THE STUDY OF RAY-KNIGHT COMPACTIFICATION ON TRANSFER FUNCTION

CHEN- LI*, FAN- ZHONG-GUANG

School of Mathematics and Statistics Zhengzhou Normal University, Zhengzhou City, Henan Province, 450044, China

Abstract: In order to construct a Markov chain of strong Markov, we need the state space of the compactification. The paper uses the properties of the resolvent operator to study some properties of Ray-Knight compactifications from a given transfer function on state space E.

Key words: transfer function; Ray resolvent; Ray-Knight Compactifications

2010 Mathematical Subject Classification: 60J10

1 Introduction

Construction of the transfer function of the given E is an important part of research of Markov chain. The state space in the canonical chain on \( E \cup \infty \) is very simple, but its orbit has only lower semicontinuity, which just keep part of the strong Markov. In order to construct a Markov chain of strong Markov, the State space of the compactification is needed. We will study the properties of Ray-Knight Compactifications from a given transfer function on state space E.

2 Preliminary knowledge

Let \( E = \{1, 2, \cdots\} \), and the topology on E is discrete topology, then E is

*Corresponding author

Received February 8, 2013
L.C.C.B (Locally compact and has a countable topological group). The elements of E are called state. $\mathcal{E}$ is a Borel algebra on E, $(E, \mathcal{E})$ is a topological space, and $C_b(E)$ represents all bounded continuous functions. If E is a compact metric space, $C_b(E)$ abbreviated $C(E)$. Remove the dense subset $\{g_m\}_{m=1}^\infty$ of $\mathcal{R}$, set $d(x, y) = \sum_{m=1}^\infty \frac{1}{2^m} \Lambda |g_m(x) - g_m(y)|$, $\forall x, y \in E$, $d(\cdot, \cdot)$ is the metric on E, completion of E under $d(\cdot, \cdot)$ written $\overline{E}$. It is obvious that $\overline{E}$ is a compact metric space. The metric on $\overline{E}$ still denoted as $d(\cdot, \cdot)$. $(U^\alpha)_{\alpha > 0}$ is the Ray resolvent operator on $\overline{E}$, D is a no branch point set.

**Definition 1:** $(\overline{E}, d)$ is called Ray-Knight compactification on E.

**Definition 2:** If $p_{ij}(t) \geq 0; \sum_{k=1}^\infty p_{ik}(t) \leq 1; p_{ij}(t+s) = \sum_{k=1}^\infty p_{ik}(t)p_{kj}(s)$ establish, then the function $P(t) = (p_{ij}(t))_{i, j \in E}$ on $[0, \infty)$ is called the transfer function on E.

If $\lim_{t \to 0} p_{ij}(t) = p_{ij}(0) = \delta_{ij}$, then $P(t)(t \geq 0)$ is called the standard. If $\forall i \in E, \sum_{k=1}^\infty p_{ik}(t) = 1$, Then $P(t)(t \geq 0)$ is said to be honest.

**Definition 3:** If $0 \leq q_{ij} < \infty, 0 \leq q_i \equiv -q_{ii} \leq \infty, \sum_{k \neq i} q_{ik} \leq q_i, \forall i, j \in E$, then $Q = (q_{ij})_{i, j \in E}$ is called density matrix of $P(t)(t \geq 0)$. If $q_i < \infty$, then $i$ is called stable state of $P(t)$. Otherwise $i$ is called instantaneous state of $P(t)$. If $\sum_{k \neq i} q_{ik} = q_i$, then $i$ is called conservative state of $P(t)$. If all States are stable, then $P(t)$ is said to be fully stabilized. If all States are conservative, then $P(t)$ is
said to be fully conservative.

Definition 4: Let $P(t)(t \geq 0)$ is the transfer function on $E$, $R_{ij}(\lambda)$ is called resolvent of $P(t)(t \geq 0)$. Among then $R_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt, i, j \in E, \lambda > 0$.

Lemma 1: $R_{ij}(\lambda)$ is the resolvent of transfer function $P(t)(t \geq 0)$, if and only if $\lambda \sum_{k \in E} R_{ik}(\lambda) \leq 1; R_{ij}(\lambda) - R_{ij}(\mu) + (\lambda - \mu) \sum_{k \in E} R_{ik}(\lambda)R_{kj}(\mu) = 0$;

$$\lim_{\lambda \to \infty} \lambda R_{ij}(\lambda) = \delta_{ij}, \lim_{\lambda \to \infty} \lambda [\lambda R_{ij}(\lambda) - \delta_{ij}] = q_{ij}.$$  

Proof: Reference[2]

Lemma 2 If $Q = (q_{ij})_{i,j \in E}$ is density matrix of $P(t)(t \geq 0)$, and $P(t)$ is fully stabilized, then

$$p_{ij}^{(1)} \geq -q_{ij} p_{ij}(t) + \sum_{k \neq i} q_{ik} p_{kj}(t)$$
$$p_{ij}^{(1)} \geq -p_{ij}(t) q_{ij} + \sum_{k \neq j} p_{ik}(t) q_{kj}$$

Proof: From the definition of density matrix and the whole stability can be directly obtained the conclusion.

Note1 If the two formulas in lemma 2 an equality, we get two group of linear differential equation group, Are called backward equations and the forward equations.

Lemma 3 Let $Q = (q_{ij})_{i,j \in E}$ is the whole stability density matrix of $P(t)(t \geq 0)$, set

$$f_{ij}^{(0)}(t) = \delta_{ij} e^{-q_{ij} t}, f_{ij}^{(n)}(t) = \sum_{k \neq i} \int_0^t e^{-q_{ik}s} q_{ik} f_{kj}^{(n-1)}(t-s)ds, n = 1, 2, \cdots$$

$$f_{ij}^{(0)}(t) = \delta_{ij} e^{-q_{ij} t}, f_{ij}^{(n)}(t) = \sum_{k \neq j} \int_0^t f_{ik}^{(n-1)}(s)q_{kj} e^{-q_{ij}(t-s)}ds, n = 1, 2, \cdots$$

$$p_{ij}^{(n)}(t) = \sum_{n=0}^{\infty} f_{ij}^{(n)}(t)$$
Then (1) \( P_{\min}(t) = (p_{ij\min}(t)) \) is transfer function on \( E \), its density matrix is \( Q \).

(2) \( P_{\min}(t) \) is minimum, that is \( p_{ij}(t) \geq p_{ij\min}(t), \forall i, j \in E, t \geq 0 \).

(3) \( P_{\min}(t) \) satisfy the forward equations and backward equations.

**Proof:** Reference[1]

**Note2:** \( P_{\min}(t) \) is called the minimum transfer function.

### 3 Main results

If the transfer function \( P(t) \) is not honest, set \( \Delta \notin E \), and

\[
E_\Delta = E \cup \{ \Delta \}, p_{\Delta \Delta}(t) = 1, p_{\Delta i}(t) = 0, p_{i\Delta} = 1 - \sum_{k \in E} p_{ik}(t), \forall i \in E
\]

Then \( P(t) = (p_{ij}(t))_{i,j \in E_\Delta} \) is the honest transfer function on \( E_\Delta \), so the \( P(t) \) can be transformed into \( P(t) \) to discuss.

Let \( P(t) = (p_{ij}(t))_{i,j \in E} \) is the honest transfer function on \( E \). \( \forall f \in M \), the function \( i \mapsto \sum_{k \in E} R_{ik}(\lambda) f(k) \) on \( E \) is noted \( R_\lambda f \).

**Theorem 1:** \( \{R_\lambda\}_{\lambda > 0} \) is the Markov resolvent on \( E \), and have:

1. \( E \) is L.C.C.B topological space.
2. \( \{R_\lambda\}_{\lambda > 0} \) is Markov, and \( R_\lambda C_b(E) \subset C_b(E), \forall \lambda > 0 \).
3. For any \( f \in C_b(E) \), and \( x \in E \), \( \lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x) \)

**Proof:** (1) and (2) is obvious. We proof (3):

\[
\lambda R_\lambda f(i) = \lambda \sum_{k \in E} R_{ik}(\lambda) f(k) = \lambda \int_0^\infty e^{-\lambda t} \sum_{k \in E} p_{ik}(t) f(k) dt
\]

\[
= \lambda \int_0^\infty e^{-\lambda t} p_{ii}(t) f(i) dt + \lambda \int_0^\infty e^{-\lambda t} \sum_{k \neq i} p_{ik}(t) f(k) dt
\]

Because \( \lim_{t \to 0} p_{ii}(t) = 1 \), when \( \lambda \to \infty \), \( \int_0^\infty e^{-\lambda t} p_{ii}(t) f(i) dt \to f(i) \);

\[
\int_0^\infty e^{-\lambda t} \sum_{k \neq i} p_{ik}(t) f(k) dt \leq \|f\| \int_0^\infty e^{-\lambda t} (1 - p_{ii}(t)) dt = 0
\]
Theorem 2: Let \((
\bar{E}, d(\cdot, \cdot))\) is Ray-Knight compactifications on \(E\), then:

1. \(E \subseteq D\)

2. For any \(i \in E, \alpha > 0, U^\alpha(i, \bar{E} \setminus E) = 0\), and for any \(k \in E,\)

\[
U^\alpha(i, \{k\}) = R_{ik}(\alpha)
\]

3. For any \(i \in E, t > 0, P_t(i, \bar{E} \setminus E) = 0\), and for any \(k \in E,\)

\[
P_t(i, \{k\}) = p_{ik}(t)
\]

Proof: (1) For any \(i \in E, \alpha > 0, f, g \in \mathcal{R}\), because \(U^\alpha f, U^\alpha g\) are \(R_\alpha f, R_\alpha g\) expansion in the \(E\), so

\[
U^\alpha(f - g)(i) = R_\alpha f(i) - R_\alpha g(i) = \sum_{k \in E} R_{ik}(\alpha)[f(k) - g(k)]
\]

For any \(f \in C(\bar{E})\), \(U^\alpha f(i) = \sum_{k \in E} R_{ik}(\alpha)f(k) = R_\alpha f(i)\), because \(R_\alpha\) meet three conclusions of the theorem 1, so we have \(\lim_{\alpha \to \infty} \alpha U^\alpha f(i) = f(i)\), so that \(i \in D\). For \(i\) is optional nature, we have \(E \subseteq D\).

(2) For bounded measurable function \(f\) on any \(\bar{E}\), it is obvious:

\[
U^\alpha f(i) = R_\alpha f(i) = \sum_{k \in E} R_{ik}(\alpha)f(k)
\]

For any \(k \in E\), Use the characteristic function \(I_k(\cdot)\) of \(\{k\}\) instead of \(f\), we get \(U^\alpha(i, \{k\}) = R_{ik}(\alpha)\). Both ends of the sum of \(k\), we get

\[
U^\alpha(i, E) = \sum_{k \in E} U^\alpha(i, \{k\}) = \sum_{k \in E} R_{ik}(\alpha) = \frac{1}{\alpha} U^\alpha(i, \bar{E})
\]
So that $U^\alpha(i, \overline{E} \setminus E) = 0$.

(3) Because $(U^\alpha)_{\alpha > 0}$ is Ray resolvent on $\overline{E}$, so for any $i, k \in E, f \in C(\overline{E}), \alpha > 0$, 
\[
\int_0^\infty e^{-\alpha t} P(t) f(i) dt = U^\alpha f(i) = \sum_{k \in E} R_k(\alpha) f(k) = \int_0^\infty e^{-\alpha t} \sum_{k \in E} p_k(t) f(k) dt
\]

For any $t_0 > 0, t > 0, h > 0$,
\[
\left| \sum_{k \in E} p_k(t_0 + t + h) f(k) - \sum_{k \in E} p_k(t_0 + t) f(k) \right| \\
\leq \| f \| \sum_{k \in E} \left| p_k(t_0 + t + h) - p_k(t_0 + t) \right| \\
= \| f \| \left| \sum_{k \in E} \left( \sum_{m \in E} p_m(t_0) \left( \sum_{l \in E} p_m(h) p_k(t) - p_k(t) \right) \right) \right| \\
\leq \| f \| \left| \sum_{k \in E, m \in E} p_m(t_0) \left( \sum_{l \in E} p_m(h) - 1 \right) p_k(t) \right| + \| f \| \left| \sum_{k \in E, m \in E} \sum_{l \in E} p_m(t_0) p_m(h) p_k(t) \right| \\
= 2 \| f \| \left| \sum_{m \in E} p_m(t_0) [1 - p_m(h)] \right|
\]

By the $p_y(t)$ standard and control convergence theorem\(^5\) to:

\[
\lim_{h \to 0} \sum_{k \in E} p_k(t_0 + t + h) f(k) - \sum_{k \in E} p_k(t_0 + t) f(k) = 0
\]

That is $t \mapsto \sum_{k \in E} p_k(t) f(k)$ is continuous function on $(0, \infty)$, For $\alpha$ is optional nature, we have : $\forall t > 0$, $\int_{\overline{E}} P_t(i, dy) f(y) = P_t f(i) = \sum_{k \in E} p_k(t) f(k)$. By the monotone class theorem\(^5\), The formula for arbitrary bounded measurable function on $\overline{E}$ is also established.

$\forall k \in E$, let $f(\cdot) = I_k(\cdot)$, substitution $U^\alpha f(i) = \sum_{k \in E} R_k(\alpha) f(k) = R_\alpha f(i)$, we get $P_t(i, \{k\}) = p_k(t)$, On both sides of the $k$ sum, then $\sum_{k \in E} P_t(i, \{k\}) = 1$, so

$P_t(i, \overline{E} \setminus E) = 0$. \(\square\)

**Note3:** Let $E_R = \{ x | x \in \overline{E}, U^1(x, E) = 1 \}$, it is clear that $E_R$ is Borel subset on $\overline{E}$. 
Theorem 3: Let $x \in E_R$, then $\forall t > 0$, $P_t(x, E) = P_t(x, E)$, and $\forall s \geq 0, k \in E$,

$$P_{t+s}(x, \{k\}) = \sum_{m \in E} P_t(x, \{m\}) p_{mk} (s).$$

Proof: because $\forall t > 0, P_t(x, E) \leq P_t(x, E) = 1$, so

$$U^1(x, E) = \int_0^{\infty} e^{-t} P_t(x, E) dt \leq \int_0^{\infty} e^{-t} P_t(x, E) dt = 1$$

By $x \in E_R$, $1 = \int_0^{\infty} e^{-t} P_t(x, E) dt = \int_0^{\infty} e^{-t} P_t(x, \overline{E}) dt$, so $P_t(x, E) = P_t(x, E) = 1$, then $\exists (t_n)_{n=1}^\infty$, so that when $t \to 0$ and $\forall t_n$, $P_{t_n}(x, E) = 1$. For any $t > 0$, Let $t_n < t$, by The properties$^{[6]}$ of semigroup of $(P_s)_{s=0}^\infty$: $P_t(x, E) = \int_E P_{t_n}(x, dy) P_{t-t_n}(y, E)$

$$= \int_E P_{t_n}(x, dy) \rho_{t-t_n}(y, E, \sum_{k \in E} P_{t_n}(x, \{k\}) p_{k} )_{t_n} k \in E,$$

$$= P_{t_n}(x, E) = 1$$

We get: $P_t(x, E) = P_t(x, \overline{E})$. $\forall t > 0, s \geq 0, k \in E$:

$$P_{t+s}(x, \{k\}) = \int_E P_t(x, dy) P_s(y, \{k\}) = \int_E P_t(x, dy) P_s(y, \{k\}) = \sum_{m \in E} P_t(x, \{m\}) p_{mk} (s)$$

$\forall x \in E_R, k \in E, t > 0, P_t(x, \{k\})$ is denoted $p_{sk}(t)$.

REFERENCES


