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ON SUMS OF CONJUGATE SECONDARY RANGE HERMITIAN MATRICES

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Abstract: Necessary and Sufficient conditions for the sum of conjugate s-EP (Con-s-EP) matrices to be Con-s-EP are discussed. As an application it is shown that sum and sum of parallel summable Con-s-EP matrices are Con-s-EP.

Keywords: Transpose, Conjugate, Moore-Penrose inverse, Conjugate s-EP matrix, Sum of conjugate s-EP.

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1. Introduction

Throughout we shall deal with $C_{n\times n}$, let A^T, A^* denote the transpose, conjugate transpose of A. Let A^- be the generalized inverse of A satisfying $AA^-A=A$ and $A^\dagger$ be the Moore-Penrose inverse of A. Any Matrix $A\in C_{nxn}$ is called Con-EP if $R(A)=R(A^T)$ or $N(A)=N(A^T)$ and is called Con-EP_r if A is Con-EP and rk(A)=r [2,3], where N(A), R(A) and rk(A) denote null space, range space and rank of A respectively the conjugate k-EP by matrix was introduced [6], for $A\in C_{nxn}$, if $R(A)=R(KA^T)$ is called Con-k-EP. If $A\in C_{nxn}$ said to we, s-EP if $N(A)=N(A^*V)$ or $R(A)=R^*(VA^*)$ [5]. Throughout V is a permutation matrix with units in the secondary diagonal. A matrix $A\in C_{nxn}$ is said to be Con-s-EP if it satisfies the condition $Av(x)=0 \Leftrightarrow A^Tv(x)=0$ or equivalently $N(A)=N(A^TV)$ [1].

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In addition to that A is Con-s-EP $\Leftrightarrow$ VA is Con-EP or AV is Con-EP. Moreover A is said to be Con-s-EP_r if A is Con-s-EP_r and rk(A)=r. In this paper, we give necessary and sufficient conditions of sums of Con-s-EP matrices to be Con-s-EP. As an application, it is shown that sum and parallel summable Con-s-EP matrices are Con-s-EP.

Theorem 1.1

Let A_i (i = 1 to m) be Con-s-EP matrices. Then $A = \sum_{i=1}^{m} A_i$ is Con-s-EP if any one of the following equivalent conditions hold:

(i) $N(A) \subseteq N(A_i)$ for all i = 1 to m.

(ii) $rk\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix} = rk(A)$

Proof

(i) $\Leftrightarrow$ (ii) :

$N(A) \subseteq N(A_i)$ for each i = 1 to m

$\Rightarrow N(A) \subseteq \cap N(A_i)$.

Since

$N(A) = N(\sum A_i) \subseteq N(A_1) \cap N(A_2) \cap N(A_3) \cap \ldots . \cap N(A_m)$,

It follows that $N(A) \subseteq \cap N(A_i)$ Hence ,

$$N(A) = \cap N(A_i) = N\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix}$$

Therefore,

$$rk(A) = rk\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix} \text{ and (ii) holds.}$$

Conversely, Since

$$N\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix} = \cap N(A_i) \subseteq N(A),$$

$$rk\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix} = rk(A)$$

$\Rightarrow N(A) = \cap N(A_i)$.Hence $N(A) \subseteq N(A_i)$ for each i and (i) holds. Since each A_i is Con-s-EP,

$N(A_i) \subseteq N(A_i^T V)$ for each i.

Now $N(A) \subseteq N(A_i)$ for each i.

$\Rightarrow N(A) \subseteq \cap N(A_i) = \cap N(A_i^T V) \subseteq N(A^T V)$ and

$rk(A) = rk(A^T V)$.

Hence $N(A) = N(A^T V)$.

Thus A is con-s-EP.

Remark 1.2

In particular, if A is non singular then the conditions automatically hold and A is con-s-EP. Theorem (1.1) fails if we relax the conditions on A_i's.

Example 1.3

Consider, $$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$VA_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ is Con-EP Therefore A_1 is Con-s-EP.

$$VA_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$ is not Con EP

Therefore A_2 is not Con-s-EP.

$$A_1 + A_2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ V(A_1 + A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$ is not Con-EP.

Therefore A_1+A_2 is not Con-s-EP.

However $N(A_1 + A_2) \subseteq N\left[A_1^T V\right] \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N\left[A_2^T V\right] \subseteq N(A_2)$,

Moreover $r_k\left[\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\right] = r_k\left[A_1 + A_2\right]$.

Remark 1.4

If rank is additive, that is $rk(A) = \sum rk(A_i)$ then by Theorem (2.8)[1],

$R(A_i) \cap R(A_j) = \{0\}$, $i \neq j$ which implies that

$N(A) \subseteq N(A_i)$ for each i

$\Rightarrow$ $N(A) \subseteq N(A_i^T V)$ for each i.

Hence A is con-s-EP.

The conditions given in Theorem (1.1) are weaker than the condition of rank additivity can been seen by the following example.

Example 1.5

Let $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ i & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

For $V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, A_1, A_2 and A_1+A_2 Con-s-EP, Matrices, conditions (i) and (ii) in Theorem (1.1) hold but $rk(A_1 + A_2)^1 r_k(A_1) + rk(A_2)$.

Theorem 1.6

Let A_i, (i = 1 to m) be con-s-EP matrices such that $\sum_{i \neq j} A_i^T A_j = 0$ then $A = \sum A_i$ is con-s-EP.

Proof

Since $\sum_{i \neq j} A_i^T A_j = 0$

$$A^T A = (\sum A_i)^T (\sum A_i)$$

$$= (\sum A_i^T)(\sum A_i)$$

$$= \sum A_i^T A_i$$

$$N(A) = N(A^T A)$$

$$= N(\sum A_i^T A_i)$$

$$= N\left(\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix}^T \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix}\right)$$

$$= N\left(\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ . \\ A_m \end{bmatrix}\right)$$

$$= N(A_1) \cap N(A_2) \cap N(A_3) \cap \ldots \cap N(A_m)$$

$$= N(A_1^T V) \cap N(A_2^T V) \cap N(A_3^T V) \cap \ldots \cap N(A_m^T V)$$

Hence $N(A) \subseteq N(A_i^T V)$ for each i.

$= N(A_i)$ for each i

Now, A is con-s-EP follows from Theorem (1.1).

Remark 1.7

Theorem (1.6) fails if we relax the condition that A_i's are Con-s-EP.

For, Let $A_1 = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ i & 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix}$

$$V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$VA_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ i & 0 & 0 \end{bmatrix} \text{ is not Con-EP.}$$

Therefore A_1 is not Con-s-EP.

$$VA_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{bmatrix} \text{ is not Con-EP.}$$

Therefore A_2 is not Con-s-EP.

$$A_1 + A_2 = \begin{bmatrix} i & i & 0 \\ i & -i & 0 \\ 0 & -i & 0 \end{bmatrix}$$

$$V(A_1 + A_2) = \begin{bmatrix} 0 & -i & 0 \\ i & -i & 0 \\ i & i & 0 \end{bmatrix} \text{ is not Con-s-EP.}$$

Therefore $A_1 + A_2$ is not Con-s-EP.

But $A_1^T A_2 + A_2^T A_1 \neq 0$.

Remark 1.8

The condition given in Theorem (1.6) implies those in Theorem (1.1) but not conversely, this can been seen by the following example.

Example 1.9

Let $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

For $V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $VA_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is Con-EP. Therefore A_1 is Con-s-EP.

$$VA_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is con-EP.}$$

Therefore A_2 is Con-s-EP. $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_2)$.

But $A_1^T A_2 + A_2^T A_1 \neq 0$.

Remark 1.10

The conditions given in Theorem (1.1) and Theorem (1.6) are only sufficient for sum of con-s-EP to be con-s-EP, but not necessary is illustrated by the following example.

Example 1.11

Let $A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ A_1 and A_2 are Con-s-EP, Neither the conditions in Theorem (1.1) Theorem (1.6) hold (A_1+A_2) is a Con-s-EP.

Theorem 1.12

Let $A^T = H_1VAV$ and $B^T = H_2VBV$ such that $(H_1 - H_2)$ is non singular and V is permutation matrix with units in the secondary diagonal, then (A + B) is Con-s-EP $\Leftrightarrow N(A + B) \subseteq N(B)$.

Proof

Since $A^T = H_1VAV$ and $B^T = H_2VBV$ by Theorem (2.8) in [1], A and B are con-s-EP matrices. Since $N (A + B) \subseteq N (B)$, by Theorem (1.1), (A + B) is Con-s-EP. Conversely, let us assume that (A + B) is con-s-EP. By Theorem (2.8) in [1] there exists a non singular matrix G such that

$$(A + B)^T = GV(A + B)V$$

$$A^T + B^T = GV(A + B)V$$

$$\Rightarrow \quad H_1VAV + H_2VBV = GV(A + B)V$$

$$\Rightarrow \quad (H_1VA + H_2VB)V = GV(A + B)V$$

$$\Rightarrow \quad (H_1 VA + H_2VB) = GV(A + B)$$

$$\Rightarrow \quad H_1 VA + H_2VB = GVA + GVB$$

$$\Rightarrow \quad (H_1V - GV)A = (GV - H_2V)B$$

$$\Rightarrow \quad (H_1 - G)VA = (G - H_2)VB$$

$$\Rightarrow \quad LVA = MVB,$$

Where, $L = H_1 - G$, $M = G - H_2$.

Now $(L + M)(VA) = LVA + MVA$

$$= MVB + MVA$$

$$= MV(A + B)$$

and similarly $(L + M)(VB) = LV(A + B)$.

By hypothesis,

$$L + M = H_1 - G + G - H_2$$

$$= H_1 - H_2 \text{ is non singular.}$$

Therefore , $N(A + B) \subseteq N(MV(A + B))$

$$= N((L + M)VA)$$

$$= N(VA)$$

$$= N(A).$$

Therefore,

$$N(A + B) \subseteq N(A).$$

Also ,

$$N(A + B) \subseteq N(LV(A + B))$$

$$= N((L + M)VB)$$

$$= N(VB)$$

$$= N(B)$$

Therefore, $N(A + B) \subseteq N(B)$

Thus $(A + B)$ is Con-s-EP.

$\Rightarrow$ $N(A + B) \subseteq N(A)$ and $N(A + B) \subseteq N(B)$.

Remark 1.13

The condition $(H_1 - H_2)$ to be non singular is essential in Theorem (1.12) is illustrated in the following example.

Example 1.14

Let $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix}$ are both Con-s-EP Matrices.

Here, $H_1=H_2$. $A_1 + A_2 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$ is Con-s-EP.

But, $N(A_1 + A_2) \not\subseteq N(A_1)$ or $N(A_2)$. Thus Theorem (1.12) fails.

PARALLEL SUMMABLE CONJUGATE s-EP MATRICES

Here, it is shown that sum and parallel sum of parallel summable Con-s-EP matrices are Con-s-EP.

Lemma 1.15

Let A_1 and A_2 be Con-s-EP matrices. Then A_1 and A_2 are p.s. if and only if $N(A_1 + A_2) \subseteq N(A_i)$ for some (and hence both) $i \in \{1, 2\}$.

Proof

A_1 and A_2 are parallel summable implies $N(A_1 + A_2) \subseteq N(A_1)$ follows from the **Definition in [7].**

Conversely, if $N(A_1 + A_2) \subseteq N(A_1)$ then $N(VA_1 + VA_2) \subseteq N(VA_1)$. Also, $N(VA_1 + VA_2) \subseteq N(VA_2)$. Since A_1 and A_2 are Con-s-EP matrices, by Theorem (2.8) in [1] VA_1 and VA_2 are Con-EP matrices.

Since VA_1 and VA_2 are Con-EP matrices, by Theorem in [4], VA_1+VA_1 is Con-EP.

Hence $N(VA_1 + VA_2)^T = N(VA_1 + VA_2)$

$$= N(VA_1) \cap (VA_2)$$

$$= N(VA_1)^T \cap (VA_2)^T$$

This implies, $N(VA_1 + VA_2)^* = N(VA_1)^* \cap N(VA_2)^*$.

Therefore $N(VA_1 + VA_2)^* \subseteq N(VA_1)^*$ and $N(VA_1 + VA_2)^* \subseteq N(VA_2)^*$. Also, by hypothesis, $N(VA_1 + VA_2) \subseteq N(VA_1)$. Hence by Definition in [7] VA_1 and VA_2 are p.s.

$$N(VA_1 + VA_2) \subseteq N(VA_1)$$
$$\Rightarrow N(V(A_1 + A_2)) \subseteq N(VA_1)$$
$$\Rightarrow N(A_1 + A_2) \subseteq N(A_1)$$

Similarly, $N(A_1 + A_2)^* \subseteq N(A_1)^*$. Therefore A_1 and A_2 are p.s.

Remark 1.16 :

Lemma (1.15) fails if we relax the condition that A_1 and A_2 are Con-s-EP. Let $A_1 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$. Let the associated permutation matrix be $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A_1 is Con-s-EP and A_2 is not Con-s-EP. $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2)$, but $N(A_1 + A_2)^* \not\subseteq N(A_1^*)$, $N(A_1 + A_2)^* \not\subseteq N(A_2^*)$. Hence A_1 and A_2 are not parallel summable.

Theorem 1.17

Let A_1 and A_2 be p.s. Con-s-EP matrices. Then A_1: A_2 and A_1+A_2 are Con-s-EP.

Proof

Since A_1 and A_2 are p. s. Con-s-EP matrices, by Lemma (1.15), $N(A_1+A_2)\subseteq N(A_1)$ and $N(A_1+A_2)\subseteq N(A_2)$. Now the fact that A_1+A_2 is Con-s-EP follows from Theorem (1.1).

Since A_1 and A_2 are p.s. Con-s-EP matrices, VA_1 and VA_2 are p.s. Con-EP matrices. Therefore $R(VA_1)^T = R(VA_1)$ and $R(VA_2)^T = R(VA_2)$.

$R(VA_1 : VA_2)^T = R((VA_1)^T : (VA_2)^T)$ (by Theorem in [7])

$= R((VA_1)^T) \cap R((VA_2)^T)$ (by Theorem in [7])

$= R(VA_1) \cap R(VA_2)$ (since VA_1 and VA_2 are Con-EP)

$= R(VA_1 : VA_2)$ (by Theorem in [7])

Thus $VA_1 : VA_2$ is Con-EP

$\Rightarrow K(A_1 : A_2)$ is Con-EP

$\Rightarrow A_1 : A_2$ is Con-s-EP (by Theorem in [1])

Thus $A_1{:}A_2$ is Con-s-EP matrix whenever A_1 and A_2 are Con-s-EP.

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