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PHASE RETRIEVAL PROBLEM IN SPACE OF $l^p(\Gamma)(0 < p < 1)$

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Abstract. In this paper, we use the injective of operator $\mathscr{A}_\Phi$ to represent the uniqueness of the phase retrieval. We show that the relationship between the complement property (CP) and phase retrieval of frames in non-Banach space $l^p(\Gamma)(0 < p < 1)$. Moreover, we also consider the stability of phase retrieval and quantify the stability by the ratio of the upper and lower bounds of the frame Φ in $l^p(\Gamma)(0 < p < 1)$.

Keywords: phase retrieval; injective; stability.

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1. Introduction and Preliminaries

In many areas, the phase of a signal is always lost during processing. Therefore, how to recover the phase of these lost signals and whether the results of the recovery are stable are particularly important. The phase retrieval problem refers to the problem of reconstructing function f from phaseless strength measurements. It appears in many areas, such as X-ray crystallography in [3], astronomy in [4], signal noise reduction in [5] and automatic speech recognition in [6]. In addition, phase retrieval also plays an important role in the fields of radar [7], signal theory [8], quantum communication [9] and so on. The concept of phaseless reconstruction in finite dimensional Hilbert space was first introduced by Balan, Casazza and Edidin in [1] and

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Sara Botelho-Andrade et al. proved in [2] that phase retrieval and phaseless reconstruction are equivalent. Therefore, we study that the phase retrieval problem can actually be converted into a phaseless reconstruction problem. James Cahlili et al. introduced phase retrieval in infinite Hilbert space in [11] and proved the stability of phase retrieval in infinite dimensional Hilbert space. Subsequently, Rima Alaifari et al. introduced phase retrieval of continuous frames in Banach space in [10] and prove that phase retrieval problems are never uniformly stable in infinite dimensional Banach space. In this paper we mainly consider some properties of phase retrieval in non-Banach space $l^p(\Gamma)(0<p<1)$. Before starting our research, we give the definition of $l^p(\Gamma)(0<p<1)$ space:

$$l^p(\Gamma) := \{x=\sum_{\gamma\in\Gamma}\xi_\gamma e_\gamma : \|x\|_p=\sum_{\gamma\in\Gamma}|\xi_\gamma|^p<\infty, \xi_\gamma\in\mathbb{K}, \gamma\in\Gamma\}.$$

where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, Γ is a non-empty index set and e_γ is a unit vector. For convenience, the $l^p(\Gamma)$ spaces below all satisfy $0<p<1$.

Obviously, the $l^p(\Gamma)$ space is not a Banach space. In fact, $l^p(\Gamma)$ is a p-normed space and $\|\cdot\|_p$ satisfies the following properties:

(1) $\|x\|_p\geq 0$ and $\|x\|_p=0$ if and only if $x=0$;

(2) $\|\lambda x\|_p=|\lambda|^p.\|x\|_p$ for all $\lambda\in\mathbb{C}$ and $x\in l^p(\Gamma)$;

(3) $\|x+y\|_p\leq\|x\|_p+\|y\|_p$ for all $x,y\in l^p(\Gamma)$.

Furthermore, it is easy to see that $\|ax\|_p=1$ with $a=1/\|x\|_p^{\frac{1}{p}}$ for any non-zero vector $x\in l^p(\Gamma)$.

We define the quotient space $P_{\mathbb{K}}l^p(\Gamma)$ of $l^p(\Gamma)$ by $P_{\mathbb{K}}l^p(\Gamma):=l^p(\Gamma)\backslash\{-1,1\}$ for $\mathbb{K}=\mathbb{R}$ and by $P_{\mathbb{K}}l^p(\Gamma):=l^p(\Gamma)\backslash S^1$ for $\mathbb{K}=\mathbb{C}$.

Therefore we can define the distance in the quotient space $P_{\mathbb{K}}l^p(\Gamma)$ by

$$\widetilde{d}_{l^p(\Gamma)}(x,y):=\min\{\|x-\tau y\|_p;\tau\in\mathbb{K},|\tau|=1\}.$$

Below we introduce the meaning of some symbols and mathematical representations commonly used in this article.

Let $l^p(\Gamma)^*$ denote the space that composed by bounded linear functionals $\varphi : l^p(\Gamma)\to\mathbb{K}$. For $x\in l^p(\Gamma)$ and $\varphi\in l^p(\Gamma)^*$, we use $[x,\varphi]:=\varphi(x)$ to represent the bounded linear functional acting on x. For a closed subspace $W\subset l^p(\Gamma)^*$ we define its annihilator space by $W_\perp:=\{x\in l^p(\Gamma):$

$[x,\varphi]=0, \forall \varphi \in W\}$. Similarly, we can define the annihilation space of the closed subspace on $l^p(\Gamma)$ space.

We fix a collection of bounded linear functionals $\Phi := (\varphi_\lambda)_{\lambda\in\Lambda} \subset l^p(\Gamma)^*$ for index set Λ and the notation $\Phi_S := (\varphi_\lambda)_{\lambda\in S}$ for $S\subset\Lambda$. For $x\in l^p(\Gamma)$ we write

$$[x,\Phi] := ([x,\varphi_\lambda])_{\lambda\in\Lambda}\ ,\ [x,\Phi_S] := ([x,\varphi_\lambda])_{\lambda\in S}$$

and

$$|[x,\Phi]| := (|[x,\varphi_\lambda]|)_{\lambda\in\Lambda}\ ,\ |[x,\Phi_S]| := (|[x,\varphi_\lambda]|)_{\lambda\in S}$$

Define the operator

$$\mathscr{A}_\Phi : \mathbb{P}_{\mathbb{K}} l^p(\Gamma) \to \mathbb{R}_+^\Lambda,\ x\mapsto |[x,\Phi]|.$$

Then the phase retrieval problem can be seen as the inversion problem of $\mathscr{A}_\Phi$.

In section 2, we are mainly studying the uniqueness properties of the inversion of $\mathscr{A}_\Phi$. The results show that the injectivity of $\mathscr{A}_\Phi$ and the complement property of Φ are equivalent in real $l^p(\Gamma)$ space. In section 3, with the help of the frame in the $l^p(\Gamma)$ space and the σ-strong complement property of frame, we consider the strong stability. Meantime, the ratio of the upper and lower bounds of the frame is used to quantify this stability.

2. INJECTIVITY

In this section, we mainly consider the uniqueness of the phase retrieval problem, that is, the injectivity of $\mathscr{A}_\Phi$. Before that, we give the definition of the complement property.

Definition 2.1. *(Complement property). The collection $\Phi=(\varphi_\lambda)_{\lambda\in\Lambda}\subset l^p(\Gamma)^*$ satisfies the complement property (CP) in $l^p(\Gamma)$ if for every subset $S\subset\Lambda$, either $(span\Phi_S)_\perp=\{0\}$, or $(span\Phi_{\Lambda\setminus S})_\perp=\{0\}$.*

The complement property of Φ is an important property for studying the injectivity of operator $\mathscr{A}_\Phi$. Jameson Cahill et al. proved that the complement property of Φ is equivalent to the injectivity of $\mathscr{A}_\Phi$ in the real Hilbert space in [11]. Next, Rima Alaifari and Philipp Grohs proved that the conclusion also holds in real Banach space in [10].

In the following theorem, we consider the relationship between the complement property (CP) of Φ and the injectivity of $\mathscr{A}_\Phi$ in the $l^p(\Gamma)$ space.

Theorem 2.2. *(a)If $\mathscr{A}_\Phi$ is injective, then Φ satisfies the CP in $l^p(\Gamma)$.*

(b)If $l^p(\Gamma)$ is a space over $\mathbb{R}$ and Φ satisfies the CP in $l^p(\Gamma)$, then $\mathscr{A}_\Phi$ is injective.

Proof. (a)Suppose that $\mathscr{A}_\Phi$ is injective for $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda}$. Let $x \in (span\Phi_S)_\perp$ and $y \in (span\Phi_{\Lambda\setminus S})_\perp$ for some $S \subset \Lambda$. Then, we have

$$|[x \pm y, \varphi_\lambda]|^2 = |[x, \varphi_\lambda]|^2 \pm 2Re(\overline{[x, \varphi_\lambda]} \cdot [y, \varphi_\lambda]) + |[y, \varphi_\lambda]|^2.$$

When $Re(\overline{[x, \varphi_\lambda]} \cdot [y, \varphi_\lambda]) = 0$ holds, we have $\mathscr{A}_\Phi(x+y) = \mathscr{A}_\Phi(x-y)$. At the same time, we have $x+y = t(x-y)$ for $|t| = 1$ by the assumption that $\mathscr{A}_\Phi$ is injective . Then we can get

$$y = \frac{t-1}{1+t}x$$

for $t \neq -1$ and $x \neq 0$. Therefore, $y \in (span\Phi_S)_\perp$ implies $\mathscr{A}_\Phi(y) = 0$. Thus, we have $y = 0$ by the injectivity of $\mathscr{A}_\Phi$. On the other hand, we can get $x = 0$ if we switch the roles of x and y. So far, we have completed the proof.

(b)Suppose that $\mathscr{A}_\Phi$ is not injective. Then, there is two vectors $u, v \in l^p(\Gamma)$ such that $\mathscr{A}_\Phi(u) = \mathscr{A}_\Phi(v)$. We consider dividing the index set Λ into two parts by the assumption of $l^p(\Gamma)$ is a space over $\mathbb{R}$, which satisfies $S := \{\lambda \in \Lambda : [u, \varphi_\lambda] = [v, \varphi_\lambda]\}$ and $\Lambda \setminus S := \{\lambda \in \Lambda : [u, \varphi_\lambda] = -[v, \varphi_\lambda]\}$. Consequently, we have $u + v \in (span\Phi_S)_\perp$ and $u - v \in (span\Phi_{\Lambda\setminus S})_\perp$. From the assumption that Φ satisfies CP, we get the contradiction. $\square$

3. Stability

In this section, we consider whether the injectivity of operator $\mathscr{A}_\Phi$ is stable and quantify the stability. When discussing the stability of the injectivity of operator $\mathscr{A}_\Phi$, we need to use the definition of frame in $l^p(\Gamma)$ space. So, we give the definition of the frame in $l^p(\Gamma)$.

Definition 3.1. *Let $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset l^p(\Gamma)^*$ is a family of bounded linear functionals. It is called a frame for $l^p(\Gamma)$ if there are positive constant $0 < A \leq B < \infty$ satisfying*

$$A\|x\|_p \leq \sum_{\lambda\in\Lambda} |[x, \varphi_\lambda]|^p \leq B\|x\|_p.$$

for every $x \in l^p(\Gamma)$. B and A are called the upper and lower bounds of the frame. Meanwhile, we use A_Φ and B_Φ to represent the best possible values of A and B respectively.

3.1 **Strong stability** Suppose $\mathscr{A}_\Phi$ is bilipschitz, that is, there exist constants $\alpha>0$ and $\beta<\infty$ such that

$$\alpha\widetilde{d}_{l^p(\Gamma)}(x,y)\leq\|\mathscr{A}_\Phi(x)-\mathscr{A}_\Phi(y)\|_p\leq\beta\widetilde{d}_{l^p(\Gamma)}(x,y).$$

where $\|\mathscr{A}_\Phi(x)-\mathscr{A}_\Phi(y)\|_p=\sum_{\lambda\in\Lambda}\big||[x,\varphi_\lambda]|-|[y,\varphi_\lambda]|\big|^p$. We use α_Φ and β_Φ to represent the best possible of $\alpha>0$ and $\beta<\infty$, respectively.

After introducing α_Φ and β_Φ, we can quantify the stability, because the stability of the injectivity of $\mathscr{A}_\Phi$ is determined by the ratio of these two constants. We denote this ratio as

$$\tau_\Phi:=\frac{\beta_\Phi}{\alpha_\Phi}.$$

First of all, we study the smallest possible β_Φ of $\beta<\infty$ and we find that it is the same as the best possible of the upper bound of the frame Φ.

Theorem 3.2. *Suppose that $\Phi=(\varphi_\lambda)_{\lambda\in\Lambda}\subset l^p(\Gamma)^*$ is a frame for space $l^p(\Gamma)$. Then, we can get*

$$\beta_\Phi=B_\Phi.$$

Proof. By the reverse triangle inequality, we have

$$|\,|a|-|b|\,|\leq\min\{|a-b|,|a+b|\},\ \forall a,b\in\mathbb{C}.$$

Then, according to the definition of frame in $l^p(\Gamma)$, we have

$$\begin{aligned}\|\mathscr{A}_\Phi(x)-\mathscr{A}_\Phi(y)\|_p&=\sum_{\lambda\in\Lambda}|\,|[x,\varphi_\lambda]|-|[y,\varphi_\lambda]|\,|^p\\&\leq\sum_{\lambda\in\Lambda}(\min\{|[x-\tau y,\varphi_\lambda]|:|\tau|=1\})^p\\&\leq\min\{\sum_{\lambda\in\Lambda}|[x-\tau y,\varphi_\lambda]|^p:|\tau|=1\}\\&\leq\min\{B\|x-\tau y\|_p:|\tau|=1\}\\&=B_\Phi\widetilde{d}_{l^p(\Gamma)}(x,y).\end{aligned}$$

Therefore, $\beta_\Phi\leq B_\Phi$. Conversely, we can pick $y=0$ and $x\in l^p(\Gamma)$ with $\|x\|_p=1$ such that $B_\Phi-\varepsilon<\sum_{\lambda\in\Lambda}|[x,\varphi_\lambda]|^p<B_\Phi$. $\square$

Next, we consider the lower Lipschitz constant α_Φ. When we analyze the lower Lipschitz boundary, we need to use the following definition. It measures the complement property we discussed earlier with accurate values, and it also prepares us to quantify the stability of phase retrieval.

Definition 3.3. *(σ-strong complement property). The system $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset l^p(\Gamma)^*$ satisfies the σ-strong complement property (σ-SCP) in $l^p(\Gamma)$ if there exists $\sigma > 0$ such that for every subset $S \subset \Lambda$, we have*

$$\max\{A_{\Phi_S}, A_{\Phi_{\Lambda\setminus S}}\} \geq \sigma.$$

We denote the supremal σ for which Φ satisfies the σ-SCP by σ_Φ.

Then we will discuss the lower Lipschitz bound α_Φ from the real case and the complex case, respectively.

3.1.1 The real case.

Theorem 3.4. *Suppose $l^p(\Gamma)$ is a space over $\mathbb{R}$ and $\Phi = (\varphi_\lambda)_{\lambda\in\Lambda} \subset l^p(\Gamma)^*$ is a frame with σ-SCP in $l^p(\Gamma)$ space. Then, we have*

$$\sigma_\Phi \leq \alpha_\Phi \leq 2\sigma_\Phi.$$

Proof. We first show that the right-hand side in the inequality. The assumption of Φ follows that

$$\sigma \leq \sigma_\Phi \leq \max\{A_{\Phi_S}, A_{\Phi_{\Lambda\setminus S}}\}\ for\ \forall S \subset \Lambda.$$

Let $\sigma_0 > \sigma_\Phi$. Then, we have

$$\sigma_0 > \max\{A_{\Phi_S}, A_{\Phi_{\Lambda\setminus S}}\}$$

for some $S \subset \Lambda$. So, there exist two vectors $u, v \in l^p(\Gamma)$ with $\|u\|_p = 1 = \|v\|_p$ such that

$$A_{\Phi_S} = A_{\Phi_S}\|u\|_p \leq \sum_{\lambda\in S} |[u, \varphi_\lambda]|^p \leq \sigma_0\|u\|_p = \sigma_0,$$

and

$$A_{\Phi_{\Lambda\setminus S}} = A_{\Phi_{\Lambda\setminus S}}\|v\|_p \leq \sum_{\lambda\in\Lambda\setminus S} |[v, \varphi_\lambda]|^p \leq \sigma_0\|v\|_p = \sigma_0.$$

Let $x := u+v$ and $y := u-v$. By the inverse triangle inequality and the assumption of $\|u\|_p = 1 = \|v\|_p$, we have

$$\begin{aligned}
\alpha_\Phi \widetilde{d}_{l^p(\Gamma)}(x,y) &\le \|\mathscr{A}_\Phi(x) - \mathscr{A}_\Phi(y)\|_p \\
&= \sum_{\lambda\in\Lambda} \big|\, |[u+v,\varphi_\lambda]| - |[u-v,\varphi_\lambda]| \,\big|^p \\
&= \sum_{\lambda\in S} \big|\, |[u+v,\varphi_\lambda]| - |[u-v,\varphi_\lambda]| \,\big|^p \\
&+ \sum_{\lambda\in\Lambda\setminus S} \big|\, |[u+v,\varphi_\lambda]| - |[u-v,\varphi_\lambda]| \,\big|^p \\
&\le 2^p \sum_{\lambda\in S} |[u,\varphi_\lambda]|^p + 2^p \sum_{\lambda\in\Lambda\setminus S} |[v,\varphi_\lambda]|^p \\
&\le 2^p \sigma_0 (\|u\|_p + \|v\|_p) \\
&= 2^{p+1}\sigma_0 \min\{\|u\|_p, \|v\|_p\} \\
&= 2\sigma_0 \min\{\|x+y\|_p, \|x-y\|_p\} \\
&= 2\sigma_0 \widetilde{d}_{l^p(\Gamma)}(x,y).
\end{aligned}$$

Therefore, $\alpha_\Phi \le 2\sigma_\Phi$.

Then we show the inequality $\sigma_\Phi \le \alpha_\Phi$. For any $\alpha_0 > \alpha_\Phi$, then there exist two vectors $x', y' \in l^p(\Gamma)$ with

$$\alpha_0 \widetilde{d}_{l^p(\Gamma)}(x',y') > \|\mathscr{A}_\Phi(x') - \mathscr{A}_\Phi(y')\|_p.$$

Set

$$S := \{\lambda \in \Lambda : sign([x',\varphi_\lambda]) = -sign([y',\varphi_\lambda])\}.$$

Let $u' := x'+y'$ and $v' := x'-y'$. From the definition of S, we can get

$$\begin{aligned}
A_{\Phi_S}\|u'\|_p &\le \sum_{\lambda\in S} |[u',\varphi_\lambda]|^p \\
&= \sum_{\lambda\in S} \big|\, [x',\varphi_\lambda] + [y',\varphi_\lambda] \,\big|^p \\
&= \sum_{\lambda\in S} \big|\, |[x',\varphi_\lambda]| - |[y',\varphi_\lambda]| \,\big|^p \\
&\le \sum_{\lambda\in\Lambda} \big|\, |[x',\varphi_\lambda]| - |[y',\varphi_\lambda]| \,\big|^p
\end{aligned}$$

$$= \|\mathscr{A}_\Phi(x') - \mathscr{A}_\Phi(y')\|_p$$

$$< \alpha_0 \widetilde{d}_{l^p(\Gamma)}(x', y')$$

$$\leq \alpha_0 \|u'\|_p.$$

By the similar method, we obtain

$$A_{\Phi_{\Lambda\backslash S}} \|v'\|_p \leq \sum_{\lambda \in \Lambda\backslash S} |[v', \varphi_\lambda]|^p = \sum_{\lambda \in \Lambda\backslash S} |\, |[x', \varphi_\lambda]| - |[y', \varphi_\lambda]|\, |^p < \alpha_0 \widetilde{d}_{l^p(\Gamma)}(x', y') \leq \alpha_0 \|v'\|_p.$$

Therefore, we have

$$\max\{A_{\Phi_S}, A_{\Phi_{\Lambda\backslash S}}\} \leq \alpha_0,$$

which completes the proof. □

In the following corollary, we give the range of values of the ratio τ_Φ under the real-value conditions.

Corollary 3.5. *Suppose $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda}$ is a frame satisfying σ-SCP in the space $l^p(\Gamma)$ over $K = \mathbb{R}$. Then we have*

$$\frac{B_\Phi}{2\sigma_\Phi} \leq \tau_\Phi \leq \frac{B_\Phi}{\sigma_\Phi}.$$

Proof. By Theorem 3.2, we can get

$$\tau_\Phi = \frac{\beta_\Phi}{\alpha_\Phi} = \frac{B_\Phi}{\alpha_\Phi}.$$

Then, the result follows from Theorem 3.4. □

3.1.2 The complex case.

Theorem 3.6. *Suppose that $\Phi = (\varphi_\lambda)_{\lambda \in \Lambda} \subset l^p(\Gamma)^*$ is a frame with σ-SCP in space $l^p(\Gamma)$. Then, there exists a constant $C > 0$ (depending on only the frame constants of Φ) such that*

$$\alpha_\Phi \leq C \sigma_\Phi.$$

The constant C can be taken to be

$$C = 2\frac{B_\Phi}{A_\Phi}.$$

Proof. Let $\sigma > \sigma_\Phi$. Similar to Theorem 3.4, there exist two vectors $u,v\in l^p(\Gamma)$ with $\|u\|_p = 1 = \|v\|_p$ for some $S\subset\Lambda$ such that

$$\sum_{\lambda\in S}|[u,\varphi_\lambda]|^p \le \sigma, \tag{1}$$

and

$$\sum_{\lambda\in\Lambda\backslash S}|[v,\varphi_\lambda]|^p \le \sigma. \tag{2}$$

Since Φ is a frame in $l^p(\Gamma)$ space, we have

$$A_\Phi\|u\|_p \le \sum_{\lambda\in\Lambda}|[u,\varphi_\lambda]|^p = \sum_{\lambda\in S}|[u,\varphi_\lambda]|^p + \sum_{\lambda\in\Lambda\backslash S}|[u,\varphi_\lambda]|^p,$$

and

$$A_\Phi\|v\|_p \le \sum_{\lambda\in\Lambda}|[v,\varphi_\lambda]|^p = \sum_{\lambda\in S}|[v,\varphi_\lambda]|^p + \sum_{\lambda\in\Lambda\backslash S}|[v,\varphi_\lambda]|^p.$$

Therefore, we have

$$\sum_{\lambda\in\Lambda\backslash S}|[u,\varphi_\lambda]|^p \ge A_\Phi - \sigma, \tag{3}$$

and

$$\sum_{\lambda\in S}|[v,\varphi_\lambda]|^p \ge A_\Phi - \sigma. \tag{4}$$

Let $x := u+v$ and $y := u-v$. Similar to the proof of Theorem 3.4, we also have

$$\begin{aligned}\alpha_\Phi\widetilde{d}_{l^p(\Gamma)}(x,y) &\le \sum_{\lambda\in\Lambda}\big|\,|[x,\varphi_\lambda]| - |[y,\varphi_\lambda]|\,\big|^p\\ &\le 2^p\sum_{\lambda\in S}|[u,\varphi_\lambda]|^p + 2^p\sum_{\lambda\in\Lambda\backslash S}|[v,\varphi_\lambda]|^p\\ &\le 2^{p+1}\sigma.\end{aligned}$$

Since $\sigma > \sigma_\Phi$, it follows that

$$\alpha_\Phi \le \frac{2^{p+1}\sigma_\Phi}{\widetilde{d}_{l^p(\Gamma)}(x,y)} \tag{5}$$

In order to prove the result, we need to give a lower bound on $\|x-\tau y\|_p$ that holds for any $\tau\in\mathbb{C}$ with $|\tau|=1$. By the frame property of Φ and equations (1) - (4), we have

$$\begin{aligned}
B_\Phi \|x-\tau y\|_p &\geq \sum_{\lambda\in\Lambda} |[x-\tau y,\varphi_\lambda]|^p \\
&= \frac{1}{2}\sum_{\lambda\in\Lambda} |[x-\tau y,\varphi_\lambda]|^p + \frac{1}{2}\sum_{\lambda\in\Lambda} |[x-\tau y,\varphi_\lambda]|^p \\
&\geq \frac{1}{2}\sum_{\lambda\in S} |[x-\tau y,\varphi_\lambda]|^p + \frac{1}{2}\sum_{\lambda\in\Lambda\setminus S} |[x-\tau y,\varphi_\lambda]|^p \\
&= \frac{1}{2}\sum_{\lambda\in S} |[(1-\tau)u+(1+\tau)v,\varphi_\lambda]|^p \\
&+ \frac{1}{2}\sum_{\lambda\in\Lambda\setminus S} |[(1-\tau)u+(1+\tau)v,\varphi_\lambda]|^p \\
&\geq \frac{|1+\tau|^p}{2}\sum_{\lambda\in S} |[v,\varphi_\lambda]|^p - \frac{|1-\tau|^p}{2}\sum_{\lambda\in S} |[u,\varphi_\lambda]|^p \\
&+ \frac{|1-\tau|^p}{2}\sum_{\lambda\in\Lambda\setminus S} |[u,\varphi_\lambda]|^p - \frac{|1+\tau|^p}{2}\sum_{\lambda\in\Lambda\setminus S} |[v,\varphi_\lambda]|^p \\
&\geq \frac{|1+\tau|^p+|1-\tau|^p}{2}\cdot(A_\Phi-2\sigma) \\
&\geq \frac{1}{2}(A_\Phi-2\sigma),
\end{aligned}$$

where the last step follows from $|a+b|^p \leq 2^p(|a|^p+|b|^p)$ for any $a,b\in\mathbb{C}$ and $p\geq 0$. Thus, by inequality (5), we can obtain

$$\alpha_\Phi \leq \sigma_\Phi \frac{2^{p+2}\cdot B_\Phi}{(A_\Phi-2\sigma_\Phi)}$$

whenever $\sigma_\Phi \leq tA_\Phi$ with $t<\frac{1}{2}$. In addition, by Theorem 3.2 we have

$$\alpha_\Phi \leq \beta_\Phi = B_\Phi \leq \sigma_\Phi \frac{B_\Phi}{tA_\Phi}$$

whenever $\sigma_\Phi \geq tA_\Phi$. To sum up, we can choose the value of C

$$C := \min_{t<1/2}\Big(\frac{2^{p+2}}{1-2t},\frac{1}{t}\Big)\frac{B_\Phi}{A_\Phi} = 2\frac{B_\Phi}{A_\Phi}.$$

□

Therefore, we have the following corollary and omit the proof.

Corollary 3.7. *Suppose $\Phi=(\varphi_\lambda)_{\lambda\in\Lambda}$ is a frame satisfying σ-SCP in the space $l^p(\Gamma)$ over $K=\mathbb{C}$. Then we have*

$$\tau_\Phi \geq \frac{A_\Phi}{2\sigma_\Phi}.$$

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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