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ON CNZ RING PROPERTY VIA IDEMPOTENT ELEMENTS

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Abstract. In this paper, the concept of $e-$CNZ rings is introduced as a generalization of symmetric rings and a particular case of $e-$reversible rings. Regarding the question of how idempotent elements affect CNZ property of rings. In this note, we show that $e-$CNZ is not left-right symmetric. We present examples of right $e-$CNZ rings that are not CNZ and basic properties of right $e-$CNZ are provided. Some subrings of matrix rings and some extensions of rings such as Jordan extension are investigated in terms of right $e-$CNZ.

Keywords: CNZ ring; $e-$reversible ring; $e-$symmetric ring; $e-$reduced ring; idempotent element.

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1. Introduction

Throughout this paper, all rings are noncommutative with identity and associative unless otherwise stated. For a ring R, let $Id(R)$, $Z(R)$ and $N(R)$ denote the set of all idempotents, the center and the set of all nilpotents of R, respectively. We denote the ring of integers (resp., modulo n) by Z(resp., Z_n). We write $Mat_n(R)$ (resp., $U_n(R)$) for the ring of all n by n full matrix (resp., upper triangular matrix) over R, and $D_n(R)$stands for the subring of $U_n(R)$ having main diagonal entries equal. $R[x]$, $R[x;x^{-1}]$, $R[[x]]$ and $R[[x;x^{-1}]]$ denote the polynomial ring,

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the Laurent polynomial ring, the power series ring and the Laurent power series ring over R, respectively. We use E_{ij} for the matrix with $(i,j)-$entry 1 and elsewhere 0.

A ring is called reduced if it has no nonzero nilpotent elements. A ring R is called CNZ [1] if $ab=0$ implies $ba=0$ for $a,b\in N(R)$. The ring R with $e\in Id(R)$ is called left $e-$reversible (resp., right $e-$reversible) if $ab=0$ implies $eba=0$ (resp., $bae=0$) for any $a,b\in R$, and R is $e-$reversible [13] if it is both left and right $e-$reversible. The ring R is called left (resp., right) $e-$reduced [16] if $eN(R)=0$ (resp., $N(R)e=0$), and also R is called $e-$symmetric [16] if $abc=0$ for all $a,b,c\in R$ implies $acbe=0$. The ring R is von Neumann regular [6] if for each $a\in R$, there is a $b\in R$ such that $a=aba$. An element r of a ring R is central if $ar=ra$ for all $a\in R$, and R is said to be abelian [2] if every idempotent is central. Also an idempotent e of R is called right (resp., left) semicentral [4] if for each $a\in R$, $ea=eae$ (resp., $ae=eae$). R is called left (resp. right) $e-$semicommutative [14] if $ab=0$ implies $eaRb=0$ (resp. $aRbe=0$) for any $a,b\in R$, and R is $e-$semicommutative if it is both left and right $e-$semicommutative. The ring R is reflexive [10] ring if $aRb=0$ then $bRa=0$ for any $a,b\in R$.

2. Properties of Right $e-$CNZ Rings

In this section we deal with the basic properties of right $e-$CNZ rings. Being with the following definition.

Definition 2.1. Let R be a ring and $e\in Id(R)$ with $e\neq 0$. Then R is called right $e-$CNZ (resp., left $e-$CNZ) if for any $a,b\in N(R)$, $ab=0$ implies $bae=0$ (resp., $eba=0$). The ring R is $e-$CNZ if it is both left and right $e-$CNZ.

It is obvious that a ring R is CNZ if and only if R is $1-$CNZ. The following example shows that $e-$CNZ property is not left-right symmetric. Also, the CNZ property of a ring with respect to an idempotent e depends on e. There are rings R and idempotents e_1 and e_2 such that R is right e_1-CNZ but not right e_2-CNZ as the following example shows.

Example 2.2. *Assume the ring* $R=U_2(Z_4)$ *with* $E_1=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $E_2=\begin{pmatrix}1&1\\0&0\end{pmatrix}\in Id(R)$, *and* $N(R)=\left\{\begin{pmatrix}a&b\\0&c\end{pmatrix}: a,c\in\{0,2\},\ b\in Z_4\right\}$ *.Then the followings hold.*

(1) *R is not CNZ.*

(2) *R is left $E_1 - CNZ$ but not right $E_1 - CNZ$.*

(3) *R is right $E_2 - CNZ$ but not left $E_2 - CNZ$*

Solution. (1) *R is not CNZ by* [1, Example 2.8].

Let $A, B \in N(R)$ with $AB = 0$. Then BA is of the form $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ *where $x \in Z_4$. Assume that $x \neq 0$.*

(2) *R is left $E_1 - CNZ$ but R is not right $E_1 - CNZ$, for* $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ *in $N(R)$ we have* $BAE_1 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$.

(3) *$BAE_2 = 0$ implies that R is right $E_2 - CNZ$ but R is not left $E_2 - CNZ$ since $E_2BA \neq 0$.*

In the next result, we give a characterization of right $e-$CNZ property in terms of subsets of rings and its proof is straightforward.

Proposition 2.3. *The following are equivalent for a ring R.*

(1) *R is right $e - CNZ$.*

(2) *For any nonempty subsets A, B of $N(R)$, being $AB = 0$ implies $BAe = 0$.*

Proof. $(1) \Rightarrow (2)$: Assume that (1) holds. Let $AB = 0$ for two subsets A, B of $N(R)$. Then $ab = 0$ for all $a \in A$ and $b \in B$, and so $bae = 0$ by (1). Therefore $BAe = \sum_{a \in A, b \in B} bae = 0$.

$(2) \Rightarrow (1)$: is straightforward. □

Proposition 2.4.

(1) *Every CNZ ring is $e - CNZ$ ring.*

(2) *Every $e -$ reversible ring is $e - CNZ$.*

(3) *Every right $e -$ reduced ring is right $e - CNZ$.*

(4) *Every $e -$ symmetric ring is right $e - CNZ$.*

Proof. (1) Let R be a CNZ ring with $e^2 = e \in R$. Suppose that $ab = 0$ for $a, b \in N(R)$. Then $ba = 0$ and so $bae = 0$ and $eba = 0$.

(2) Let R be an $e-$reversible ring with $e^2 = e \in R$. Suppose that $ab = 0$ for $a, b \in N(R)$. Since $N(R) \subseteq R$ and R is $e-$reversible then $bae = 0$ and $eba = 0$.

(3) Since every right $e-$reduced ring is right $e-$reversible by [13, Example 2.5(2)], and every right $e-$reversible is right $e-$CNZ by (2) above. Then every right $e-$reduced is right $e-$CNZ.

(4) Assume that R is an $e-$symmetric ring. Let $a,b \in N(R)$ with $ab=0$. R is with identity $1ab=0$. By assumption $1bae=0$ implies $bae=0$. Therefore R is right $e-$CNZ. □

There are right $e-$CNZ rings R for some $e \in Id(R)$ but not CNZ. This yields that the converse of Proposition 2.4.(1) need not be true in general, as shown in Example 2.2.

Also the converse of Proposition 2.4.(2) is not true by the following example.

Example 2.5. *Let A be a reduced ring, consider the ring $R=U_2(A)$. Then R is CNZ by* [1, Theorem 2.7]. *Therefore R is $e-$CNZ by Proposition 2.4.(1). Now for $a=\begin{pmatrix}0&1\\0&1\end{pmatrix}$, $b=\begin{pmatrix}1&0\\0&0\end{pmatrix}\in U_2(A)$, and let $e=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ we have $ab=0$ but $bae=\begin{pmatrix}0&1\\0&0\end{pmatrix}\neq 0$. Therefore R is not $e-$reversible.*

The following properties of right $e-$CNZ rings do an essential roles throughout this paper.

Proposition 2.6.

(1) *The class of right $e-$CNZ ring is closed under subrings.*

(2) *For a family $\{R_\lambda : \lambda \in \Delta\}$ of rings, the following statements are equivalent:*

(i) *R_λ is right $e_\lambda-$CNZ;*

(ii) *The direct product $\prod_{\lambda\in\Delta} R_\lambda$ of R_λ is right $e-$CNZ;*

(iii) *The direct sum $\bigoplus_{\lambda\in\Delta} R_\lambda$ of R_λ is right $e-$CNZ.*

(3) *Let R be an abelian ring and $e \in Id(R)$. Then R is right $e-$CNZ if and only if both eR and $(1-e)R$ are right $e-$CNZ.*

Proof. (1) Let S be a subring of a ring R. Note that $Id(S)=Id(R)\cap S$ for any subring S of a ring R. Now let $s_1,s_2 \in N(S)$ with $s_1s_2=0$. Since $N(S)\subseteq N(R)$ and by assumption $s_2s_1e=0$. Thus S is right $e-$CNZ.

(2) $(i)\Rightarrow(ii)$, let $a=(a_\lambda),\ b=(b_\lambda)\in N(\prod_{\lambda\in\Delta} R_\lambda)\subseteq \prod_{\lambda\in\Delta}(N(R_\lambda))$ with $ab=0$ for each $\lambda\in\Delta$. By assumption $b_\lambda a_\lambda e_\lambda=0$ for each $\lambda\in\Delta$. Then $bae=0$.

$(ii) \Rightarrow (i)$, $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (i)$. The proof comes from (1) and the fact that $Id(\underset{\lambda\in\Delta}{\oplus} R_\lambda) = \underset{\lambda\in\Delta}{\oplus} Id(R_\lambda)$.

(3) This follows (2). Since $R \cong eR \oplus (1-e)R$. □

Lemma 2.7. *The following are equivalent for a ring R and $e \in Id(R)$:*

(i) *R is a CNZ ring.*

(ii) *R is both an $e-$CNZ and $(1-e)-$CNZ ring.*

Proof. $(i) \Rightarrow (ii)$: clear.

$(ii) \Rightarrow (i)$: Let $a, b \in N(R)$ such that $ab = 0$. Since R is $(1-e)-$CNZ ring, then $ba(1-e) = 0$. We get that $ba = 0$ as R is an $e-$CNZ ring. Hence R is a CNZ ring. □

Theorem 2.8. *Let R be a ring and $e \in Id(R)$ is left semicentral. Then R is a right $e-$CNZ ring if and only if eRe is an CNZ ring.*

Proof. Let R be a right $e-$CNZ ring with $(exe)(eye) = 0$, for $x, y \in N(R)$. By assumption $(eye)(exe) = 0$. Thus eRe is CNZ.

Conversely, Suppose that eRe is CNZ and $ab = 0$ for $a, b \in N(R)$. Since e is left semicentral and $abe = 0$ this implies $(eae)(ebe) = 0$, and hence $(ebe)(eae) = 0$ by assumption. We get $bae = 0$, since e is left semicentral. Thus R is right $e-$CNZ. □

Similar to Theorem 2.8., we have the following result.

Proposition 2.9. *Let R be a ring and $e \in Id(R)$ is right semicentral in R. Then R is left $e-$CNZ if and only if eRe is CNZ.*

Proposition 2.10. *Let e be a left semicentral then the following are equivalent for a ring R.*

(1) *R is right $e-$CNZ.*

(2) *For any $a, b \in N(R)$, if $ab \in Id(R)$, then $bae \in Id(R)$.*

Proof. $(1) \Rightarrow (2)$: Let R be a right $e-$CNZ ring and $a, b \in N(R)$ with $ab \in Id(R)$. Being $ab \in Id(R)$ implies $a(1-ba)b = 0$. Then $(1-ba)bae = 0$ by (1). Since e is left semicentral, $bae = baebae$. So $bae \in Id(R)$.

$(2) \Rightarrow (1)$: Assume that for any $a,b \in N(R)$ being $ab \in Id(R)$ implies $bae \in Id(R)$. Let $a,b \in N(R)$ with $ab = 0$. Then $ab \in Id(R)$ entailing $bae \in Id(R)$. Hence $bae = baebae = babae = 0$ by the facts that $ab = 0$ and e is left semicentral. Thus $bae = 0$. So R is right $e-$CNZ. □

It is not necessary for the ring R to be right $e-$CNZ even if the ring R/I is right $\bar{e}-$CNZ. We illustrate this by the following example.

Example 2.11. *Consider the ring $R = U_3(Z)$ and its idempotent $E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. R is not right $E-CNZ$. For $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in $N(U_3(Z))$ we have $AB = 0$ but $BAE = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. Now, for a nonzero proper ideal $I = \begin{pmatrix} 0 & 0 & Z \\ 0 & 0 & Z \\ 0 & 0 & Z \end{pmatrix}$ of R. $R/I \cong U_2(Z)$ is $e-CNZ$ by* [1, Theorem 2.7] *and Proposition 2.4.(1).*

Proposition 2.12. *Let R be a ring with an ideal I and $e \in Id(R)$ with e is central. If R/I is a right $\bar{e}-CNZ$ ring and Iis reduced as a ring without identity, then R is right $e-CNZ$.*

Proof. Let $a,b \in N(R)$ with $ab = 0$. Then $\bar{a},\bar{b} \in N(R/I)$ and $\bar{a}\bar{b} = 0$. Since R/I is right $\bar{e}-$CNZ, $bae \in I$ by assumption. Then $(bae)^2 = baebae = beabae = 0$. Since I is reduced implies $bae = 0$. Therefore R is right $e-$CNZ. □

In the next proposition, we show that R being right $e-$CNZ not necessary imply R/I being right $\bar{e}-$CNZ.

Proposition 2.13. *Let R be an $e-symmetric$ ring and I an ideal of R with $I = r_R(S)$ for some subset S of $N(R)$. Then R/I is right $\bar{e}-CNZ$.*

Proof. Let $\bar{a},\bar{b} \in N(R/I)$ such that $\bar{a}\bar{b} = 0$. It follows that $Sab = 0$. The $e-$symmetricity of R implies that $Sbae = 0$. Therefore $bae \in I$. This leads to $\bar{b}\bar{a}\,\bar{e} = 0$. Thus R/I is right $\bar{e}-$CNZ. □

Proposition 2.14. *Every right $e-semicommutative$ reflexive ring is right $e-CNZ$.*

Proof. Let R be a right $e-$semicommutative reflexive ring and $a,b \in N(R)$ such that $ab=0$ then $aRbe=0$ since R is right $e-$semicommutative and reflexivity yields $beRa=0$. Since $bea=0$, we have $bRea=0$, and so $bReae=0$. By [14, Theorem 2.4(1)], e is left semicentral. This leads $bRae=0$. Thus $bae=0$. □

Proposition 2.15. *Let R be a Von Neumann regular abelian ring. Then R is a right $e-$CNZ ring.*

Proof. Let $a,b \in N(R)$ with $ab=0$. There exists an $c \in R$ such that $a=aca$. Multiply this equality by b from left, we get $ba=b(aca)$. Also, ca is idempotent. Since R is abelian, we have $bae=(ca)\ bae=c\ (ab)\ ae=0$. Therefore R is a right $e-$CNZ ring. □

3. Relations to Right $e-$CNZ Rings

In this section we study the related rings and the extension rings of right $e-$CNZ rings, concentrating on matrix rings, polynomial rings, Jordan extension and some other kinds of extensions.

For a reduced ring R, we now show that $Mat_n(R)$ is neither right $E-$CNZ nor left $E-$CNZ for some $E \in Id(Mat_n(R))$.

Example 3.1. *Let R be a reduced and E_{ij} denote the matrix unit in $Mat_n(R)$ whose $(i,j)-th$ entry is 1 and the others are zero. Consider $A=E_{23}$, $B=E_{12}$ in $N(Mat_n(R))$ and $E=E_{11}+E_{33} \in Id(Mat_n(R))$. Then $AB=0$ and $BA \neq 0$. Also, $BAE \neq 0$ and $EBA \neq 0$. Thus $Mat_n(R)$ is not right (resp., left) $E-$CNZ. Therefore $Mat_n(R)$ is neither right $E-$CNZ nor left $E-$CNZ.*

If R is a reduced ring, then both $U_2(R)$ and $D_2(R)$ are $e-$CNZ for every idempotent e of R by [1, Theorem 2.7] and Proposition 2.4.(1).

The next example shows that the ring $U_2(R)$ is not $e-$CNZ when we replace the condition "R is reduced" with condition "R is $e-$CNZ".

Example 3.2. *Let $R=Z_4$ be the ring of integers modulo 4. Then R is $e-$CNZ but not reduced. By* [1, Example 2.8].

For $a=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b=\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in N(U_2(Z_4))$ *we have* $a^2=0$ *and* $b^3=0$. *Then* $ab=0$. *But for*

$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, *bae* $= \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$. *Therefore* $U_2(Z_4)$ *is not right* $e-CNZ$.

For a reduced ring R, $D_3(R)$ need not be CNZ by the same argument as in the proof as noted in [1, Remark 2.6(2)]. We note also the following.

In case $n \geq 3$, the ring R being reduced and $e \in Id(R)$ need not imply $D_n(R)$ being $eI_n - CNZ$ as illustrated below.

Example 3.3. *Let R be a reduced ring and $e \in Id(R)$ with $E = eI_3 \in Id(D_3(R))$.*

Consider $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N(D_3(R))$.

Then $AB = 0$ *and* $BAE = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$

However, there are CNZ rings and idempotents E in $D_3(R)$ such that $D_3(R)$ is right $E-$CNZ.

Theorem 3.4. *Let R be a reduced ring and n any positive integer such that $n \geq 3$. Then $D_n(R)$ is right $E_{22} - CNZ$ ring.*

Proof. Consider $n = 3$. Let $A = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & b' & c' \\ 0 & 0 & d' \\ 0 & 0 & 0 \end{pmatrix} \in N(D_3(R))$ with $AB = 0$.

Being R reduced gives rise to $BA = \begin{pmatrix} 0 & 0 & bd' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence $BAE_{22} = 0$. □

For any ring R, $U_n(R)$, $D_n(R)$ need not be $e-$CNZ for $n \geq 3$ by the following example.

Example 3.5. *Take $U_3(Z)$ for* $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in N(U_3(Z))$ *we get* $AB = 0$, *but* $BAE \neq 0$ *for* $E = eI_3 \in Id(U_3(Z))$.

For any ring R and $n \geq 2$, $V_n(R)$ is the subring of $Mat_n(R)$.

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{pmatrix} : a_i \in R,\ 1 \leq i \leq n \right\}.$$

The nilpotents of $V_n(R)$ is given by

$$N(V_n(R)) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{pmatrix} : a_1 \in N(R),\ a_2, \dots, a_n \in R \right\}$$

and $R[x]/(x^n) \cong V_n(R)$.

Theorem 3.6. *Let R be a reduced ring, $e \in Id(R)$ and any positive integer $n \geq 3$. Then $V_n(R)$ is right $eI_n - CNZ$.*

Proof. Clearly $V_n(R)$ is reversible by [12, Theorem 2.5] and so it is CNZ. This yields $V_n(R)$ is right $eI_n -$CNZ. □

Given a ring R and an $(R,R) -$ bimodule M, the trivial extension of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the next multiplication: $(c_1, m_1)(c_2, m_2) = (c_1c_2, c_1m_2 + m_1c_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} c & m \\ 0 & c \end{pmatrix}$, where $c \in R$ and $m \in M$ and the usual matrix operations are used. Note $D_2(R) = T(R,R)$.

Proposition 3.7. *Let R be a reduced ring. Then R is a right $e - CNZ$ ring if and only if the trivial extension $T(R,R)$ is a right $E - CNZ$ for each idempotent $E$of $T(R,R)$.*

Proof. It is enough to prove the necessity. Assume that $AB=0$, for $A=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $B=\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\in N(T(R,R))$, and $E=\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\in Id(T(R,R))$ with $e\in Id(R)$. Then $ac=0$ and $ad+bc=0$. Since R is right $e-$CNZ, we get $cae=0$ and $0=c(ad+bc)$ since R is reduced implies R is reversible and so $0=cad+cbc$. $0=cbc=c^2b$. $0=cb$ implies $0=cbe$ which leads to $dae=0$. Therefore $BAE=0$. Hence $T(R,R)$ is a right $E-$CNZ ring. □

The condition "R is reduced" is not superfluous. By the following example.

Example 3.8. *$R=U_2(Z)$, R is CNZ by* [1, Theorem 2.7] *thus R is right $E-CNZ$. But R is not reversible (and so not reduced).*

For $A=\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$ *and* $B=\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}\in N(T(R,R))$, *with* $A^3=0$ *and* $B^2=0$, *and consider the idempotent* $E=\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}\in T(R,R)$. *Then* $AB=0$, *but* $BAE\neq 0$. *Therefore* $T(R,R)$ *is not right* $E-CNZ$.

A ring R is Armendariz [18] if whenever any polynomials $f(x)=\sum_{i=0}^{m} a_i x^i$, $g(x)=\sum_{j=0}^{n} b_j x^j\in R[x]$ satisfy $f(x)g(x)=0$, $a_ib_j=0$ for all i,j. And also a ring R is called power-serieswise Armendariz [11] if for all $a\in C_{f(x)}$ and $b\in C_{g(x)}$, $ab=0$ whenever $f(x),g(x)\in R[[x]]$ satisfy $f(x)g(x)=0$. Every power-serieswise Armendariz ring is definitely Armendariz, but not conversely by [11, Example 2.1].

Proposition 3.9. *Let R be an Armendariz ring, then R is right $e-CNZ$ with $e\in Id(R)$ if and only if $R[x]$ is right $e-CNZ$ with $e\in Id(R[x])$.*

Proof. It is enough to show that $R[x]$ is right $e-$CNZ when so is R. Assume that R is right $e-$CNZ with $e\in Id(R)$. Let $f(x)g(x)=0$ for $f(x)=\sum_{i=0}^{m} a_i x^i$, $g(x)=\sum_{j=0}^{n} b_j x^j\in N(R[x])$ because

$N(R[x]) = N(R)[x]$ by [3, Corollary 5.2]. By hypothesis, $a_ib_j = 0$, for all i and j implies that , $b_ja_ie = 0$ for $0 \leq i \leq m$, $0 \leq j \leq n$. This leads to $g(x)f(x)e = 0$. Thus $R[x]$ is right $e-$CNZ. □

Note that: $Id(R) = Id(R[x]) = Id(R[[x]])$ by [15, Lemma 2.1(1)].

Proposition 3.10. *If R is a power-serieswise Armendariz ring, then the following conditions equivalent:*

(i) *R is right $e-$CNZ with $e \in Id(R)$.*

(ii) *$R[x]$ is right $e-$CNZ with $e \in Id(R[x])$.*

(iii) *$R[[x]]$ is right $e-$CNZ with $e \in Id(R[[x]])$.*

Proof. Let R be a power-serieswise Armendariz ring. Then it is sufficies to prove that $R[[x]]$ is right $e-$CNZ when so is R. Note that $N(R[[x]]) \subseteq N(R)[[x]]$ for a power-serieswise Armendariz ring R by [11, Lemma 2.3(2)] and [8, Lemma 2]. Therefore, it can be shown that $R[[x]]$ is right $e-$CNZ if R is right $e-$CNZ by similar computation to the proof of Proposition 3.9. □

For a ring R with an endomorphism α, we denote $R[x;\alpha]$ a skew polynomial ring (or an Ore extension of endomorphism type) whose elements are the polynomials $\sum_{i=0}^{n} a_ix^i$, $a_i \in R$, with usual addition and the multiplication related to $xa = \alpha(a)x$ for any $a \in R$. the set $\{x^j\}_{j\geq 0}$ is immediately recognized as a left Ore subset of $R[x;\alpha]$, so that $R[x;\alpha]$ can be found and the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$ can be formed. Elements of $R[x,x^{-1};\alpha]$ are finite sums of elements of the form $x^{-j}ax^i$ for nonnegative integers i and j and $a \in R$. The skew power series ring is denoted by $R[[x;\alpha]]$, and whose elements are the series $\sum_{i=0}^{\infty} a_ix^i$ for nonnegative integers i and $a_i \in R$. The skew Laurent power series ring $R[[x,x^{-1};\alpha]]$ which contains $R[[x;\alpha]]$ as a subring, arises as the localization of $R[[x;\alpha]]$ with respect to the Ore set $\{x^j\}_{j\geq 0}$, also when α is an automorphism of R, it consists elements of the form $x^sa_s + x^{s+1}a_{s+1} + \cdots + a_0 + a_1x + \cdots$, for $a_i \in R$ and integers $s \leq 0$ and $i \geq s$, with usual addition and the multiplication is defined by $xa = \alpha(a)x$ for any $a \in R$.

A ring R with an endomorphism α is called skew power-serieswise Armendariz (simply, SPA) [17, Definition 2.1] if for all skew power series $p(x) = \sum_{i=0}^{\infty} a_ix^i$, $q(x) = \sum_{j=0}^{\infty} b_jx^j \in R[[x;\alpha]]$, $p(x)q(x) = 0 \Leftrightarrow a_ib_j = 0$ for all i, j.

Theorem 3.11. *Let R be an SPA ring and α an automorphism of R. Then the following are equivalent.*

(1) *R is right $e-CNZ$ for each $e \in Id(R)$.*

(2) *$R[x;\alpha]$ is right $e-CNZ$ for each $e \in Id(R[x;\alpha])$.*

(3) *$R[x,x^{-1};\alpha]$ is right $e-CNZ$ for each $e \in Id(R[x,x^{-1};\alpha])$.*

(4) *$R[[x;\alpha]]$ is right $e-CNZ$ for each $e \in Id(R[[x;\alpha]])$.*

(5) *$R[[x,x^{-1};\alpha]]$ is right $e-CNZ$ for each $e \in Id(R[[x,x^{-1};\alpha]])$.*

Proof. The proof is similar to [1, Theorem 3.11]. □

Corollary 3.12. *Let R be a power-serieswise Armendariz ring. The following are equivalent:*

(1) *R is right $e-CNZ$.*

(2) *$R[x]$ is right $e-CNZ$.*

(3) *$R[x,x^{-1}]$ is right $e-CNZ$.*

(4) *$R[[x]]$ is right $e-CNZ$.*

(5) *$R[[x,x^{-1}]]$ is right $e-CNZ$.*

An element a of a ring R is called right regular if $ac=0$ implies $c=0$ for $c \in R$. In the same way, left regular is defined and it is regular if it is both left and right regular (and hence non-zero divisor).

Note that: $Id(\Delta^{-1}R) = \{u^{-1}e : e \in Id(R) \text{ and } u \in \Delta\}$

Theorem 3.13. *Let R be a ring, Δ be a multiplicatively closed subset of R consisting of central regular elements, $1 \in \Delta$ and $e \in Id(R)$. Then R is right $e-CNZ$ if and only if $\Delta^{-1}R$ is right $(1^{-1}e)-CNZ$.*
Note that: $N(\Delta^{-1}R) = \Delta^{-1}N(R)$.

Proof. Assume that R is right $e-$CNZ, and let $a,b \in N(R)$, $s,t \in \Delta$ such that $(s^{-1}a)(t^{-1}b)=0$. Since $\Delta \subseteq Z(R)$. We get $0=(s^{-1}t^{-1})(ab)= (st)^{-1}(ab)$, and so $ab=0$. Thus $bae=0$ by assumption so we have $0=(ts1)^{-1}(bae)= (t^{-1}b)(s^{-1}a)(1^{-1}e)$. Hence $\Delta^{-1}R$ is right $(1^{-1}e)-$CNZ.

Conversely, Let $a,b\in N(R)$ with $ab=0$. Then $(1^{-1}a)(1^{-1}b)=0$ This implies $(1^{-1}b)(1^{-1}a)(1^{-1}e)=0$ since $\Delta^{-1}R$ is $(1^{-1}e)-$CNZ, and $1^{-1}(bae)=0$. Hence $bae=0$. Thus R is right $e-$CNZ. □

Corollary 3.14. *For a ring R with $e\in Id(R)$. $R[x]$ is right $e-$CNZ if and only if $R[x,x^{-1}]$ is right $e-$CNZ.*

Proof. Assume that $R[x]$ is right $e-$CNZ and $\Delta=\{1,x,x^2,\ldots\}$. Since $R[x,x^{-1}]=\Delta^{-1}R[x]$. It follows that $R[x,x^{-1}]$ is right $e-$CNZ by Theorem 3.13. The sufficient is obvious, since the subrings of $e-$CNZ rings are also $e-$CNZ. □

Let R be a ring and α a monomorphism of R. Now we consider the Jordan's construction of an over-ring of R by α [9]. Let $A(R,\alpha)$ be the subset $\{x^{-i}rx^i:\ r\in R \text{ and } i\geq 0\}$of the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$. Notice that for $j\geq 0$, $x^jr=\alpha^j(r)x^j$ implies $rx^{-j}=x^{-j}\alpha^j(r)$ for $r\in R$. This yields that $x^{-i}rx^i=x^{-(i+j)}\alpha^j(r)x^{i+j}$ for each $j\geq 0$. It follows that $A(R,\alpha)$ forms a subring of $R[x,x^{-1};\alpha]$ with the natural operations that follow: $x^{-i}rx^i+x^{-j}sx^j=x^{-(i+j)}(\alpha^j(r)+\alpha^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j)=x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}$ for $i,j\geq 0$ and $r,s\in R$.

Note that $A(R,\alpha)$ is an over-ring of R, and $\bar{\alpha}:A(R,\alpha)\to A(R,\alpha)$ defined by $\bar{\alpha}(x^{-i}rx^i)=x^{-i}\alpha(r)x^i$ is an automorphism of $A(R,\alpha)$. Jordan demonstrated, with the use of left localization of the skew polynomial $R[x;\alpha]$ with respect to the set of powers of x, that for any pair (R,α), such an extension $A(R,\alpha)$ always exists in [9]. This ring $A(R,\alpha)$ is usually called the Jordan extension of R by α. Note that: $Id(A)=\{x^{-i}rx^i:\ r\in Id(R) \text{ and } i\geq 0\}$.

Theorem 3.15. *Let R be an Abelian ring with a monomorphism α. Then R is a right $e-$CNZ if and only if the Jordan extension $A=A(R,\alpha)$ of R by α is a right $x^{-k}ex^k-$CNZ.*

Proof. It is enough to show the necessity by Proposition 2.6 (1). Suppose that R is right $e-$CNZ and $ab=0$ for $a=x^{-i}rx^i$, $b=x^{-j}sx^j\in N(A)$ for $i,j\geq 0$ and for $x^{-k}ex^k\in Id(A)$, $e\in Id(R)$. Then $r,s\in N(R)$ obviously and so $\alpha^m(r),\ \alpha^n(s)\in N(R)$ for any non-negative integers m and n, since $\alpha(N(R))\subseteq N(R)$. From $ab=0$, we have $\alpha^j(r)\alpha^i(s)=0$ and hence $0=\alpha^i(s)\alpha^j(r)\,e$ by assumption. This implies $bae=(x^{-j}sx^j)(x^{-i}rx^i)(x^{-k}ex^k)=x^{-(k+j+i)}\alpha^i(s)\alpha^j(r)\,e\,x^{k+j+i}=0$. Thus the Jordan extension A of R by α is right $x^{-k}ex^k-$CNZ. □

Let R be an algebra over a commutative ring C. Due to Dorroh [5], consider the abelian group $R \oplus C$ with multiplication defined by $(a,b)(c,d) = (ac + da + bc,\ bd)$ where $a,c \in R$, $b,d \in S$. By this operation $R \oplus C$becomes a ring called Dorroh extension of R by C and denoted by $D(R,C)$. By definition, C is isomorphic to a subring of R. By this reason we may assume that C is contained in the center of R and use this fact in the sequel.

Lemma 3.16. [13, Lemma 3.1] *Let $(a,b) \in D(R,C)$. Then $(a,b) \in Id(D(R,C))$ if and only if $a + b \in Id(R)$, $b \in Id(C)$.*

Theorem 3.17. *Let R be an algebra over a commutative domain C and D is the Dorroh extension of R by C. Then R is right $e-CNZ$ with $e \in Id(R)$ if and only if $D(R,C)$ is right $(e,0)-CNZ$.*

Proof. By assumption, let R be right $e-$CNZ with $(a,0)(c,0) = 0$ for $(a,0),\ (c,0) \in N(D) = (N(R),0)$. Then $(ac,0) = 0$ we get $ac = 0$ since $a,c \in N(R)$ and R is right $e-$CNZ we have $cae = 0$. Hence $(c,0)(a,0)(e,0) = (cae,0) = (0,0)$. Therefore $D(R,C)$ is right $(e,0)-$CNZ.

Conversely, By supposition, suppose that $D(R,C)$ is right $(e,0)-$CNZ and $ab = 0$ for $a,b \in N(R)$. Then $(a,0)(b,0) = 0$. By supposition $(b,0)(a,0)(e,0) = 0$ This implies $(bae,0) = 0$. Thus $bae = 0$. Therefore R is right $e-$CNZ. □

4. Related Topics

As consequences of our observation in our previous sections, we introduce further results related to our main concept.

Let T and S be any rings. Take M as $(T,S)-$bimodule and R the formal triangular matrix $R = \begin{pmatrix} T & M \\ 0 & S \end{pmatrix}$. Note that: $N(R) = \begin{pmatrix} N(T) & M \\ 0 & N(S) \end{pmatrix}$.

Proposition 4.1. *Let $R = \begin{pmatrix} T & M \\ 0 & S \end{pmatrix}$ where T and S are rings, and M an $(T,S)-$bimodule. If R is a right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} -CNZ$ ring, with $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in Id(R)$, then:*

(1) *T is a right $e-CNZ$ ring,*

(2) *S is a right $g-$CNZ ring.*

Proof. Assume that R is right $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix}-$CNZ, where $\begin{pmatrix} e & k \\ 0 & g \end{pmatrix} \in Id(R)$. Then by simple computation we can check that $e \in Id(T)$, $g \in Id(S)$.

(1) For the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, eRe is right $e-$CNZ, since the subrings of $e-$CNZ are $e-$CNZ and $eRe \cong T$, so T is right $e-$CNZ.

(2) A similar discussion to (1) we show that S is right $g-$CNZ with $g = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. □

The converse of Proposition 4.1. is not true in general.

Example 4.2. *Take $A = U_2(Z)$ be right $e-$CNZ and $R = \begin{pmatrix} T_2(A) & M_2(A) \\ 0 & S_2(A) \end{pmatrix}$.*

For $a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in N(R)$ we have $ab = 0$, but for $e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in Id(R)$, $bae \neq 0$. Hence R is not right $e-$CNZ.

Let R be a ring and S a subring of R and $T[R,S] = \{(r_1, r_2, \ldots, r_n, s, s, \ldots) : r_i \in R,\ s \in S, n \geq 1, 1 \leq i \leq n\}$. Then $T[R,S]$ is a ring under the componentwise addition and multiplication. In the following we give necessary and sufficient conditions for $T[R,S]$ to be right $E-$CNZ.

Proposition 4.3. *Let R be a ring and S a subring of R with the same identity as that of R. Let $e \in Id(S)$ and $E = (e, e, e, \ldots) \in Id(T[R,S])$. Then the following are equivalent.*

(1) *$T[R,S]$ is a right $E-$CNZ ring.*

(2) *R and S are right $e-$CNZ rings.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in R$ be nilpotents with $ab = 0$. Let $A = (a, 0, 0, 0, \ldots)$, $B = (b, 0, 0, 0, \ldots)$. Then A and B are nilpotents in $T[R,S]$ and $AB = 0$. By (1) $BAE = 0$ which implies $bae = 0$. Thus R is right $e-$CNZ. Let $s,t \in S$ be nilpotents with $st = 0$. Let $X = (0,s,s,s,\ldots)$, $Y = (0,t,t,t,\ldots) \in T[R,S]$. Then X and $Y \in N(T[R,S])$ and $XY = 0$. By (1) $YXE = 0$ witch implies $tse = 0$. Therefore S is right $e-$CNZ.

(2) $\Rightarrow$ (1) Let $a = (a_1,a_2,\ldots,a_n,b,b,b,\ldots)$ and $c = (c_1,c_2,\ldots,c_m,d,d,\ldots) \in T[R,S]$ be nilpotents with $ac = 0$. Then all components of a and c are nilponents. We may assume that $n \leq m$. Then $a_ic_i = 0$ where $1 \leq i \leq n$, so $c_ia_ie = 0$. If $n+1 \leq i$, then $bc_i = 0$ and $bd = 0$. Hence $c_ibe = 0$ and $dbe = 0$. It follows that $BAE = 0$. Similarly, if $m > n$, then we have $BAE = 0$. So $T[R,S]$ is right $E-$CNZ. □

Similar to the proof of Proposition 4.3., we have the next proposition:

Proposition 4.4. *Let R be a ring and S a subring of R with the same identity as that of R. Then the following hold.*

(1) *Let $e \in Id(R)$. Then R is right $e-$CNZ if and only if $T[R,S]$ is right $(\underbrace{e,e,\ldots,e}_{n\ times},0,0,\ldots)-$CNZ for every integer $n \geq 1$.*

(2) *Let $e_0 \in Id(S)$ and $e_1,e_2,\ldots,e_n \in Id(R)$. Then R is right e_i-CNZ for every $i = 0,1,\ldots,n$ if and only if $T[R,S]$ is right $(e_1,e_2,\ldots,e_n,e_0,e_0,\ldots)-$CNZ.*

The rings $H_{(x,y)}(R)$ [7]: Let R be a ring, and let $x,y \in Z(R)$ be invertible in R.

Let $H_{(x,y)}(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & f \\ 0 & 0 & g \end{pmatrix} \in Mat_3(R) \mid a,c,d,f,g \in R,\ a-d = xc,\ d-g = yf \right\}$. Then $H_{(x,y)}(R)$ is a subring of $Mat_3(R)$.

Theorem 4.5. *Let R be a ring with $e \in Id(R)$ and $E = eI_3 \in H_{(x,y)}(R)$. Then R is right $e-$CNZ if and only if $H_{(x,y)}(R)$ is right $E-$CNZ.*

Proof. For the necessity, assume that R is right $e-$CNZ.

Let $A = \begin{pmatrix} a & 0 & 0 \\ c & d & f \\ 0 & 0 & g \end{pmatrix}$, $B = \begin{pmatrix} s & 0 & 0 \\ t & u & v \\ 0 & 0 & w \end{pmatrix} \in N(H_{(x,y)}(R))$ with $AB = 0$. Then a,d,g and s$,u,w$ are nilpotents in R and $as = 0$, $cs+dt = 0$, $du = 0$, $dv+fw = 0$, $gw = 0$. By assumption $sae = 0$, $ude = 0$, $wge = 0$. We must show that $(ta+uc)e = 0$ and $(uf+vg)e = 0$ to get $BAE = 0$.

We use $a-d=xc$, $d-g=yf$, $s-u=xt$ and $u-w=yv$ in the sequel without reference. By using these equalities, we have $ta+uc=t^{-1}(s-u)a+uc= x^{-1}sa-x^{-1}ua+x^{-1}(xuc)= x^{-1}sa-x^{-1}u(a-xc)= x^{-1}sa-x^{-1}ud= x^{-1}(sa-ud)$. Multiplying the latter equalities on the right by e yields $(ta+uc)e=x^{-1}(sa-ud)e=x^{-1}sae-x^{-1}ude=0$, since $sae=0$ and $ude=0$. By the same way, $uf+vg=y^{-1}(uyf)+y^{-1}(u-w)g= y^{-1}u(yf+g)-y^{-1}wg= y^{-1}ud-y^{-1}wg$. Multiplying the latter equalities on the right by e we get $(uf+vg)e=y^{-1}ude-y^{-1}wge=0$ since $ude=0$ and $wge=0$. It follows that $BA=0$. Thus $H_{(x,y)}(R)$ is right $E-$CNZ.

Conversely, Let $a,b\in N(R)$ with $ab=0$. Let $A=aI_3$, $B=bI_3\in N(H_{(x,y)}(R))$. Then $AB=0$ and $BAE=0$ since $H_{(x,y)}(R)$ is right $E-$CNZ we get $bae=0$. Therefore R is right $e-$CNZ. $\square$

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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