SOME PROPERTIES OF CERTAIN SUBCLASSES OF \( p \)-VALENT FUNCTIONS DEFINED BY A LINEAR DERIVATIVE OPERATOR

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Abstract. In the present paper, we introduce new classes of \( p \)-valent functions defined by using a generalized linear derivative operator with negative coefficients in the unit disk. The results presented here include coefficient estimates, extreme points and distortion properties for the aforementioned classes.

Key words. \( p \)-valent functions, starlike, convex, distortion theorems, linear derivative operator.

AMS Mathematics Subject Classification (2000): 30C45.

1. Definition and Preliminaries

Let \( A_p \) denote the class of functions of the form :

\[
    f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}).
\]

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Received November 27, 2011
which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$. A function $f \in A_p$ is called $p$-valent starlike of order $\beta$ and type $\gamma$, if it satisfies

$$\left| \frac{zf'(z)}{f(z)} - p \right| \left| \frac{zf''(z)}{f(z)} + p - 2\gamma \right| < \beta,$$

(1.2)

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in N$. We denote by $S^*(p, \gamma, \beta)$ the class of $p$-valent starlike functions of order $\gamma$ and type $\beta$. A function $f \in A_p$ is called $p$-valent convex functions of order $\beta$ and type $\gamma$, if it satisfies

$$\left| \frac{1 + zf''(z)}{f'(z)} - p \right| \left| \frac{1 + zf''(z)}{f'(z)} + p - 2\gamma \right| < \beta,$$

(1.3)

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in N$. We denote by $K(p, \gamma, \beta)$ the class of $p$-valent convex functions of order $\gamma$ and type $\beta$.

From (1.2) and (1.3), we note that: $f(z) \in K(p, \gamma, \beta)$ if, and only if,

$$\frac{zf'}{p} \in S^*(p, \gamma, \beta).$$

The classes $S^*(p, \gamma, \beta)$ and $K(p, \gamma, \beta)$ were considered by Aouf [2] and Hossen [3]. For $\beta = 1$, reduced to the class $S^*(p, \gamma, 1) = S^*(p, \gamma)$ which was studied by Patil and Thakare [4], and the class $K(p, \gamma, 1) = K(p, \gamma)$ given by Owa [5].

Let $T_p$ denote the subclass of $A_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}).$$

(1.4)

We denote by $T^*(p, \gamma, \beta)$ and $C(p, \gamma, \beta)$, the classes obtained by taking intersections, respectively, of the classes $S^*(p, \gamma, \beta)$ and $K(p, \gamma, \beta)$ with the class $T_p$. Thus we have

$$T^*(p, \gamma, \beta) = S^*(p, \gamma, \beta) \cap T_p,$$

and

$$C(p, \gamma, \beta) = K(p, \gamma, \beta) \cap T_p.$$
The classes $T^*(p, \gamma, \beta)$ and $C(p, \gamma, \beta)$ were studied by Aouf [2] and Hossen [3]. In particular, the classes $T^*(p, \gamma, 1) = T^*(p, \gamma)$ and $C(p, \gamma, 1) = C(p, \gamma)$ were introduced by Owa [5]. Also the classes $T^*(1, \gamma, 1) = T^*(\gamma)$ and $C(1, \gamma, 1) = C(\gamma)$ were studied by Silverman [6].

For functions $f \in A_p$, given by (1.1), and $g$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of functions $f$ and $g$ is defined by

$$(f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g \ast f)(z), \quad (p \in \mathbb{N}).$$

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by $(x)_k =

$$\begin{cases} 
1 & \text{for } k = 0, \\
(x+1)(x+2)\ldots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots\}. 
\end{cases}$$

The authors in [1] have recently introduced a new generalized linear derivative operator $D_p^{\alpha,\delta}(\mu, q, \gamma)$, as the following:

**Definition 1.1.** For $f \in A_p$, the linear operator $D_p^{\alpha,\delta}(\mu, q, \gamma)$ is defined by $D_p^{\alpha,\delta}(\mu, q, \gamma) : A_p \to A_p$ as:

$$D_p^{\alpha,\delta}(\mu, q, \gamma)f(z) = z^p + \sum_{k=p+1}^{\infty} \binom{k}{p}^{\alpha} \left(1 + \frac{k - p}{p + q} \lambda\right)^\mu c(\delta, k)a_k z^k, \quad (1.5)$$

where $\lambda, \mu, q \geq 0$, $k, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$,

$$c(\delta, k) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+\delta)}{(k)!\Gamma(p+\delta)} z^k.$$ 

Next we define the following new subclasses of $p$-valent functions as follows:
Definition 1.2. Let $f \in T_p$ be given by (1.4). Then $f$ is said to be in the class $T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$ if, and only if,

$$\left| \frac{z(D_p^{\alpha,\delta}(\mu, q, \gamma)f'(z))'}{D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)} - p \right| + \frac{z(D_p^{\alpha,\delta}(\mu, q, \gamma)f'(z))'}{D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)} + p - 2\gamma < \beta,$$

where $D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)$ is given by (1.5) and $\lambda, \mu, q \geq 0, k, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in \mathbb{N}$.

Further, a function $f \in T_p$ is said to be in the class $C_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$ if, and only if,

$$\frac{zf'}{p} \in T_p^{\alpha,\delta}(\mu, q, \gamma, \beta).$$

We note that, by specializing the parameters $\alpha, \delta, \mu, \lambda, \beta$ and $p$, we shall obtain the following subclasses which were studied by various authors:

1. For $\alpha = \delta = \mu = 0$ we get $T_p^{0,0}(0, q, \gamma, \beta) = T^*(p, \gamma, \beta)$, is the class of $p$-valent starlike function of order $\gamma$ and type $\beta$ which was studied by Aouf [2] and Hossen [3].

2. For $\alpha = \delta = \mu = 0$ and $p = 1$, we have $T_1^{0,0}(0, q, \gamma, \beta) = S^*(\gamma, \beta)$, is the class of starlike function of order $\gamma$ and type $\beta$ which was studied by Gupta and Jain [7].

3. For $\alpha = \delta = \mu = 0$ and $\beta = 1$, we obtain the class $T_p^{0,0}(0, q, \gamma, 1) = T^*(p, \gamma)$, which was introduced by Owa [5].

4. For $\alpha = \delta = \mu = 0$, $p = 1$ and $\beta = 1$ we obtain the class $T_1^{0,0}(0, q, \gamma, 1) = T^*(\gamma)$, which was studied by Silverman [6].

5. For $\alpha = \delta = q = 0, \mu = 1$ and $p = 1$, we have the class $C_1^{0,0}(1, 0, \gamma, \beta) = C^*(\gamma, \beta)$, which was studied by Gupta and Jain [7].

6. For $\alpha = \delta = q = 0, \mu = 1$, we have the class $C_p^{0,0}(1, 0, \gamma, \beta) = C(p, \gamma, \beta)$, is the class of $p$-valent convex function of order $\gamma$ and type $\beta$, studied by Aouf [2] and Hossen [3].

7. For $\alpha = \delta = q = 0, \mu = 1$, and $\beta = 1$, we have the class $C_p^{0,0}(1, 0, \gamma, 1) = C(p, \gamma)$, studied by Owa [5].
8. For $\alpha = \delta = q = 0, \mu = 1, \beta = 1$, and $p = 1$, we obtain the class $C^{0,0}_1(1, 0, \gamma, 1) = C(\gamma)$, studied by Silverman [6].

2. COEFFICIENT ESTIMATES

**Theorem 2.1.** A function $f$ belongs to the class $T^{\alpha,\delta}_p(\mu, q, \gamma, \beta)$ if, and only if,

$$
\sum_{k=p+1}^{\infty} \left( ((k - p) + \beta(k + p - 2\gamma)) \left( \frac{k}{p} \right)^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} a_k z^k \right) \leq 2\beta(p - \gamma). \quad (2.1)
$$

**Proof:** Let the function $f$ be in the class $T^{\alpha,\delta}_p(\mu, q, \gamma, \beta)$. Then we have

$$
\left| \frac{z(D^\alpha_p(\mu, q, \gamma)f)'(z)}{D^\alpha_p(\mu, q, \gamma)f(z)} + p - 2\gamma \right| = \left| \frac{p^2 - \sum_{k=p+1}^{\infty} (\frac{k}{p})^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} a_k z^k}{z^p - \sum_{k=p+1}^{\infty} (\frac{k}{p})^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} a_k z^k} - p \right| \leq \beta.
$$

Since $|Re(z)| \leq |z|$ for all $z$, we have

$$
\mathcal{R} \left\{ \frac{\sum_{k=p+1}^{\infty} (k - p) (\frac{k}{p})^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} a_k z^k}{\sum_{k=p+1}^{\infty} (k + p - 2\gamma) (\frac{k}{p})^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} a_k z^k + (2p - 2\gamma)} \right\} \leq \beta.
$$

Choosing values of $z$ on the real axis, so that $\frac{z(D^\alpha_p(\mu, q, \gamma)f)'(z)}{D^\alpha_p(\mu, q, \gamma)f(z)}$ is real, and letting $z \to 1^-$, through real axis, we get

$$
\sum_{k=p+1}^{\infty} (k - p) (\frac{k}{p})^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{(k)!\Gamma(p + \delta)} a_k z^k \leq -\beta \left( \sum_{k=p+1}^{\infty} (k + p - 2\gamma) (\frac{k}{p})^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} a_k z^k + \beta(2p - 2\gamma) \right),
$$

which implies the assertion (2.1). Conversely, let the inequality (2.1) holds true, then

$$
\left| z(D^\alpha_p(\mu, q, \gamma)f)'(z) - p(D^\alpha_p(\mu, q, \gamma)f(z)) \right| - \beta
$$

$$
\left| z(D^\alpha_p(\mu, q, \gamma)f)'(z) + (p - 2\gamma) D^\alpha_p(\mu, q, \gamma)f(z) \right|,
$$

$$
\sum_{k=p+1}^{\infty} \left( ((k - p) + \beta(k + p - 2\gamma)) (\frac{k}{p})^\alpha (1 + \frac{k - p}{p + q})^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} \right) - \beta(2p - 2\gamma) \leq 0,
$$

by the assumption. This implies that $f \in T^{\alpha,\delta}_p(\mu, q, \gamma, \beta)$. 
Corollary 2.1. Let the function \( f \) be in the class \( T_{\alpha,\delta}^{\alpha,\beta}(\mu, q, \gamma, \beta) \), then

\[
a_k \leq \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))(\frac{k}{p})^\alpha(1 + \frac{k-p}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!(p+\delta)}}.
\]  

(2.2)

The result (2.2) is sharp for the function \( f \) of the form

\[
f(z) = z^p - \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))(\frac{k}{p})^\alpha(1 + \frac{k-p}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!(p+\delta)}} z^k.
\]

(2.3)

By using the same arguments as in the proof of Theorem 2.1, we can establish the next theorem.

Theorem 2.2. A function \( f \) belongs to the subclass \( C_{\alpha,\delta}^{\alpha,\beta}(\mu, q, \gamma, \beta) \), if, and only if,

\[
\sum_{k=p+1}^{\infty} \left( k[(k - p) + \beta(k + p - 2\gamma)](\frac{k}{p})^\alpha(1 + \frac{k-p}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!(p+\delta)} a_k z^k \right) \leq 2\beta p(p - \gamma),
\]

Corollary 2.2. Let the function \( f \) be in the class \( C_{\alpha,\delta}^{\alpha,\beta}(\mu, q, \gamma, \beta) \). Then

\[
a_k \leq \frac{2\beta p(p - \gamma)}{k[(k - p) + \beta(k + p - 2\gamma)](\frac{k}{p})^\alpha(1 + \frac{k-p}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!(p+\delta)}},
\]

with equality only for functions of the form

\[
f(z) = z^p - \frac{2\beta p(p - \gamma)}{k[(k - p) + \beta(k + p - 2\gamma)](\frac{k}{p})^\alpha(1 + \frac{k-p}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!(p+\delta)}} z^k.
\]
3. Distortion Properties

In this section, we obtain distortion bounds for the classes $T_{\alpha,\delta}^{\alpha,\delta}(\mu, q, \gamma, \beta)$ and $C_{\alpha,\delta}^{\alpha,\delta}(\mu, q, \gamma, \beta)$.

**Theorem 3.1.** If $f \in T_{\alpha,\delta}^{\alpha,\delta}(\mu, q, \gamma, \beta)$, then

$$|f(z)| \geq r^p - \frac{2\beta(p-\gamma)}{(1 + \beta(1 + 2p - 2\gamma))(p+1)^{\alpha}(1 + \frac{\lambda}{p+q})^\mu \Gamma(p+1+\delta)(p+1)!\Gamma(p+\delta)} r^{p+1}$$

(3.1)

and

$$|f'(z)| \geq pr^{p-1} - \frac{2\beta(p-\gamma)(p+1)}{(1 + \beta(1 + 2p - 2\gamma))(p+1)^{\alpha}(1 + \frac{\lambda}{p+q})^\mu \Gamma(p+1+\delta)(p+1)!\Gamma(p+\delta)} r^p$$

(3.3)

for $z \in \mathcal{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

**Proof:** Since $f \in T_{\alpha,\delta}^{\alpha,\delta}(\mu, q, \gamma, \beta)$, and in view of inequality (2.1) of Theorem 2.1, we have

$$(1 + \beta(1 + 2p - 2\gamma))(p+1)^{\alpha}(1 + \frac{\lambda}{p+q})^\mu \Gamma(p+1+\delta)(p+1)!\Gamma(p+\delta) \sum_{k=p+1}^{\infty} a_k \leq \sum_{k=p+1}^{\infty} \left( (k-p) + \beta(k+p-2\gamma) \frac{k}{p}(1 + \frac{k-p}{p+q}) \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k \right) \leq 2\beta(p-\gamma),$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{2\beta(p-\gamma)}{(1 + \beta(1 + 2p - 2\gamma))(p+1)^{\alpha}(1 + \frac{\lambda}{p+q})^\mu \Gamma(p+1+\delta)(p+1)!\Gamma(p+\delta)}.$$

(3.5)

Since

$$r^p - r^{p+1} \sum_{k=p+1}^{\infty} a_k \leq |f(z)| \leq r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k,$$

(3.6)

on using (3.5) and (3.6), we easily arrive at the desired results of (3.2) and (3.1). Furthermore, we observe that
\[ pr^{p-1} - (p + 1)r^p \sum_{k=p+1}^{\infty} a_k \leq |f'(z)| \leq pr^{p-1} + (p + 1)r^p \sum_{k=p+1}^{\infty} a_k, \quad (3.7) \]

On using (3.5) and (3.7), we easily arrive at the desired results of (3.3) and (3.4). Finally, we can see that the estimates for \(|f(z)|\) and \(|f'(z)|\) are sharp for the function,

\[ f(z) = z^p - \frac{2\beta(p - \gamma)}{(1 + (1 + 2p - 2\gamma))(1 + \frac{\lambda}{p+q})}(\frac{2^p(p + 1)}{(p+1)!}\Gamma(p+\delta)). \]

Similarly, we can prove the following theorem.

**Theorem 3.2.** If \( f \in C^\alpha_\delta(p, q, \gamma, \beta) \), then

\[ |f(z)| \geq r^p - \frac{2\beta p(p - \gamma)}{(p + 1)[1 + \beta(1 + 2p - 2\gamma)](\frac{p+1}{p})^\alpha(1 + \frac{\lambda}{p+q})\frac{\Gamma(p+1+\delta)}{(p+1)!}\Gamma(p+\delta)}r^{p+1}, \]

and

\[ |f'(z)| \geq pr^{p-1} - \frac{2\beta p(p - \gamma)(p + 1)}{[1 + \beta(1 + 2p - 2\gamma)](\frac{p+1}{p})^\alpha(1 + \frac{\lambda}{p+q})\frac{\Gamma(p+1+\delta)}{(p+1)!}\Gamma(p+\delta)}r^p, \]

for \( z \in U. \) The estimates for \(|f(z)|\) and \(|f'(z)|\) are sharp.

4. **Extreme Points**

**Theorem 4.1.** Let \( f_p(z) = z^p \) and,

\[ f_k(z) = z^p - \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))(\frac{k}{p})^\alpha(1 + \frac{(k-p)}{p+q})\frac{\Gamma(k+\delta)}{k!}\Gamma(p+\delta)}z^k. \]

Then \( f \) is in the class \( T^\alpha_\delta(p, q, \gamma, \beta) \), if, and only if, it can be expressed in the form

\[ f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z), \]
where
\[\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1.\] (4.1)

**Proof:** Let \( f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z) \)
\[f(z) = z^p - \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))(k)^{\alpha}(1 + \frac{(k-p)}{p+q} \lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} \omega_k z^k.\]

Then, in view of (4.1), it follows that
\[\sum_{k=p+1}^{\infty} \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))(k)^{\alpha}(1 + \frac{(k-p)}{p+q} \lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} \omega_k = \sum_{k=1}^{\infty} \omega_k = 1 - \omega_1 \leq 1.\]

Thus \( f \in T_{\alpha,\delta}^{(\mu, q, \gamma, \beta)}. \)

Conversely, assume that a function \( f \) defined by (1.4) belongs to class \( T_{\alpha,\delta}^{(\mu, q, \gamma, \beta)}. \) Then
\[a_k \leq \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))(k)^{\alpha}(1 + \frac{(k-p)}{p+q} \lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}.\]

We set
\[\omega_k = \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))(k)^{\alpha}(1 + \frac{(k-p)}{p+q} \lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} \omega_k\]
and \( \omega_k = 1 - \sum_{k=1}^{\infty} \omega_k. \) Then we have \( f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z) \), and hence completes the proof.

Similarly, we can prove the following result:

**Theorem 4.2.** Let \( f_p(z) = z^p \) and,
\[f_k(z) = z^p - \frac{2\beta p(p - \gamma)}{k[(k - p) + \beta(k + p - 2\gamma)](k)^{\alpha}(1 + \frac{(k-p)}{p+q} \lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} z^k.\]
Then $f$ is in the class $\mathcal{C}_p^{\alpha,\beta}(\mu, q, \gamma, \beta)$, if, and only if, it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z),$$

where

$$\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1.$$

Many other work on $p$-valent functions related to derivative operator and integral operator can be read in [8]-[10] and [11], respectively.

**Acknowledgement** The work here is supported by MOHE: UKM-ST-06-FRGS0244-2010.

**References**


