PIECEWISE DEFINED RECURSIVE SEQUENCES WITH APPLICATION IN MATRIX THEORY

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Abstract. In this paper we determine periodicity and boundedness of orbits of a piecewise defined difference equation. A corollary is that the real eigenvalues of certain arbitrarily large, sparse matrices may be computed exactly.

Keywords: piecewise defined sequence; eventually periodic; strictly increasing; sparse matrix; eigenvalue.

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1. Introduction

The geometric sequence \( S_k = hS_{k-1} \) is a basic component of exponential growth models [2, 3, 5, 10]. In this note, we consider a limiting or harvesting condition on \( S_k \) and describe the resulting piecewise defined sequence. Precisely, we make the

Definition 1. Let \( h, v \) and \( w \) be natural numbers and let \( S_0 \) be an integer. Define for \( k > 0 \)

\[
S_k = \begin{cases} 
  hS_{k-1} & \text{if } S_{k-1} \leq v \\
  hS_{k-1} - w & \text{if } S_{k-1} > v 
\end{cases}
\]

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If $S_r \leq 0$ for some $r$, then $S_k$ is geometric for $k \geq r$. Hence we restrict our attention to positive initial values.

We begin with a characterization of $S_k$, then discuss feasibility when hypotheses fail. A central condition in the treatment is $w = 2v + 1$. For example, suppose that $h = 2$, $v = 5$, $w = 11$ and $S_0 = 1$. Then $S_k$ is periodic and all integers from 1 to $w - 1$ appear in the sequence:

$$1, 2, 4, 8, 5, 10, 9, 7, 3, 6, 1, 2, \ldots .$$

However, if $h = 2$, $v = 3$, $w = 7$ and $S_0 = 1$, then $S_k$ is again periodic but not all such integers appear: 1, 2, 4, 1, 2, \ldots . On the other hand, if $h = 3$, $v = 2$, $w = 5$ and $S_0 = 2$, then $S_k$ is strictly increasing (to infinity): 2, 6, 13, 34, 97, 286, 853, 2554, 7657, \ldots .

2. The Main Result

In general we have the following.

**Theorem 2.1.** Let $h$, $v$, $w$ and $S_0$ be positive integers.

1. If $h = 1$, then $S_k$ reaches a point of equilibrium, i.e., there is a nonnegative integer $r$ such that $S_k = S_0 - rw$ for all $k \geq r$.

2. Let $h \geq 2$.

   a. If $S_0 = w > v$, then

   $$S_k = \frac{h^k(h - 2) + 1}{h - 1} S_0 \quad \text{for all} \quad k.$$

   b. If $S_0 > w \geq v$, then $S_k$ is strictly increasing.

3. Let $h = 2$.

   a. Suppose that $w = 2v + u$ for some $u = 0$, 1 or 2, and $1 \leq S_0 \leq w - 1$.

      i. If $u = 1$, then $S_k$ is periodic and $1 \leq S_k \leq w - 1$ for all $k$. 

ii. If \( u = 0 \), then \( S_k \) either reaches equilibrium at \( w \) or is eventually periodic such that \( 1 \leq S_k \leq w - 1 \) for all \( k \).

iii. If \( u = 2 \), then \( S_k \) either reaches equilibrium at \( 0 \) or is eventually periodic such that \( 1 \leq S_k \leq w - 1 \) for all \( k \).

\[ S_1 = S_0 - w > v, \ldots, \quad S_r = S_0 - (r - 1)w > v, \]

b. If \( w > 2v + 2 \), then \( S_0 \geq w \) (in which case, see 2) or \( S_k \) is either eventually nonpositive or eventually periodic.

c. If \( w \leq v \), then \( S_k \) is strictly increasing.

4. If \( h \geq 3 \) and \( w \leq 2v + 1 \), then \( S_k \) is strictly increasing.

**Proofs.**

1. Assume \( h = 1 \). There is a nonnegative integer \( r \) such that \( S_0 \) satisfies

\[ 0 < v < v + w < \cdots < v + (r - 1)w < S_0 \leq v + rw. \]

If \( r = 0 \), then \( S_k = S_0 \) for all \( k \). And if \( r > 0 \), then

\[ S_1 = S_0 - w > v, \ldots, \quad S_r = S_0 - (r - 1)w > v, \]

but

\[ S_r = S_0 - rw \leq v \]

and it follows that \( S_k = S_r \) for \( k \geq r \).

2. Assume that \( h \geq 2 \).

a. If \( S_0 = w > v \), then

\[ S_1 = hS_0 - w = (h - 1)w \geq w > v \]

\[ S_2 = hS_1 - w = [h(h - 1) - 1]w \geq w > v \]

\[ S_3 = hS_2 - w = \{h[h(h - 1) - 1] - 1\}w \geq w > v, \text{ etc.} \]

Therefore in general

\[ S_k = (h^k - h^{k-1} - h^{k-2} - \cdots - 1)w = (h^k - \frac{h^k - 1}{h - 1})w = \frac{h^k(h - 2) + 1}{h - 1} w. \]
b. Assume $S_0 > w \geq v$. Then

$$S_1 = hS_0 - w = S_0 + [(h - 1)S_0 - w] > S_0 > w \geq v$$

and similarly by induction

$$S_k = hS_{k-1} - w = S_{k-1} + [(h - 1)S_{k-1} - w] > S_{k-1} > w \geq v$$

for every $k \geq 2$. Thus $S_{k+1} > S_k > v$ for all $k$.

3a. Let $h = 2$, $1 \leq S_0 \leq w - 1$, and $w = 2v + u$ for some $u = 0, 1$ or $2$. If $1 \leq S_0 \leq v$, then

$$2 \leq S_1 = 2S_0 \leq 2v = w - u. \quad (1)$$

On the other hand, if $v < S_0 \leq w - 1$, then $2v + 2 \leq 2S_0 \leq 2w - 2$ and

$$2 - u \leq S_1 = 2S_0 - w \leq w - 2 < w - 1. \quad (2)$$

i. Suppose that $u = 1$. By (1) and (2), it follows that $1 \leq S_1 \leq w - 1$, and by induction, $1 \leq S_k \leq w - 1$ for all $k$. Since $S_k$ is a sequence of natural numbers, we have that some term $S_r$ must repeat. We show that $S_r$ must repeat: Suppose that $r > 0$ is the least integer such that $S_r = S_{r+s}$ for some $s > 0$. If $S_r$ is even, then $S_r = 2S_{r-1}$, since the other possibility $2S_{r-1} - w$ is odd; and since $S_r = S_{r+s}$, it follows that $S_{r+s} = 2S_{r+s-1}$ so that $S_{r+1} = S_{r+s-1}$ in this case. If $S_r$ is odd, then similarly $S_{r+1} = S_{r+s+1}$. Therefore $S_{r+1} = S_{r+s-1}$ in either case, which contradicts the minimality of $r$. Hence $S_k$ is periodic when $u = 1$.

ii. Assume $u = 0$. If $S_N = w$ for some $N$, then since $h = 2$ and $w > v$, we have that $S_k = w$ for all $k \geq N$. Thus suppose that $S_k \neq w$ for all $k$. Hence by (1) and (2), $1 \leq S_1 \leq w - 1$, and an induction argument shows that $1 \leq S_k \leq w - 1$ for all $k$. As above, some term $S_r$ must repeat so $S_k$ is eventually periodic.

iii. Similar to (ii).

3b. Suppose that $h = 2$, $w > 2v + 2$, and $S_N \geq w$ for some $N$. Assume that $N > 0$. If $S_{N-1} \leq v$, then

$$S_N = 2S_{N-1} \leq 2v < w$$
which is impossible. Hence \( S_{N-1} > v \) and \( S_N = 2S_{N-1} - w \geq w \) so \( S_{N-1} \geq w \).

Continuing by induction, we have that \( N = 0 \) is the only possibility if \( S_N \geq w \).

Therefore, if \( S_0 \leq w - 1 \), then \( S_k \leq w - 1 \) for all \( k \). Hence either \( S_N \leq 0 \) for some \( N \) or \( 1 \leq S_k \leq w - 1 \) for all \( k \). It follows that either \( S_k \leq 0 \) for all \( k \geq N \) or \( S_k \) is eventually periodic as in the proof of \( (3a) \).

3c. Assume \( h = 2 \) and \( w \leq v \). Let \( l \) be the least integer such that \( 2^l S_0 > v \). Then \( S_l > S_{l-1} > \cdots > S_0 \), \( S_l = 2^l S_0 \), and

\[
S_{l+1} = 2^{l+1} S_0 - w \geq 2^{l+1} S_0 - v > 2^l S_0 = S_l
\]

since \( 2^l (2 - 1) S_0 > v \). Hence

\[
S_{l+1} > S_l > v.
\]

By induction

\[
S_{k+1} > S_k > v \quad \text{for all} \quad k \geq l
\]

and thus \( S_k \ (k \geq 0) \) is strictly increasing.

4. Assume \( h \geq 3 \) and \( w \leq 2v + 1 \). We show \( S_{k+1} > S_k \) for all \( k \). Let \( l \) denote the least integer such that

\[
S_l = h^l S_0 > v.
\]

Then \( S_l > S_{l-1} > \cdots > S_0 \), and

\[
S_{l+1} = h S_l - w \geq 3S_l - w > S_l.
\]

The last inequality follows since

\[
S_l \geq v + 1 \quad \text{and} \quad 2S_l \geq (2v + 1) + 1 \geq w + 1 > w.
\]

Hence

\[
S_{l+1} > S_l > v
\]

and \( S_k \) is strictly increasing as in \( (3c) \). \( \square \)
Corollary 2.2. Let $v$ be a natural number, $h = 2$, $w = 2v + 1$, and let $R$ be the relation defined on the set \{1, 2, $\ldots$, $n = 2v$\} by: $xRy$ if and only if there exist terms $S_0$ and $S_k$ such that $x = S_0$ and $y = S_k$ for some $k > 0$. Then $R$ is an equivalence relation on \{1, 2, $\ldots$, $n$\}.

Proof. By (3ai) of Theorem 2.1, for any given $S_0$ in \{1, 2, $\ldots$, $n$\} there exists a unique integer $p = p(S_0) > 0$ such that the sequence $S_k$ is

$$S_0, S_1, \ldots, S_{p-1}, S_p = S_0, S_{p+1} = S_1, \ldots \tag{3}$$

and $S_0, S_1, \ldots, S_{p-1}$ are distinct.

(i) Reflexive: Let $x = S_0$. By (3), $x = S_p$ so $xRx$.

(ii) Symmetric: Suppose $xRy$. Thus $x = S_0$ and $y = S_k$ where by (3) we may assume $1 \leq k < p$. Redefine $y = S'_0$ (another starting value). Then by (3), $x = S'_{p-k}$ where $p - k > 0$ so $yRx$.

(iii) Transitive: Assume $xRy$ and $yRz$. As above, by the definition of the sequence $S_k$,

$$x = S_0, \ y = S_k = S'_0, \ z = S'_l = S_{k+l}$$

for some positive integers $k$ and $l$. Therefore $x = S_0$ and $z = S_{k+l}$ where $k + l > 0$, thus $xRz$. □

The following is a fundamental result from Algebra [8, 9]:

Let $R$ be an equivalence relation on a set $S$. For any $s$ in $S$, the equivalence class of $s$ under $R$, denoted $[s]$, is the subset of $S$ consisting of all elements $t$ of $S$ such that $tRs$. Then every element of $S$ is in exactly one equivalence class under $R$. That is, the equivalence classes partition $S$ into a family of mutually disjoint nonempty subsets.

The equivalence classes of Corollary 2.2 are, moreover, ordered sets

$$[S_0] = \{S_0, S_1, \ldots, S_{p-1}\}$$

and we have for example
\{1, 2, \ldots, 14\} = [1] \cup [3] \cup [7],
\{1, 2, \ldots, 16\} = [1] \cup [3],
\{1, 2, \ldots, 18\} = [1],

An interesting problem in this algebraic context is to determine all even values \(n\) such that \(\{1, 2, \ldots, n\} = [1]\).

**Example 1.** The following table illustrates possible situations in (3a) and (3b) of Theorem 2.1 where \(h = 2\):

<table>
<thead>
<tr>
<th>(S_0)</th>
<th>(v)</th>
<th>(w)</th>
<th>sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>8</td>
<td>1, 2, 4, 0, 0, 0, \ldots</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>20</td>
<td>3, 6, 12, 4, 8, 16, 12, 4, \ldots</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>20</td>
<td>5, 10, 20, 20, 20, \ldots</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>7</td>
<td>8, 9, 11, 15, 23, 39, 71, 135, \ldots</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>24</td>
<td>1, 2, 4, 8, 16, 8, 16, 8, \ldots</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>22</td>
<td>1, 2, 4, 8, 16, 10, \ldots</td>
</tr>
</tbody>
</table>

3. **Pathology**

Theorem 2.1 describes piecewise defined recursive sequences when either \(w \leq v\), \(w = 2v\) or \(w = 2v + 1\) for any \(h\) and \(S_0\). The specific cases not covered are

1. \(h = 2\) and \(v < w < 2v\)
2. \(h \geq 3\) and \(w > 2v + 1\).

In each case, \(w > v\), so by the reasoning in the proofs of (2) and (3) of Theorem 2.1, one of the following situations must hold for some \(N\):

i. \(S_N = w\) \(\text{(and } S_{N+k} = \frac{h^{k(h-2)+1}}{h-1}w \text{ for every } k)\)
ii. \(S_N > w\) \(\text{(and } S_k \text{ is strictly increasing for } k \geq N)\)
iii. \(1 \leq S_k \leq w - 1\) for all \(k \geq N\) \(\text{(and } S_k \text{ is eventually periodic)}\)
iv. \(S_N \leq 0\) \(\text{(and } S_{N+k} = h^kS_N \text{ for every } k)\)
We illustrate these possibilities as follows.

**Example 2.** \((h = 2 \text{ and } v < w < 2v)\) In this case, we show that \(S_k \geq 2\) for all \(k \geq 1\) so (iv) is not feasible: Let \(l\) be the least natural number such that \(S_l = h^l S_0 > v\). Then

\[
S_{l+1} = h^{l+1} S_0 - w > h^{l+1} S_0 - 2v = 2(h^l S_0 - v) \geq 2.
\]

Let \(l'\) be the least natural number such that \(S_{l+l'} = h^{l'-1}(h^{l+1} S_0 - w) > v\). Then

\[
S_{l+l'+1} = hS_{l+l'} - w > 2(S_{l+l'} - v) \geq 2.
\]

Continuing similarly by induction, the result follows.

The other situations are possible: with \(S_0 = 1\), we have

<table>
<thead>
<tr>
<th>(v)</th>
<th>(w)</th>
<th>(S_k) type</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>16</td>
<td>i</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>ii</td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>iii</td>
</tr>
</tbody>
</table>

For case 2, any of (i) - (iv) are feasible:

**Example 3.** \((h = 3 \text{ and } w > 2v + 1)\) Choosing \(S_0 = 1\) again, we calculate the table

<table>
<thead>
<tr>
<th>(v)</th>
<th>(w)</th>
<th>(S_k) type</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>27</td>
<td>i</td>
</tr>
<tr>
<td>7</td>
<td>19</td>
<td>ii</td>
</tr>
<tr>
<td>16</td>
<td>72</td>
<td>iii</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>iii</td>
</tr>
<tr>
<td>7</td>
<td>30</td>
<td>iv</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>iv</td>
</tr>
</tbody>
</table>

It is easy to generate periodic \(S_k\) with arbitrary initial values from known examples. If \(S_k\) is periodic and \(\alpha\) is a positive integer, the piecewise defined sequence \(S'_k\) with \(h' = h\), \(v' = \alpha v\) and \(w' = \alpha w\) is periodic and \(S'_0 = \alpha S_0\). For \(h = 2\), periodic sequences \(S_k\) with \(S_0 = 1\) are given by the theorem (3a) where \(w = 2v + u\) \((u = 0, 1, 2)\). If
\( \alpha > 2 \) and \( u > 0 \), then \( S' \) is periodic and satisfies (3b). We can similarly modify the following results.

**Example 4.** For \( h \geq 2 \), there are general choices of \( v \) and \( w \) such that \( w > 2v + 1 \), \( S_0 = 1 \), and \( S_k \) is either periodic, strictly increasing or reaches an equilibrium point:

a. For any positive integers \( r \) and \( s \), let \( t = (r + 1)s \) and define

\[
v = \frac{(h - 1)(h^{r+t} - h^r)}{h^{r+1} - 1} \quad \text{and} \quad w = hv + h - 1.
\]

We begin by showing \( h^{t-1} < v < h^t \). The right inequality is equivalent to

\[
h^r + h^t < h^{r+1} + h^{r+t}
\]

which is trivial. The left inequality is equivalent to

\[
h^{t-1}(h^{r+1} - 1) \leq h^r(h^t - 1)(h - 1)
\]

which clearly holds when \( s = 1 \) since \( t = (r + 1)s \) and \( h \geq 2 \). Thus assume that \( s \geq 2 \). Then the inequality becomes

\[
2h^{r+t} + h^{r+1} \leq h^{r+t+1} + h^r + h^{t-1}
\]

where \( 2h^{r+t} \leq h \) \( h^{r+t} \) since \( h \geq 2 \), and \( h^{r+1} \leq h^r + h^{t-1} \) (or \( h \leq 1 + h^{t-(r+1)} \)) since \( t = (r + 1)s \geq 2(r + 1) \) and \( r \geq 0 \). Thus \( h^{t-1} < v < h^t \) and straightforward calculations show that

\[
w = \frac{(h - 1)(h^{r+t+1} - 1)}{h^{r+1} - 1} \quad \text{and} \quad h^{t+1} - w > v.
\]

More generally, for \( 1 \leq k < r \),

\[
h^{t+k} - \frac{h^k - 1}{h - 1}w = h^{t+k} - \frac{(h - 1)(h^{r+t+1} - 1)}{h^{r+1} - 1} > v
\]

if and only if

\[
h^{r+t} + h^{r+1} + h^k > h^{k+t} + h^r + 1,
\]

which holds since \( h \geq 2 \).
It follows that since $S_0 = 1$,

$$S_{t+k} = h^{t+k} - \frac{h^k - 1}{h - 1} w \quad (k = 1, \ldots, r).$$

Since $S_{t+r} = v + 1$, we have that $S_{t+r+1} = h(v + 1) - (hv + h - 1) = 1$ and $S_k$ is periodic of period $t + r + 1 = (r + 1)(s + 1)$.

For example, if $h = 3$, $r = 1$ and $s = 3$, then $v = 546$, $w = 1640$ and the sequence is

$$1, 3, 9, 27, 81, 243, 729, 547, 1, \ldots$$

b. Let $t$ be a positive integer, $v = h^t$ and $w = h^{t+2} - h^{t+1} + h - 1$ (which satisfy (3ai) of Theorem 2.1 if $h = 2$). The sequence $S_k$ is then computed as follows:

$$1, h, \ldots, h^t, h^t + h + 1, h^t + h - 2 + 1, \ldots, h^t + h^t - 1, 1, \ldots.$$  

For example, if $h = 4$, $v = 64$ and $w = 771$, then the sequence is

$$1, 4, 16, 64, 256, 253, 241, 193, 1, \ldots.$$  

c. Let $t$ and $v$ be any positive integers that satisfy

$$h^{t-1} < v \leq h^t - h^{t-1} - 1,$$

and let $w = hv + h - 1$ as above. Then $S_k = h^k \ (0 \leq k \leq t)$, $S_t > v$, and

$$S_{t+1} = hS_t - w = h^{t+1} - hv - h + 1 \geq h^t + 1 > S_t > v.$$  

By induction,

$$S_{t+k+1} > S_{t+k} > v$$  

for all $k$ and thus $S_{t+k}$, and therefore $S_k$, are strictly increasing.

d. Let $t$ and $v$ be positive integers such that

$$h^{t-1} \leq v < h^t,$$
and let \( w = h^{t+1} - h^t \). Then \( S_k = h^k \) for \( k \leq t \) and \( S_{t+1} = h^{t+1} - w = h^t \). Thus \( S_k = h^t \) for all \( k \geq t \).

4. Application in Matrix Theory

The following sparse matrices arose in [1] while considering certain vector spaces of magic squares.

**Definition 2.** The C-matrix \( A = (a_{ij}) \) is the square matrix of order \( n \) such that its nonzero elements are defined as follows where either \( n = 2k \) or \( n = 2k + 1 \):

\[
a_{ij} = \begin{cases} 
1 & \text{for } j = 2i \text{ when } 1 \leq i \leq k \\
1 & \text{for } j = 2i - (n + 1) \text{ when } n - k < i \leq n \\
-2 & \text{for } j = i
\end{cases}
\]

The ones appear as the moves of the knight on a chessboard. Odd ordered C-matrices are distinguished from even ones by a middle row without ones.

According to Gerschgorin’s Disk Theorem (see [7]), the eigenvalues of C-matrices lie in the unit circle with centre \((-2, 0)\) in the complex plane. We show that the real bounds \(-1\) and \(-3\) of the circle will indeed be eigenvalues in many cases. Moreover we note that \(0\) is not contained in the Gerschgorin disk so C-matrices are invertible. (This also follows since they are strictly diagonally dominant.)

For any C-matrix of odd order we note that \(-2\) is an eigenvalue of \( A \) since the matrix \( A + 2I \) has the zero row as its middle row. We conjecture that \(-2\) is the only eigenvalue when the order of \( A \) is \( n = 2^l - 1 \). This is illustrated in the following

**Example 5.** Let \( A \) be the C-matrix of order 15. Suppose by way of contradiction that \( \beta \neq -2 \) is a (real or complex) eigenvalue of \( A \) and let \( \alpha = \beta + 2 \). Since \( \alpha \neq 0 \), by the definition of C-matrix, if \( x = (x_1, x_2, \ldots, x_{15})^t \) is a nonzero vector in the kernel
of $A - \beta I$, then $x_8 = 0$ and $x_i = x_{i+8}$ ($1 \leq i \leq 7$). Moreover, if $x_{2i} = 0$ for some $i = 1, 2, \ldots, 7$, then $x_i = 0$. It follows in order that

$$0 = x_4 = x_2 = x_1 = x_{4+8} = x_{2+8} = x_{1+8} = x_6 = x_3 = x_5 = x_{6+8} = x_{3+8} = x_{5+8} = x_7 = x_{7+8}.$$ 

Thus, $x$ is the zero vector, a contradiction. Therefore, $-2$ is the only eigenvalue of $A$.

Another possible eigenvalue of odd ordered C-matrices is $-1$:

Proposition 4.1. Let $A$ be a C-matrix of order $n = 4l + 1$. Then $-1$ is an eigenvalue of $A$.

Proof. Let $n = 4l + 1$. For each row of the matrix $A + I$ except the middle row we have one entry 1, one entry $-1$, and the other entries 0. The ones occur in even numbered cells. Column $2l + 1$ is the middle column so it contains no ones. If we sum the columns of $A + I$ except the middle column, then we obtain the zero vector. Hence, the determinant of $A + I$ is zero. □

We now turn to the eigenvalues of C-matrices of even order. The following is similar to the above result.

Proposition 4.2. Let $A$ be a C-matrix of order $n = 6l + 2$. Then $-3$ is an eigenvalue of $A$.

Proof. Let $n = 6l + 2$. We show in this case that rows $4l + 2$ and $2l + 1$ of $A + 3I$ are identical. By the definition of C-matrix, the main diagonal of $A + 3I$ consists of ones, and row $2l + 1$ has one in the entries $(2l + 1, 2l + 1)$ and $(2l + 1, 2(2l + 1))$ since $2l + 1 < 3l + 1$. On the other hand, row $4l + 2$ has one in the entries $(4l + 2, 4l + 2)$ and $(4l + 2, 2(4l + 2) - (n + 1)) = (4l + 2, 2l + 1)$. □
We can extend the idea behind the above proof for other C-matrices of even order. We consider matrices where the sum of several rows of \( A + 3I \) is identical to the sum of another set of rows. For example, let \( A \) be the C-matrix of order 4. We obtain

\[
A + 3I = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

The sum of the first and fourth rows is the same as the sum of the second and third rows. Thus \( -3 \) is an eigenvalue of \( A \). In general, we have the following procedure.

**Theorem 4.3.** Let \( A \) be a C-matrix of even order \( n = 2k \). Then \(-1\) is an eigenvalue of \( A \).

Conversely, if \( \lambda \) is a real eigenvalue of \( A \), then \( \lambda = -1 \) or \( \lambda = -3 \).

Define the sequence \( Q_l \) of positive integers as follows: let \( Q_0 = 1 \) and for \( l \geq 0 \) let \([Q_l, Q_{l+1}]\) denote the positions in row \( Q_l \) of the ones in the matrix \( A + 3I \). Then the sequence \( Q_l \) is periodic. If the period of \( Q_l \) is even, then \(-3\) is an eigenvalue of \( A \).

**Proof.** Let \( A \) be a C-matrix of order \( n = 2k \). Then each column of \( A + I \) has exactly one entry 1, one entry \(-1\) and the remaining entries 0. Hence, the sum of all rows of \( A + I \) is the zero row so \( |A + I| = 0 \).

Let \( \lambda \) be a real eigenvalue of \( A \). We argue indirectly. Assume that \( |\lambda + 2| < 1 \) by Gerschgorin’s theorem. We rearrange the columns \( C_i \) of \( A - \lambda I \) in the order

\[
C_2, C_4, ..., C_n, C_1, C_3, ..., C_{n-1}.
\]

The resulting matrix is strictly diagonally dominant, and is therefore invertible with nonzero determinant, a contradiction.

Let \( Q_l \) be given as above. Note that rows with ones in positions \([Q_l, Q_{l+1}]\) and \([Q_{l+1}, Q_{l+2}]\) have a one in the same position \( Q_{l+1} \). Thus, for \( l \geq 1 \), row \( Q_l \) also
shares a one with row $Q_{l-1}$. By the definition of C-matrix of order $2k$, with $Q_0 = 1$, for $l \geq 1$,

$$Q_l = \begin{cases} 
2Q_{l-1} & \text{if } Q_{l-1} \leq k \\
2Q_{l-1} - (2k + 1) & \text{if } Q_{l-1} > k
\end{cases}$$

By (3ai) of Theorem 2.1, $Q_l$ is periodic. Hence if the period of $Q_l$ is $p$, then $Q_p = 1$ and row 1 with ones in $[Q_0, Q_1]$ and row $Q_{p-1}$ with ones in $[Q_{p-1}, Q_p]$ have position $Q_p = Q_0$ in common. Since $Q_0, Q_1, \ldots, Q_{p-1}$ are distinct, we have that if $p$ is even, then the sum of the rows $Q_0, Q_2, \ldots, Q_{p-2}$ coincides with the sum of the rows $Q_1, Q_3, \ldots, Q_{p-1}$ and hence the determinant of $A + 3I$ is zero. \hfill \Box

**Example 6.** We can readily list the sequences $Q_l$. Two of the first eleven even ordered C-matrices have sequences $Q_l$ with odd periods:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Q_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1, 2, 1, \ldots</td>
</tr>
<tr>
<td>4</td>
<td>1, 2, 4, 3, 1, \ldots</td>
</tr>
<tr>
<td>6</td>
<td>1, 2, 4, 1, \ldots</td>
</tr>
<tr>
<td>8</td>
<td>1, 2, 4, 8, 7, 5, 1, \ldots</td>
</tr>
<tr>
<td>10</td>
<td>1, 2, 4, 8, 5, 10, 9, 7, 3, 6, 1, \ldots</td>
</tr>
<tr>
<td>12</td>
<td>1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, \ldots</td>
</tr>
<tr>
<td>14</td>
<td>1, 2, 4, 8, 1, \ldots</td>
</tr>
<tr>
<td>16</td>
<td>1, 2, 4, 8, 16, 15, 13, 9, 1, \ldots</td>
</tr>
<tr>
<td>18</td>
<td>1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, 6, 12, 5, 10, 1, \ldots</td>
</tr>
<tr>
<td>20</td>
<td>1, 2, 4, 8, 16, 11, 1, \ldots</td>
</tr>
<tr>
<td>22</td>
<td>1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12, 1, \ldots</td>
</tr>
</tbody>
</table>

For example, the period of $Q_l$ is even for $n = 16$ and by Theorem 4.3, the sum of the rows 1, 4, 16 and 13 of $A + 3I$ is identical to the sum of the rows 2, 8, 15 and 9.
We can compute the period of \( Q_l \) in some general cases:

**Corollary 4.4.** Let \( l \geq 2 \) be an even integer and let \( A \) be a C-matrix of order \( n = 2^l - 2 \). Then \( -3 \) is an eigenvalue of \( A \).

**Proof.** We have \( Q_0 = 1, Q_1 = 2, Q_2 = 4, \ldots, Q_{l-1} = 2^{l-1} \) since \( 2^{l-2} \leq \frac{n}{2} \). But \( 2^{l-1} = \frac{n}{2} + 1 > \frac{n}{2} \) so \( Q_l = 2\left(\frac{n}{2} + 1\right) - (n + 1) = 1 \). \( \Box \)

**Corollary 4.5.** Let \( A \) be a C-matrix of order \( n = 2^l \) where \( l \geq 2 \). Then \( -3 \) is an eigenvalue of \( A \).

**Proof.** Assume that \( A \) is a C-matrix of order \( n = 2^l \) where \( l \geq 2 \). Then \( Q_0 = 1, Q_1 = 2, Q_2 = 4, \ldots, Q_l = 2^l \). We prove by induction that

\[
Q_{l+i} = Q_l - 2^i + 1 > 2^{l-1} = \frac{n}{2}
\]

for \( i = 0, 1, \ldots, l - 1 \). The initialization \( i = 0 \) is clear. Assume the statement holds for some \( i < l - 1 \). Then

\[
Q_{l+i+1} = 2(Q_l - 2^i + 1) - (n + 1) = Q_l - 2^{i+1} + 1 > 2^{l-1}
\]

since \( i + 1 \leq l - 1 \) and \( 2^{l-1} + 1 > 2^{i+1} \).

In particular,

\[
Q_{2l-1} = Q_{l+(l-1)} = Q_l - 2^{l-1} + 1 = \frac{n}{2} + 1 > \frac{n}{2}
\]

so \( Q_{2l} = 2\left(\frac{n}{2} + 1\right) - (n + 1) = 1 \). Therefore the period of \( Q_l \) is even. \( \Box \)

If the period of \( Q_l \) is odd, then we can not deduce any information about the value \(-3\). For example, the period of \( Q_l \) is three for the C-matrix \( A \) of order six and its eigenvalues are

\[
-\frac{5}{2} \pm \frac{\sqrt{3}}{2}i, \quad -\frac{5}{2} \pm \frac{\sqrt{3}}{2}i, \quad -1, \quad -1.
\]
On the other hand, \(-3\) is an eigenvalue of the C-matrix of order 366 although \(Q_t\) has period 183.

We computed the eigenvalues of the C-matrices of even orders up to order 4780 and found the following orders for which \(-3\) is not an eigenvalue:

6, 22, 30, 46, 48, 70, 72, 78, 88, 102, 126, 150, 160, 166, 190, 198, 216, 222, 232, 238, 262, 270, 310, 328, 336, 342, 358, 430, 438, 496, 510, 552, 600, 622, 630, 712, 720, 880, 888, 910, 918, 936, 960, 1056, 1102, 1288, 1392, 1432, 1456, 1518, 1560, 1678, 1800, 1896, 2046, 2088, 2142, 2200, 2262, 2350, 2358, 2592, 2686, 2758, 2920, 3016, 3190, 3390, 3478, 3472, 3576, 3936, 4056, 4176, 4206, 4512, 4576, 4680.

References