k} (w) \ast (1)^{k-2} (R_{2(k-2)}^e (x))^{(m)},$$

for $x \in \partial \Omega$ and $k = 2, 3, 4, \ldots$

Moreover, for $q = 0$ then (25) becomes

$$\Delta^4 u(x) = f(x, \Delta^4 u(x)),$$

with boundary condition

$$\Delta^{4k-1} u(x) = 0, \ for \ x \in \partial \Omega.$$
where \( \Delta^4_k \) is the Laplacian of \( p \)-dimension iterated \( 4k \)-times. we have

\[
(39) \\
\quad u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * R_{6k}^e(x) * W(x)
\]

as a solution of (38) where \( |x| = x_1^2 + x_2^2 + \ldots + x_p^2 \).

On the other hand, we can also find (39) from (37), since \( q = 0 \), we have \( R^H_{2k}(v) \) reduces to \( R_{2(k)}^e(x) \), \( S_{2k}(w) \) reduces to \( R_{2(k)}^e(x) \) and \( T_{2k}(z) \) reduces to \( R_{2(k)}^e(x) \), where \( |x| = x_1^2 + x_2^2 + \ldots + x_p^2 \).

Thus, by (37) for \( q = 0 \), we obtain

\[
(37) \\
\quad u(x) = (-1)^{k-1} R_{2(k-1)}^e(x) * R_{2k}^e(x) * R_{2k}^e(x) * R_{6k}^e(x) * W(x) \\
\quad = (-1)^{k-1} R_{2(k-1)}^e(x) * R_{6k+2k+2k}^e(x) * W(x) \\
\quad = (-1)^{k-1} R_{2(k-1)}^e(x) * R_{6k}^e(x) * W(x).
\]

This completes the proof. \( \square \)

REFERENCES


