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ON FUZZY ORDERED SEMIGROUPS

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Abstract. The results on ordered semigroups can be transferred to fuzzy ordered semigroups, and from the results on fuzzy ordered semigroups one can get the corresponding results on ordered semigroups in the way indicated in the present paper. So not only ordered semigroups give information about fuzzy ordered semigroups, but fuzzy ordered semigroups give information about ordered semigroups as well.

Keywords: ordered semigroup, fuzzy ordered semigroup, left ideal, bi-ideal, left regular, regular, intra-regular, fuzzy left ideal, fuzzy ideal, fuzzy bi-ideal, simple, semiprime, fuzzy simple, fuzzy semiprime.

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1. Introduction and prerequisites

Let $(S, ., \leq)$ be an ordered semigroup. For a subset H of S, $(H]$ denotes the subset of S defined by $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$. Clearly $S = (S]$, and for any subsets A, B of S, we have $A \subseteq (A]$, if $A \subseteq B$ then $(A] \subseteq (B]$, and $((A)] = (A]$. A nonempty subset A of S is called a *left* (resp. *right*) *ideal* of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$, that is $(A] = A$. A is called an *ideal* of S if it is both a left and a right ideal of S. A left (resp. right) ideal A of S is clearly a subsemigroup of S i.e. $A^2 \subseteq A$. A nonempty subset A of S is called a *bi-ideal* of S if (1) $ASA \subseteq A$ and 2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. For an

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element a of S, $L(a)$, $R(a)$, $I(a)$, $B(a)$ denote the left ideal, right ideal, the ideal, and the bi-ideal of S, respectively, generated by a, and we have $L(a)=(a\cup Sa]$, $R(a)=(a\cup aS]$, $I(a)=(a\cup Sa\cup aS\cup aSa]$, $B(a)=(a\cup aSa]$. For any nonempty subset A of S, the set $(SA]$ (resp. $(AS]$) is a left (resp. right) ideal of S, the set $(SAS]$ is an ideal, and the set $(ASA]$ is a bi-ideal of S. Indeed, the sets $(SA]$, $(AS]$, $(SAS]$ and $(ASA]$ are nonempty subsets of S, $S(SA]=(S](SA]\subseteq(S^2A]\subseteq(SA]$, $(AS]S\subseteq(AS]$, $S(SAS]=(S](SAS]\subseteq(S^2AS]\subseteq(SAS]$, $(SAS]S\subseteq(SAS]$, $(ASA]S(ASA]=(ASA](S](ASA]\subseteq(ASASASA]\subseteq(ASA]$, $((SA]]=(SA]$, $((AS]]=(AS]$, $((SAS]]=(SAS]$ and $((ASA]]=(ASA]$. For $A=\{a\}$, we write $(Sa]$, $(aS]$, $(SaS]$, $(aSa]$ instead of $(S\{a\}]$, $(\{a\}S]$, $(S\{a\}S]$, $(\{a\}S\{a\}]$ thus, for any $a\in S$, $(Sa]$ (resp. $(aS]$) is a left (resp. right) ideal of S, $(SaS]$ is an ideal, and $(aSa]$ is a bi-ideal of S.

An ordered semigroup $(S,.,\leq)$ is called *left regular* if for every $a\in S$ there exists $x\in S$ such that $a\leq xa^2$, that is $a\in(Sa^2]$ for all $a\in S$ or $A\subseteq(SA^2]$ for all $A\subseteq S$. It is called *right regular* if for every $a\in S$ there exists $x\in S$ such that $a\leq a^2x$, that is $a\in(a^2S]$ for all $a\in S$ or $A\subseteq(A^2S]$ for all $A\subseteq S$. An ordered semigroup $(S,.,\leq)$ is called *regular* if for any $a\in S$ there exists $x\in S$ such that $a\leq axa$ i.e. $a\in(aSa]$ for every $a\in S$ or $A\subseteq(ASA]$ for every $A\subseteq S$. It is called *intra-regular* if for each $a\in S$ there exist $x,y\in S$ such that $a\leq xa^2y$, that is $a\in(Sa^2S]$ for all $a\in S$ or $A\subseteq(SA^2S]$ for all $A\subseteq S$. We say that S is *left* (resp. *right*) *duo* if every left (resp. right) ideal of S is two-sided. It is called *duo* if it is both left and right duo. An ordered semigroup S is called *left* (resp. *right*) *simple* if there is no proper left (resp. right) ideals in S i.e. if A is a left (resp. right) ideal of S, then $A=S$. S is called *simple* if S does not contain proper ideals. An ordered semigroup S is left (resp. right) simple if and only if $(Sa]=S$ (resp. $(aS]=S$) for every $a\in S$. It is simple if and only if $S=(SaS]$ for all $a\in S$ which is equivalent to saying that for all $a,b\in S$, we have $b\in(SaS]$. A subset T of S is called *semiprime* if for any $a\in S$ such that $a^2\in T$, we have $a\in T$, equivalently, if $A\subseteq S$ such that $A^2\subseteq T$, then $A\subseteq T$.

A *fuzzy subset* of S (or *fuzzy set*) in S is a mapping f of S into the real closed interval $[0,1]$ of real numbers. For a subset A of S, denote by f_A the characteristic function on A, that is the mapping of S into the set $\{0, 1\}(\subseteq [0, 1])$ defined by

$$f_A : S \to \{0, 1\} \mid f_A(x) := \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

which is clearly a fuzzy subset of S. A fuzzy subset f of S is called a *fuzzy left ideal* of S if

(1) $f(xy) \geq f(y)$ for all $x, y \in S$ and

(2) if $x \leq y$, then $f(x) \geq f(y)$.

It is called *fuzzy right ideal* of S if

(1) $f(xy) \geq f(x)$ for all $x, y \in S$ and

(2) if $x \leq y$, then $f(x) \geq f(y)$.

A fuzzy subset of S which is both a fuzzy left and a fuzzy right ideal of S is called a fuzzy *ideal* of S. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if

(1) $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in S$ and

(2) if $x \leq y$, then $f(x) \geq f(y)$.

We say that S is *fuzzy left duo* if every fuzzy left ideal of S is at the same time a fuzzy right ideal of S. It is *fuzzy right duo* if every fuzzy right ideal of S is a fuzzy left ideal of S. S is called *fuzzy duo* if it is both fuzzy left duo and fuzzy right duo.

By a fuzzy ordered semigroup we mean an ordered semigroup having a fuzzy set. Our aim is to show the relation between ordered semigroups and fuzzy ordered semigroups and show that not only ordered semigroups give information about fuzzy ordered semigroups but fuzzy ordered semigroups give information about ordered semigroups as well. A subset A of an ordered semigroup S has a property if and only if the characteristic function f_A has its fuzzy analogous. A is a left (right) ideal, bi-ideal, quasi-ideal, semiprime subset, prime subset of S, etc. if and only if f_A is so. But if S has a property, then S does not always equivalently have its fuzzy analogous. For example, if S is fuzzy left duo, then S is left duo, but if S is left regular and left duo (or regular and left duo), then it is fuzzy left duo. If every fuzzy bi-ideal of S is a fuzzy right ideal of S, then every bi-ideal of S is a right ideal of S, but if S is regular and every bi-ideal of S is a right ideal of S, then every fuzzy bi-ideal of S is a fuzzy right ideal of S. We prove that an ordered semigroup is left (resp. right) simple if and only if it is fuzzy left (resp. fuzzy right) simple. In left simple ordered semigroups, the fuzzy bi-ideals are fuzzy right ideals. As a consequence, in left simple ordered semigroups, the bi-ideals are right ideals. Denote by N the set $\{1, 2, 3, ..., \}$ of natural numbers. Assuming that S is an ordered semigroup and k a natural number such that $k \geq 2$, we prove that for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^k y$ if and only if for every fuzzy ideal f of S and every $a \in S$, we have $f(a) = f(a^k)$ which is a generalized form of intra-regularity. Using fuzzy left (resp. fuzzy right) ideals, we characterize the ordered semigroups in which $a \leq xa^k$ (resp. $a \leq a^k x$) for some $N \ni k \geq 2$. As a result, if an ordered semigroup S is intra-regular and f a fuzzy ideal of S, then $f(ab) = f(ba)$ for all $a, b \in S$. An ordered semigroup S is intra-regular if and only if for every fuzzy ideal f of S and any $a \in S$, we have $f(a) = f(a^2)$. An ordered semigroup S is left (resp. right) regular if and only if for every fuzzy left (resp. fuzzy right) ideal f of S and any $a \in S$, we have $f(a) = f(a^2)$. We show that if S is an archimedean ordered semigroup, then every fuzzy semiprime fuzzy ideal of S is a constant function. The archimedean ordered semigroups do not contain proper semiprime ideals. An ordered semigroup S is intra-regular if and only if every fuzzy ideal of S is fuzzy semiprime. As a consequence, an ordered semigroup S is intra-regular if and only if every ideal of S is semiprime. Using fuzzy sets, we prove that an ordered semigroup is simple if and only if it is intra-regular and archimedean. Finally, we characterize the semilattices of left (and right) simple (ordered) semigroups in terms of fuzzy sets. An ordered semigroup S is a semilattice of left (resp. right) simple semigroups if and only if it is decomposable into pairwise disjoint left (resp. right) simple components S_α indexed by a semilattice (: idempotent and commutative semigroup) Y such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for every $\alpha, \beta \in Y$. Fuzzy semigroups (without order) have been systematically studied by N. Kuroki. We refer, for example, to his papers in [6, 7].

2. Main results

Lemma 1. (cf. also [1; Proposition 2]) *Let S be an ordered groupoid. If A is a left (resp. right) ideal of S, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal* *of S. Conversely, if A is a nonempty set and f_A a fuzzy left (resp. fuzzy right) ideal of S, then A is a left (resp. right) ideal of S.*

Theorem 2. *Let S be an ordered semigroup. We consider the statements:*

(1) *S is fuzzy left duo.*

(2) *S is left duo.*

Then (1) $\Rightarrow$ (2). In particular, if S is left regular (or regular), then (1) $\Leftrightarrow$ (2).

Proof. (1) $\Longrightarrow$ (2). Let S be fuzzy left duo and A a left ideal of S. By Lemma 1, f_A is a fuzzy left ideal of S. By hypothesis, f_A is a fuzzy right ideal of S. Since A is nonempty, by Lemma 1, A is a right ideal of S, so S is left duo.

(2) $\Longrightarrow$ (1). Let S be left regular and left duo, f a fuzzy left ideal of S and $a, b \in S$. As $(Sa]$ is a left ideal of S, by hypothesis, it is a right ideal of S as well. Since S is left regular, we have $ab \in (Sa^2]b \subseteq (Sa]S \subseteq (Sa]$. Then $ab \leq xa$ for some $x \in S$. Since f is a fuzzy left ideal of S, we have $f(ab) \geq f(xa) \geq f(a)$ i.e. f is a fuzzy right ideal of S. Thus S is fuzzy left duo. □

The right analogue of Theorem 2 also holds, and we have

Corollary 3. *The fuzzy duo ordered semigroups are duo.*

Lemma 4. (cf. also [2; Theorem 1]) *Let S be an ordered semigroup. If A is a bi-ideal of S, then the characteristic function f_A is a fuzzy bi-ideal of S. Conversely, if A is a nonempty set and f_A a fuzzy bi-ideal of S, then A is a bi-ideal of S.*

Theorem 5. *Let $(S, ., \leq)$ be an ordered semigroup. We consider the following statements:*

(1) *Every fuzzy bi-ideal of S is a fuzzy right ideal of S.*

(2) *Every bi-ideal of S is a right ideal of S.*

Then (1) $\Rightarrow$ (2). In particular, if S is regular, then (1) $\Leftrightarrow$ (2).

Proof. (1) $\Longrightarrow$ (2). Let A be a bi-ideal of S. By Lemma 4, f_A is a fuzzy bi-ideal of S. By hypothesis, f_A is a fuzzy right ideal of S. Since A is nonempty, by Lemma 1, A is a right ideal of S.

(2) $\Longrightarrow$ (1). Let S be regular, f a fuzzy bi-ideal of S and $a, b \in S$. Since $(aSa]$ is a bi-ideal of S, by hypothesis, it is a right ideal of S. Since S is regular, we have $ab \in (aSa]S \subseteq (aSa]$. Then there exists $x \in S$ such that $ab \le axa$. Since f is a fuzzy bi-ideal of S, we have $f(ab) \ge f(axa) \ge \min\{f(a), f(a)\} = f(a)$, so f is a fuzzy right ideal of S. $\square$

Remark 6. Theorem 5 remains true if we replace the word "right" by "left". As a consequence, if S is a regular ordered semigroup, then the bi-ideals of S are ideals of S if and only if the fuzzy bi-ideals of S are fuzzy ideals of S.

We characterize the ordered semigroups in which, for any element a of S, we have $a \le xa^k y$ (or $a \le xa^k$) for some $x, y \in S$.

Theorem 7. *Let $(S, ., \le)$ be an ordered semigroup, $a \in S$ and $N \ni k \ge 2$. The following are equivalent:*

(1) *There exist $x, y \in S$ such that $a \le xa^k y$.*

(2) *For every fuzzy ideal f of S, we have $f(a) = f(a^k)$.*

Proof. (1) $\Longrightarrow$ (2). Let f be a fuzzy ideal of S. By hypothesis, there exist $x, y \in S$ such that $a \le xa^k y$. Since f is a fuzzy ideal of S, we have $f(a) \ge f(xa^k y) \ge f(a^k) = f(aa^{k-1}) \ge f(a)$, so $f(a) = f(a^k)$.

(2) $\Longrightarrow$ (1). Since $I(a^k)$ is an ideal of S, by Lemma 1, $f_{I(a^k)}$ is a fuzzy ideal of S. By hypothesis, $f_{I(a^k)}(a) = f_{I(a^k)}(a^k) = 1$, then $a \in I(a^k) = (a^k \cup Sa^k \cup a^k S \cup Sa^k S]$. If $a \le a^k$, then $a \le aaa^{k-2} \le aa^k a^{k-2}$, where $a, a^{k-2} \in S$, and property (1) is satisfied. If $a \le xa^k$ for some $x \in S$, then $a \le xaa^{k-1} \le x(xa^k)a^{k-1} = x^2 a^k a^{k-1}$, where $x^2, a^{k-1} \in S$, and (1) holds. If $a \le a^k x$ for some $x \in S$, then $a \le a^{k-1}ax \le a^{k-1}(a^k x)x = a^{k-1}a^k x^2$, again property (1) is true. Finally, we have the case $a \le xa^k y$ for some $x, y \in S$, which is condition (1). $\square$

As an immediate consequence of Theorem 7, we have the following theorem.

Theorem 8. *For an ordered semigroup S and a natural number k such that $k \ge 2$, the following are equivalent:*

(1) *For every $a \in S$, there exist $x, y \in S$ such that $a \le xa^k y$.*

(2) *For every fuzzy ideal f of S and every $a \in S$, we have $f(a) = f(a^k)$.*

Corollary 9. [8; Theorem 5.3] *An ordered semigroup S is intra-regular if and only if for every fuzzy ideal f of S and any $a \in S$, we have $f(a) = f(a^2)$.*

Proposition 10. *Let $(S, ., \leq)$ be an ordered semigroup and $N \ni k \geq 2$. If for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^k y$, then for every fuzzy ideal f of S and every $a, b \in S$, we have $f(ab) \geq f((ba)^{k-1})$.*

Proof. Let f be a fuzzy ideal of S and $a, b \in S$. By Theorem 8, we have

$$f(ab) = f((ab)^k) = f(a(ba)^{k-1}b) \geq f((ba)^{k-1}).$$

□

Corollary 11. [8; Theorem 5.4] *If S is an intra-regular ordered semigroup, f a fuzzy ideal of S and $a, b \in S$, then $f(ab) = f(ba)$.*

Proof. By Proposition 10, we have $f(ab) \geq f(ba)$. By symmetry, we get $f(ba) \geq f(ab)$, thus we have $f(ab) = f(ba)$. □

In a similar way we prove the following theorem.

Theorem 12. *Let $(S, ., \leq)$ be an ordered semigroup, $a \in S$ and $N \ni k \geq 2$. The following are equivalent:*

(1) *There exists $x \in S$ such that $a \leq xa^k$ (resp. $a \leq a^k x$).*

(2) *For every fuzzy left (resp. right) ideal f of S, we have $f(a) = f(a^k)$.*

Theorem 13. *For an ordered semigroup S and a natural number k such that $k \geq 2$, the following are equivalent:*

(1) *For every $a \in S$, there exists $x \in S$ such that $a \leq xa^k$ (resp. $a \leq a^k x$).*

(2) *For every fuzzy left (resp. right) ideal f of S and every $a \in S$, we have $f(a) = f(a^k)$.*

Corollary 14. *An ordered semigroup S is left (resp. right) regular if and only if for every fuzzy left (resp. fuzzy right) ideal f of S, we have $f(a) = f(a^2)$.*

We deal now with semiprime subsets of ordered semigroups and with simple, archimedean and intra-regular ordered semigroups.

Definition 15. An ordered semigroup S is called *fuzzy left* (resp. *right*) *simple* if every fuzzy left (resp. right) ideal f of S is a constant function, that is

$$f(a) = f(b) \text{ for every } a, b \in S.$$

S is called *fuzzy simple* if for any fuzzy ideal f of S and any $a, b \in S$, we have $f(a) = f(b)$.

Theorem 16. *An ordered semigroup S is left simple if and only if it is fuzzy left simple.*

Proof. $\Longrightarrow$. Let f be a fuzzy left ideal of S and $a, b \in S$. Since $(Sa]$, $(Sb]$ are left ideals of S and S is left simple, we have $(Sa] = S$ and $(Sb] = S$. Then $b \leq xa$ and $a \leq yb$ for some $x, y \in S$. Since f is a fuzzy left ideal of S, we have $f(b) \geq f(xa) \geq f(a) \geq f(yb) \geq f(b)$, thus $f(a) = f(b)$.

$\Longleftarrow$. Let A be a left ideal of S and $a \in S$. Take an element $b \in A$ ($A \neq \emptyset$). Since f_A is a fuzzy left ideal of S, by hypothesis, we have $f_A(a) = f_A(b) = 1$, then $a \in A$. So $A = S$, and S is left simple. $\square$

Theorem 17. [3; Theorem 3.7] *An ordered semigroup S is simple if and only if it is fuzzy simple.*

Definition 18. Let S be an ordered semigroup. A fuzzy subset f of S is called *fuzzy semiprime* if

$$f(a) \geq f(a^2) \text{ for every } a \in S.$$

Lemma 19. *Let S be an ordered semigroup, f a fuzzy semiprime fuzzy left ideal of S and $a \in S$. Then*

$$f(a) = f(a^n) \textit{ for every } n \in N.$$

Proof. For $n = 2$, we have $f(a^2) = f(aa) \geq f(a) \geq f(a^2)$, so $f(a) = f(a^2)$. Suppose $f(a) = f(a^n)$ for some $N \ni n \geq 2$. Then

$$f(a^{n+1}) = f(aa^n) \geq f(a^n) \geq f(a^{2n}) = f(a^{n-1}a^{n+1}) \geq f(a^{n+1}).$$

Thus we have $f(a^{n+1}) = f(a^n) = f(a)$. $\square$

For the sake of completeness, we give the following lemma.

Lemma 20. *Let S be an ordered semigroup. A subset A of S is semiprime if and only if the characteristic function f_A is fuzzy semiprime.*

Proof. $\Longrightarrow$. Let A be a semiprime subset of S and $a \in S$. If $a^2 \in A$ then, since S is semiprime, we have $a \in A$, and $1 = f_A(a) = f_A(a^2)$. If $a^2 \notin A$, then $f_A(a^2) = 0 \leq f_A(a)$. Thus we have $f_A(a) \geq f_A(a^2)$ for every $a \in S$, and f_A is fuzzy semiprime.
$\Longleftarrow$. Let A be a subset of S such that f_A is fuzzy semiprime, and let $a \in S$ such that $a^2 \in A$. Since f_A is fuzzy semiprime, we have $f_A(a) \geq f_A(a^2) = 1$. Since f_A is a fuzzy subset of S, we have $f_A(a) \leq 1$. Then $f_A(a) = 1$, and $a \in A$. Thus S is semiprime. □

The results on ordered semigroups can be also obtained from corresponding results obtained via fuzzy sets. In the rest of the paper, for a statement on ordered semigroups, we first give an independent proof, then we prove the same using fuzzy sets.

Theorem 21. *Let S be a left simple ordered semigroup. Then every bi-ideal of S is a right ideal of S.*

Proof. Let B be a bi-ideal of S. Since $(SB]$ is a left ideal of S and S is left simple, we have $(SB] = S$. Then we have $BS = B(SB] \subseteq (B](SB] \subseteq (BSB] \subseteq (B] = B$, and B is a right ideal of S. □

Theorem 22. *Let S be a left simple ordered semigroup. Then every fuzzy bi-ideal of S is a fuzzy right ideal of S.*

Proof. Let f be a fuzzy bi-ideal of S and $a, b \in S$. Since $(Sa]$ is a left ideal of S and S is left simple, we have $(Sa] = S$, then $b \leq xa$ for some $x \in S$, and $ab \leq axa$. Since f is a fuzzy bi-ideal of S, we obtain $f(ab) \geq f(axa) \geq \min\{f(a), f(a)\} = f(a)$, so f is a fuzzy right ideal of S. □

Second proof of Theorem 21 using fuzzy sets

Theorem 21 can be obtained as a Corollary of Theorem 22, as follows:

Let B be a bi-ideal of S. By Lemma 4, f_B is a fuzzy bi-ideal of S. By Theorem 22, f_B is a fuzzy right ideal of S. Since B is nonempty, by Lemma 1, B is a right ideal of S. □

Theorem 23. *An ordered semigroup S is intra-regular if and only if every ideal of S is semiprime.*

Proof. $\Longrightarrow$. Let A be an ideal of S, $a \in S$ and $a^2 \in A$. Since S is intra-regular, we have $a \in (Sa^2S] \subseteq (SAS] \subseteq (A] = A$, and $a \in A$. Thus S is semiprime.
$\Longleftarrow$. Let $a \in S$. Since $(Sa^2S]$ is an ideal of S, by hypothesis, $(Sa^2S]$ is semiprime. Since $a^4 \in (Sa^2S]$, we have $a^2 \in (Sa^2S]$, and $a \in (Sa^2S]$. Thus S is intra-regular. □

Theorem 24. *An ordered semigroup S is intra-regular if and only if every fuzzy ideal of S is fuzzy semiprime.*

Proof. $\Longrightarrow$. Let f be a fuzzy ideal of S and $a \in S$. Since S is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Since f is a fuzzy ideal of S, we get $f(a) \geq f(xa^2y) \geq f(a^2)$.
$\Longleftarrow$. Let $a \in S$. Since $I(a^2)$ is an ideal of S, by Lemma 1, $f_{I(a^2)}$ is a fuzzy ideal of S. By hypothesis, we have $f_{I(a^2)}(a) \geq f_{I(a^2)}(a^2) = 1$. Then $f_{I(a^2)}(a) = 1$, by easy calculation we have $a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S] = (Sa^2S]$, then $a \in (Sa^2S]$, and S is intra-regular. □

Second proof of Theorem 23 using fuzzy sets

Theorem 23 can be also obtained as an application of Theorem 24, using fuzzy sets, as follows:

$\Longrightarrow$. Let S be intra-regular and A an ideal of S. By Lemma 1, f_A is a fuzzy ideal of S, by Theorem 24, f_A is fuzzy semiprime, by Lemma 20, A is semiprime.
$\Longleftarrow$. Suppose every ideal of S is semiprime. Then every fuzzy ideal of S is fuzzy semiprime. Indeed: Let f be a fuzzy ideal of S and $a \in S$. Since $(Sa^2S]$ is an ideal of S, by hypothesis, $(Sa^2S]$ is semiprime. Since $a^4 \in (Sa^2S]$, we have $a^2 \in (Sa^2S]$, and $a \in (Sa^2S]$. Then $a \leq xa^2y$ for some $x, y \in S$. Since f is a fuzzy ideal of S, we get $f(a) \geq f(xa^2y) \geq f(a^2)$, so f is fuzzy semiprime. By Theorem 24, S is intra-regular. □

Definition 25. (cf., for example [4]) An ordered semigroup S is called *archimedean* if for every $a, b \in S$ there exists $n \in N$ such that $b^n \in (SaS]$ (or $a^n \in (SbS]$).
That is, for every $a, b \in S$ there exists $n \in N$ such that $b^n \leq xay$ for some $x, y \in S$.

Theorem 26. *Let S be an archimedean ordered semigroup. Then S does not contain proper semiprime ideals.*

Proof. Let A be a semiprime ideal of S and $a \in S$. Take an element $b \in A$ $(A \neq \emptyset)$. Since $b, a \in S$ and S is archimedean, there exists $n \in N$ such that $a^n \in (SbS] \subseteq (SAS] \subseteq (A] = A$, so $a^n \in A$. If $n = 1$, then $a \in A$. If $n = 2$ then, since S is semiprime, we have $a \in A$. Let us prove it, for example for $n = 17$, and this is the way we work for any n. Let $a^{17} \in A$. Then, since A is a subsemigroup of S, we have $a^{17}a^{15} \in A$, then $(a^{16})^2 \in A$. Since A is semiprime, we have $(a^8)^2 = a^{16} \in A$, $(a^4)^2 = a^8 \in A$, $(a^2)^2 = a^4 \in A$, $a^2 \in A$, and $a \in A$. □

Theorem 27. *Let S be an archimedean ordered semigroup. Then every fuzzy semiprime fuzzy ideal of S is a constant function.*

Proof. Let f be a fuzzy semiprime fuzzy ideal of S and $a, b \in S$. Since S is archimedean, there exist $n \in N$ and $x, y \in S$ such that $a^n \leq xby$. By Lemma 19, we have $f(a) = f(a^n) \geq f(xby) \geq f(b)$. By symmetry, we get $f(b) \geq f(a)$, so $f(a) = f(b)$. □

Second proof of Theorem 26 using fuzzy sets

Theorem 26 can be also obtained as a Corollary of Theorem 27 using fuzzy sets, as follows:

Let A be a semiprime ideal of S and $b \in S$. By Lemmas 1 and 20, f_A is a fuzzy semiprime fuzzy ideal of S. Take an element $a \in A$ $(A \neq \emptyset)$. By Theorem 27, we have $1 = f_A(a) = f_A(b)$, and $b \in A$. Thus we have $A = S$. □

Theorem 28. *An ordered semigroup S is simple if and only if it is intra-regular and archimedean.*

Proof. $\Longrightarrow$. Let $a \in S$. Since $a, a^2 \in S$ and S is simple, we have $a \in (Sa^2S]$, and S is intra-regular. Let now $a, b \in S$. Since S is simple, we have $a^1 = a \in (SbS]$, thus S is archimedean.

$\Longleftarrow$. Let $a, b \in S$. By hypothesis, $a \in (Sa^2S]$ and there exists $n \in N$ such that $a^n \in (SbS]$. If $n = 1$, then $a \in (SbS]$, so S is simple. If $n = 2$, then $a \in (Sa^2S] \subseteq (S(SbS]S] = (S(SbS)S] \subseteq (SbS]$. If $n = 3$ then, since S is intra-regular, we have $a \in (Sa^2S] \subseteq$

$(S(Sa^4S]S] \subseteq (S(Sa^4S)S] \subseteq (Sa^4S] \subseteq (S(SbS]aS] \subseteq (SbS]$. If $n = 4$, then $a \in (Sa^2S] \subseteq (Sa^4S] \subseteq (S(SbS]S] \subseteq (SbS]$. If $n = 5$ then, since S is intra-regular, we have $a \in (Sa^2S] \subseteq (Sa^4S] \subseteq (S(Sa^8S]S] \subseteq (Sa^8S] \subseteq (S(SbS]a^3S] \subseteq (SbS]$. If $n = 6$, then $a \in (Sa^2S] \subseteq (Sa^4S] \subseteq (Sa^8S] \subseteq (S(SbS]a^2S] \subseteq (SbS]$. If $n = 7$ then, since S is intra-regular, we have $a \in (Sa^2S] \subseteq (Sa^4S] \subseteq (Sa^8S] \subseteq (S(SbS]aS] \subseteq (SbS]$. Suppose, for example, $n = 17$. Then, since S is intra-regular, we have

$$\begin{aligned} a \in (Sa^2S] &\subseteq (Sa^4S] \subseteq (Sa^8S] \subseteq (Sa^{16}S] \subseteq (Sa^{32}S] = (Sa^{17}a^{15}S] \\ &\subseteq (S(SbS]a^{15}S] = (S(SbS)a^{15}S] \subseteq (SbS]. \end{aligned}$$

Continuing this way, for every $n \in N$, $a^n \in (SbS]$ implies $a \in (SbS]$, and so S is simple. $\square$

Second proof of Theorem 28 using fuzzy sets

$\Longrightarrow$. Let S be simple. By Theorem 17, it is fuzzy simple, that is, for any fuzzy ideal f of S and any $a, b \in S$, we have $f(a) = f(b)$. Then for any fuzzy ideal f of S and any $a \in S$, we have $f(a) = f(a^2)$. By Corollary 9, S is intra-regular. Let now $a, b \in S$. Since S is simple, we have $b^1 := b \in (SaS]$, and S is archimedean.

$\Longleftarrow$. By Theorem 17, it is enough to prove that S is fuzzy simple. For this purpose, let f be a fuzzy ideal of S. Since S is intra-regular, by Corollary 9, we have $f(a) = f(a^2)$ for every $a \in S$, then f is fuzzy semiprime. Since S is archimedean and f a fuzzy semiprime fuzzy ideal of S, by Theorem 27, f is a constant function, which means that S is fuzzy simple. $\square$

Finally, we characterize the ordered semigroups which are left regular and left duo in terms of fuzzy left ideals. We need the following lemma.

Lemma 29. (cf. [5; Theorem 6]) *Let S be ordered semigroup. The following are equivalent:*

(1) *S left regular and left duo.*

(2) *If A, B are left ideals of S, then $(AB] = (BA]$ and $(A^2] = A$.*

Theorem 30. *An ordered semigroup S is left regular and left duo if and only if for every fuzzy left ideal f of S and every $a, b \in S$, we have $f(a) = f(a^2)$ and $f(ab) = f(ba)$.*

Proof. $\Longrightarrow$. Let f be a fuzzy left ideal of S and $a, b \in S$. Since S is left regular, by Corollary 14, we have $f(a) = f(a^2)$ and $f(ab) = f((ab)^2)$. Since S is left regular and left duo, by Theorem 2, S is fuzzy left duo. Thus we have

$$\begin{aligned} f(ab) &= f((ab)^2) = f((aba)b) \\ &\geq f(aba) \text{ (since } f \text{ is a fuzzy right ideal of } S) \\ &= f(a(ba)) \geq f(ba) \text{ (since } f \text{ is a fuzzy left ideal of } S). \end{aligned}$$

By symmetry, we get $f(ba) \geq f(ba)$, so $f(ab) = f(ba)$.

$\Longleftarrow$. Let A, B be left ideals of S, $a \in A$ and $b \in B$. Since $L(ba)$ is a left ideal of S, by Lemma 1, $f_{L(ba)}$ is a fuzzy left ideal of S. By hypothesis, we have $f_{L(ba)}(ab) = f_{L(ba)}(ba) = 1$, so $ab \in L(ba) = (ba \cup Sba] \subseteq (BA \cup (SB)A] = (BA]$. Thus we have $AB \subseteq (BA]$, then $(AB] \subseteq ((BA]] = (BA]$. By symmetry, we get $(BA] \subseteq (AB]$, so $(AB] = (BA]$. Since $L(a^2)$ is a left ideal of S, by Lemma 1, $f_{L(a^2)}$ is a fuzzy left ideal of S. Again by hypothesis, we obtain $f_{L(a^2)}(a) = f_{L(a^2)}(a^2) = 1$, hence

$$a \in L(a^2) = (a^2 \cup Sa^2] \subseteq (A^2 \cup (SA)A] = (A^2].$$

Therefore we have $A \subseteq (A^2] \subseteq (SA] \subseteq A$, and $(A^2] = A$. By Lemma 29, S is left regular and left duo. $\square$

The right analog of the results of this paper also hold.

As the ordered semigroups which are both left regular and left duo and the semilattices of left regular and left duo (ordered) semigroups are the same (cf. [5; Theorem 6]), we have the following corollary which gives a necessary sufficient condition under which an ordered semigroup is decomposable into left (or right) simple components.

Corollary 31. *An ordered semigroup S is a semilattice of left (resp. right) simple semigroups if and only if for every fuzzy left (resp. fuzzy right) ideal f of S and every $a, b \in S$, we have $f(a) = f(a^2)$ and $f(ab) = f(ba)$.*

Problem 1. Using computer find an example of an ordered semigroup S of order, preferable 5, which is not regular, it is left duo but it is not fuzzy left duo (that is, there exists a fuzzy left ideal of S which is not a fuzzy right ideal of S).

Problem 2. Using computer find an example of an ordered semigroup S of order, preferable 5, which is is not regular, every bi-ideal of S is a right ideal of S and there is a fuzzy bi-ideal of S which is not a fuzzy right ideal of S.

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