ON SUMS OF CONJUGATE SECONDARY RANGE HERMITIAN MATRICES

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Abstract: Necessary and Sufficient conditions for the sum of conjugate s-EP (Con-s-EP) matrices to be Con-s-EP are discussed. As an application it is shown that sum and sum of parallel summable Con-s-EP matrices are Con-s-EP.

Keywords: Transpose, Conjugate, Moore-Penrose inverse, Conjugate s-EP matrix, Sum of conjugate s-EP.

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1. Introduction

Throughout we shall deal with $C_{n \times n}$, let $A^T$, $A^*$ denote the transpose, conjugate transpose of A. Let $A^-$ be the generalized inverse of A satisfying $AA^-A=A$ and $A^\dagger$ be the Moore-Penrose inverse of A. Any Matrix $A \in C_{n \times n}$ is called Con-EP if $R(A)=R(A^T)$ or $N(A)=N(A^T)$ and is called Con-EP$_r$ if A is Con-EP and $rk(A)=r$ [2,3], where $N(A)$, $R(A)$ and $rk(A)$ denote null space, range space and rank of A respectively the conjugate k-EP by matrix was introduced [6], for $A \in C_{n \times n}$, if $R(A)=R(KA^T)$ is called Con-k-EP. If $A \in C_{n \times n}$ said to we, s-EP if $N(A)=N(A^*V)$ or $R(A)=R^* (VA^*)$ [5]. Throughout $V$ is a permutation matrix with units in the secondary diagonal. A matrix $A \in C_{n \times n}$ is said to be Con-s-EP if it satisfies the condition $Av(x)=0 \iff A^Tv(x)=0$ or equivalently $N(A)=N(A^TV)$ [1].

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In addition to that A is Con-s-EP ⇔ VA is Con-EP or AV is Con-EP. Moreover A is said to be Con-s-EP, if A is Con-s-EP, and rk(A)=r. In this paper, we give necessary and sufficient conditions of sums of Con-s-EP matrices to be Con-s-EP. As an application, it is shown that sum and parallel summable Con-s-EP matrices are Con-s-EP.

**Theorem 1.1**

Let $A_i$ (i = 1 to m) be Con-s-EP matrices. Then \[ A = \sum_{i=1}^{m} A_i \] is Con-s-EP if any one of the following equivalent conditions hold:

(i) \[ N(A) \subseteq N(A_i) \] for all i = 1 to m.

(ii) \[ \text{rk} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \text{rk}(A) \]

**Proof**

( i ) $\iff$ ( ii ):

\[ N(A) \subseteq N(A_i) \] for each i = 1 to m

\[ \Rightarrow N(A) \subseteq \bigcap N(A_i). \]

Since

\[ N(A) = N(\sum A_i) \subseteq N(A_1) \cap N(A_2) \cap N(A_3) \cap \ldots \cap N(A_m), \]

It follows that \(N(A) \subseteq \bigcap N(A_i)\). Hence,

\[ N(A) = \bigcap N(A_i) = N\left[\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array}\right] \]
Therefore, 

\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_m \\
\end{bmatrix}
\]

\[\text{rk}(A) = \text{rk} \begin{bmatrix}A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_m \end{bmatrix}\]

and (ii) holds.

Conversely, Since

\[
N \begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_m \\
\end{bmatrix} = \cap \text{N}(A_i) \subseteq \text{N}(A),
\]

\[
\text{rk} \begin{bmatrix}A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_m \end{bmatrix} = \text{rk}(A)
\]

\[\Rightarrow \text{N}(A) = \cap \text{N}(A_i)\]. Hence \(\text{N}(A) \subseteq \text{N}(A_i)\) for each i and (i) holds. Since each \(A_i\) is Con-s-EP,

\(N(A_i) \subseteq N(A_i^T V)\) for each i.

Now \(N(A) \subseteq N(A_i)\) for each i.

\[\Rightarrow N(A) \subseteq \cap N(A_i) = \cap N(A_i^T V) \subseteq N(A^T V)\] and

\[\text{rk}(A) = \text{rk}(A^T V)\].

Hence \(N(A) = N(A^T V)\).

Thus A is con-s-EP.
Remark 1.2

In particular, if A is non singular then the conditions automatically hold and A is con-s-EP. Theorem (1.1) fails if we relax the conditions on \( A_i \)'s.

Example 1.3

Consider, \[ A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

\[ V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

\[ VA_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

is Con-EP Therefore \( A_1 \) is Con-s-EP.

\[ VA_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \]

is not Con EP

Therefore \( A_2 \) is not Con-s-EP.

\[ A_1 + A_2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

\[ VA_1 + A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \]

is not Con-EP.

Therefore \( A_1 + A_2 \) is not Con-s-EP.

However \( N(A_1 + A_2) \subseteq N \left[ A_1^T V \right] \subseteq N(A_1) \) and \( N(A_1 + A_2) \subseteq N \left[ A_2^T V \right] \subseteq N(A_2) \),

Moreover \( r_k \left[ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right] = r_k \left[ A_1 + A_2 \right] \).

Remark 1.4

If rank is additive, that is \( rk(A) = \sum rk(A_i) \) then by Theorem (2.8)[1],
R(A_i) \cap R(A_j) = \{ 0 \}, i \neq j \text{ which implies that}

N(A) \subseteq N(A_i) \quad \text{for each } i

\Rightarrow N(A) \subseteq N(A_i^T V) \text{ for each } i.

Hence A is con-s-EP.

The conditions given in Theorem (1.1) are weaker than the condition of rank additivity can been seen by the following example.

**Example 1.5**

Let \( A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ i & 0 & 1 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \)

For \( V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \), \( A_1, A_2 \) and \( A_1 + A_2 \) Con-s-EP, Matrices, conditions (i) and (ii) in Theorem (1.1) hold but \( \text{rk}(A_1 + A_2)^T r_k(A_1) + \text{rk}(A_2) \).

**Theorem 1.6**

Let \( A_i, \ (i = 1 \text{ to } m) \text{ be con-s-EP matrices such that } \sum_{i \neq j} A_i^T A_j = 0 \text{ then } A = \sum A_i \) is con-s-EP.

**Proof**

Since \( \sum_{i \neq j} A_i^T A_j = 0 \)

\( A^T A = ( \sum A_i )^T ( \sum A_i ) \)

\( = ( \sum A_i^T ) ( \sum A_i ) \)

\( = \sum A_i^T A_i \)

\( N(A) = N(A^T A) \)
\[ \mathcal{N}( \sum A_i^T A_i ) = \mathcal{N}( A_{1}^T V ) \cap \mathcal{N}( A_{2}^T V ) \cap \mathcal{N}( A_{3}^T V ) \cap \ldots \cap \mathcal{N}( A_{m}^T V ) \]

Hence \( \mathcal{N}( A ) \subseteq \mathcal{N}( A_i^T V ) \) for each \( i \).

Now, \( A \) is con-s-EP follows from Theorem (1.1).

**Remark 1.7**

Theorem (1.6) fails if we relax the condition that \( A_i \)'s are Con-s-EP.

For, Let \( A_1 = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ i & 0 & 0 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & -i & 0 \end{bmatrix} \)

\[ V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]
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\[ V A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ i & 0 & 0 \end{bmatrix} \] is not Con-EP.

Therefore \( A_1 \) is not Con-s-EP.

\[ V A_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{bmatrix} \] is not Con-EP.

Therefore \( A_2 \) is not Con-s-EP.

\[ A_1 + A_2 = \begin{bmatrix} i & i & 0 \\ i & -i & 0 \\ 0 & -i & 0 \end{bmatrix} \]

\[ V(A_1 + A_2) = \begin{bmatrix} 0 & -i & 0 \\ i & -i & 0 \\ i & i & 0 \end{bmatrix} \] is not Con-s-EP.

Therefore \( A_1 + A_2 \) is not Con-s-EP.

But \( A_1^T A_2 + A_2^T A_1 \neq 0 \).

**Remark 1.8**

The condition given in Theorem (1.6) implies those in Theorem (1.1) but not conversely, this can be seen by the following example.

**Example 1.9**

Let \( A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \)

For \( V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \), \( V A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \) is Con-EP. Therefore \( A_1 \) is Con-s-EP.
VA_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}
is con-EP.

Therefore A_2 is Con-s-EP. N(A_1 + A_2) \subseteq N(A_1) and N(A_2).

But A_1^T A_2 + A_2^T A_1 \neq 0.

**Remark 1.10**

The conditions given in Theorem (1.1) and Theorem (1.6) are only sufficient for sum of con-s-EP to be con-s-EP, but not necessary is illustrated by the following example.

**Example 1.11**

Let \( A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \)

\( V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \)

A_1 and A_2 are Con-s-EP. Neither the conditions in Theorem (1.1) Theorem (1.6) hold \( (A_1 + A_2) \) is a Con-s-EP.

**Theorem 1.12**

Let \( A^T = H_1 V A \) and \( B^T = H_2 V B \) such that \( (H_1 - H_2) \) is non singular and \( V \) is permutation matrix with units in the secondary diagonal, then \( (A + B) \) is Con-s-EP \( \iff N(A + B) \subseteq N(B) \).

**Proof**

Since \( A^T = H_1 V A \) and \( B^T = H_2 V B \) by Theorem (2.8) in [1], A and B are con-s-EP matrices. Since \( N(A + B) \subseteq N(B) \), by Theorem (1.1), \((A + B)\) is Con-s-EP.

Conversely, let us assume that \((A + B)\) is con-s-EP. By Theorem (2.8) in [1] there exists a non singular matrix \( G \) such that
(A + B)^T = GV(A + B)V
A^T + B^T = GV(A + B)V

⇒ H_1VAV + H_2VBV = GV(A + B)V
⇒ (H_1VA + H_2VB)V = GV(A + B)V
⇒ (H_1VA + H_2VB) = GV(A + B)
⇒ H_1VA + H_2VB = GVA + GVB
⇒ (H_1V - GV)A = (GV - H_2V)B
⇒ (H_1 - G)VA = (G - H_2)VB
⇒ LVA = MVB,

Where, \( L = H_1 - G, \quad M = G - H_2. \)

Now \((L + M)(VA) = LVA + MVA = MVB + MVA = MV(A + B)\)

and similarly \((L + M)(VB) = LV(A + B)\).

By hypothesis,

\( L + M = H_1 - G + G - H_2 = H_1 - H_2 \) is non singular.

Therefore, \( N(A + B) \subseteq N(MV(A + B)) = N((L + M)VA) = N(VA) = N(A). \)

Therefore,

\( N(A + B) \subseteq N(A). \)

Also,

\( N(A + B) \subseteq N(LV(A + B)) = N((L + M)VB) = N(VB) = N(B). \)

Therefore, \( N(A + B) \subseteq N(B) \)

Thus \((A + B)\) is Con-s-EP.
\[ \Rightarrow \quad N(A + B) \subseteq N(A) \quad \text{and} \quad N(A + B) \subseteq N(B). \]

**Remark 1.13**

The condition \((H_1 - H_2)\) to be non singular is essential in Theorem (1.12) is illustrated in the following example.

**Example 1.14**

Let \( V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)

\[ A_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} \]

are both Con-s-EP Matrices.

Here, \( H_1 = H_2 \). \( A_1 + A_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \) is Con-s-EP.

But, \( N(A_1 + A_2) \not\subseteq N(A_1) \) or \( N(A_2) \). Thus Theorem (1.12) fails.

**PARALLEL SUMMABLE CONJUGATE s-EP MATRICES**

Here, it is shown that sum and parallel sum of parallel summable Con-s-EP matrices are Con-s-EP.

**Lemma 1.15**

Let \( A_1 \) and \( A_2 \) be Con-s-EP matrices. Then \( A_1 \) and \( A_2 \) are p.s. if and only if \( N(A_1 + A_2) \subseteq N(A_i) \) for some (and hence both) \( i \in \{1, 2\} \).

**Proof**

\( A_1 \) and \( A_2 \) are parallel summable implies \( N(A_1 + A_2) \subseteq N(A_i) \) follows from the **Definition in [7]**.

Conversely, if \( N(A_1 + A_2) \subseteq N(A_i) \) then \( N(VA_1 + VA_2) \subseteq N(VA_i) \). Also, \( N(VA_1 + VA_2) \subseteq N(VA_2) \). Since \( A_1 \) and \( A_2 \) are Con-s-EP matrices, by Theorem (2.8) in [1] \( VA_1 \) and \( VA_2 \) are Con-EP matrices.
Since $V_{A_1}$ and $V_{A_2}$ are Con-EP matrices, by Theorem in [4], $V_{A_1} + V_{A_2}$ is Con-EP.

Hence $N(V_{A_1} + V_{A_2})^T = N(V_{A_1} + V_{A_2})$

$= N(V_{A_1}) \cap (V_{A_2})$

$= N(V_{A_1})^T \cap (V_{A_2})^T$

This implies, $N(V_{A_1} + V_{A_2})^* = N(V_{A_1})^* \cap N(V_{A_2})^*$.

Therefore $N(V_{A_1} + V_{A_2})^* \subseteq N(V_{A_1})^*$ and $N(V_{A_1} + V_{A_2})^* \subseteq N(V_{A_2})^*$. Also, by hypothesis, $N(V_{A_1} + V_{A_2}) \subseteq N(V_{A_1})$. Hence by Definition in [7] $V_{A_1}$ and $V_{A_2}$ are p.s.

$$N(V_{A_1} + V_{A_2}) \subseteq N(V_{A_1})$$

$$\Rightarrow N(V_{A_1} + V_{A_2}) \subseteq N(V_{A_1})$$

Similarly, $N(V_{A_1} + V_{A_2})^* \subseteq N(V_{A_1})^*$. Therefore $A_1$ and $A_2$ are p.s.

**Remark 1.16:**

**Lemma (1.15)** fails if we relax the condition that $A_1$ and $A_2$ are Con-s-EP. Let

$$A_1 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \quad \text{Let the associated permutation matrix be}$$

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad A_1 \text{ is Con-s-EP and } A_2 \text{ is not Con-s-EP. } N(A_1 + A_2) \subseteq N(A_1) \text{ and } N(A_1 + A_2) \subseteq N(A_2), \quad \text{but } N(A_1 + A_2)^* \not\subseteq N(A_1^*), \quad N(A_1 + A_2)^* \not\subseteq N(A_2^*).$$

Hence $A_1$ and $A_2$ are not parallel summable.

**Theorem 1.17**

Let $A_1$ and $A_2$ be p.s. Con-s-EP matrices. Then $A_1, A_2$ and $A_1 + A_2$ are Con-s-EP.

**Proof**
Since $A_1$ and $A_2$ are p. s. Con-s-EP matrices, by Lemma (1.15), $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2)$. Now the fact that $A_1 + A_2$ is Con-s-EP follows from Theorem (1.1).

Since $A_1$ and $A_2$ are p.s. Con-s-EP matrices, $VA_1$ and $VA_2$ are p.s. Con-EP matrices. Therefore $R(VA_1)^T = R(VA_1)$ and $R(VA_2)^T = R(VA_2)$.

$$R(VA_1 : VA_2)^T = R((VA_1)^T : (VA_2)^T)$$

(by Theorem in [7])

$$= R((VA_1)^T) \cap R((VA_2)^T)$$

(by Theorem in [7])

$$= R(VA_1) \cap R(VA_2)$$

(since $VA_1$ and $VA_2$ are Con-EP)

$$= R(VA_1 : VA_2)$$

(by Theorem in [7])

Thus $VA_1 : VA_2$ is Con-EP

$$\Rightarrow K(A_1 : A_2)$$ is Con-EP

$$\Rightarrow A_1 : A_2$$ is Con-s-EP

(by Theorem in [1])

Thus $A_1 : A_2$ is Con-s-EP matrix whenever $A_1$ and $A_2$ are Con-s-EP.

REFERENCE:


