MAPS AND FUZZY CONNECTIONS

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Abstract. In this paper, we investigate the relations between maps and residuated (dual residuated, residuated, Galois, dual Galois) connections in complete residuated lattices.

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1. Introduction

In this paper, we investigate the relations between maps and residuated (dual residuated, residuated, Galois, dual Galois) connections in complete residuated lattices. We give their examples.

**Definition 1.1.** [1,7] An algebra \((L, \land, \lor, \odot, \to, 0, 1)\) is called a complete residuated lattice if it satisfies the following conditions:

1. \((L, \leq, \lor, \land, 1, 0)\) is a complete lattice with the greatest element 1 and the least element 0;
2. \((L, \odot, 1)\) is a commutative monoid;
3. \(x \odot y \leq z\) iff \(x \leq y \to z\) for \(x, y, z \in L\).

In this paper, we assume \((L, \land, \lor, \odot, \to, *, 0, 1)\) is a complete residuated lattice with the law of double negation; i.e. \(x^{**} = x\).

**Lemma 1.2.** [1,7] For each \(x, y, z, x_i, y_i \in L\), we have the following properties.

1. If \(y \leq z\), \((x \odot y) \leq (x \odot z)\), \(x \to y \leq x \to z\) and \(z \to x \leq y \to z\).
2. \((x \to (\land_{i \in \Gamma} y_i)) = \land_{i \in \Gamma} (x \to y_i)\).
3. \((\lor_{i \in \Gamma} x_i) \to y = \lor_{i \in \Gamma} (x_i \to y)\).
4. \(\land_{i \in \Gamma} y_i^* = (\lor_{i \in \Gamma} y_i)^*\) and \(\lor_{i \in \Gamma} y_i^* = (\land_{i \in \Gamma} y_i)^*\).
5. \((x \odot y) \to z = x \to (y \to z) = y \to (x \to z)\).
6. \(x \odot y = (x \to y^*)^*\) and \(x \to y = y^* \to x^*\).
7. \(x \odot (x \to y) \leq y\).
8. \((x \to y) \odot (y \to z) \leq x \to z\).
9. \(x \leq y \to z\) iff \(y \leq x \to z\).

**Definition 1.3.** [1-3] Let \(X\) be a set. A function \(e_X : X \times X \to L\) is called:

1. **reflexive** if \(e_X(x, x) = 1\) for all \(x \in X\),
2. **transitive** if \(e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)\), for all \(x, y, z \in X\),
3. **if** \(e_X(x, y) = e_X(y, x) = 1\), then \(x = y\).

If \(e\) satisfies (E1) and (E2), \((X, e_X)\) is a fuzzy preorder set. If \(e\) satisfies (E1), (E2) and (E3), \((X, e_X)\) is a fuzzy partially order set (simply, fuzzy poset).

**Remark 1.4.** (1) We define a function \(e_{L^X} : L^X \times L^X \to L\) as \(e_{L^X}(A, B) = \land_{x \in X} (A(x) \to B(x))\). Then \((L^X, e_{L^X})\) is a fuzzy poset from Lemma 1.2 (8).
(2) We denote $e_X^{-1}(x,y) = e_X(y,x)$, $(e_X)_x(y) = e_X(x,y)$ and $(e_X)_y^{-1} = e_X(x,y)$. Moreover, $1_x$ is a characteristic function such that $1_x(x) = 0$, $1_x(y)$, for otherwise.

**Definition 1.5.**[1-3] Let $(X, e_X)$ and $(Y, e_Y)$ be a fuzzy poset and $f : X \to Y$ and $g : Y \to X$ maps.

1. $(e_X, f, g, e_Y)$ is called a Galois connection if for all $x \in X, y \in Y$,
   $$e_Y(y, f(x)) = e_X(x, g(y)).$$

2. $(e_X, f, g, e_Y)$ is called a dual Galois connection if for all $x \in X, y \in Y$,
   $$e_Y(f(x), y) = e_X(g(y), x).$$

3. $(e_X, f, g, e_Y)$ is called a residuated connection if for all $x \in X, y \in Y$,
   $$e_Y(f(x), y) = e_X(x, g(y)).$$

4. $(e_X, f, g, e_Y)$ is called a dual residuated connection if for all $x \in X, y \in Y$,
   $$e_Y(y, f(x)) = e_X(g(y), x).$$

5. A map $f : (X, e_X) \to (Y, e_Y)$ is called an isotone map if for all $x, z \in X$, $e_X(x, z) \leq e_Y(f(x), f(z))$.

6. A map $f : (X, e_X) \to (Y, e_Y)$ is called an antitone map if for all $x, z \in X$, $e_X(x, z) \leq e_Y(f(z), f(x))$.

2. Maps and fuzzy connections

**Theorem 2.1.** Let $(X, e_X)$ and $(Y, e_Y)$ be a fuzzy poset and $f : X \to Y$ and $g : Y \to X$ maps. For each $A \in L^X$ and $B \in L^Y$, we define operations as follows:

$$F_1(A)(y) = \bigwedge_{x \in X} (A(x) \to e_Y(y, f(x))), \quad F_2(A)(y) = \bigwedge_{x \in X} (A(x) \to e_Y(f(x), y)),$$

$$G_1(B)(x) = \bigwedge_{y \in Y} (B(y) \to e_X(x, g(y))), \quad G_2(B)(x) = \bigwedge_{y \in Y} (B(y) \to e_X(g(y), x)),$$

$$H_1(B)(x) = \bigvee_{y \in Y} (e_X(x, g(y)) \odot B(y)), \quad H_2(B)(x) = \bigvee_{y \in Y} (e_X(g(y), x) \odot B(y)).$$
\[ I_1(A)(y) = \bigvee_{x \in X} (A(x) \circ e_Y(y, f(x))), \quad I_2(A)(y) = \bigvee_{x \in X} (A(x) \circ e_Y(f(x), y)), \]

\[ J_1(B)(x) = \bigwedge_{y \in Y} (e_X(x, g(y)) \rightarrow B(y)), \quad J_2(B)(x) = \bigwedge_{y \in Y} (e_X(g(y), x) \rightarrow B(y)), \]

\[ K_1(A)(y) = \bigwedge_{x \in X} (e_Y(y, f(x)) \rightarrow A(x)), \quad K_2(A)(y) = \bigwedge_{x \in X} (e_Y(f(x), y) \rightarrow A(x)). \]

\[ L_1(B)(x) = \bigvee_{y \in Y} (B^*(y) \circ e_X(x, g(y))), \quad L_2(B)(x) = \bigvee_{y \in Y} (B^*(y) \circ e_X(g(y), x)), \]

\[ M_1(A)(y) = \bigvee_{x \in X} (A^*(x) \circ e_Y(y, f(x))), \quad M_2(A)(y) = \bigvee_{x \in X} (A^*(x) \circ e_Y(f(x), y)). \]

Then the following statements hold:

1. If \( F_1(1_x) = (e_Y)^{1}_{f(x)}, \quad F_2(1_x) = (e_Y)_{f(x)}, \quad K_1(1_x^*) = ((e_Y)^{1}_{f(x)})^*, \quad K_2(1_x^*) = (e_Y)_{f(x)}, \)
   
   \( M_1(1_x^*) = (e_Y)^{1}_{f(x)}, \quad M_2(1_x^*) = (e_Y)_{f(x)}, \quad I_1(1_x) = (e_Y)^{1}_{f(x)} \) and \( I_2(1_x) = (e_Y)_{f(x)}. \)

2. If \( G_1(1_y) = (e_X)^{1}_{g(y)}, \quad G_2(1_y) = (e_X)_{g(y)}, \quad H_1(1_w) = (e_X)^{1}_{g(y)}, \quad H_2(1_w) = (e_X)_{g(y)}, \)
   
   \( J_1(1_y^*) = ((e_X)^{-1}_{g(y)})^*, \quad J_2(1_y^*) = (e_X)_{g(y)}, \quad L_1(1_y^*) = (e_X)_{g(y)} \) and \( L_2(1_y^*) = (e_X)_{g(y)}. \)

3. If \((e_X, f, g, e_Y)\) is a Galois connection iff \((e_{LX}, F_1, G_1, e_{LY})\) is a Galois connection with antitone maps \( f \) and \( g \) iff \((e_{LX}, K_1, H_1, e_{LY})\) is a dual residuated connection with antitone maps \( f \) and \( g \) iff \((e_{LX}, M_1, L_1, e_{LY})\) is a dual Galois connection with antitone maps \( f \) and \( g \) iff \((e_{LX}, I_1, J_1, e_{LY})\) is a residuated connection with antitone maps \( f \) and \( g \).

4. If \((e_X, f, g, e_Y)\) is a residuated connection iff \((e_{LX}, F_2, G_1, e_{LY})\) is a Galois connection with isotone maps \( f \) and \( g \) iff \((e_{LX}, K_2, H_1, e_{LY})\) is a dual residuated connection with isotone maps \( f \) and \( g \) iff \((e_{LX}, M_2, L_1, e_{LY})\) is a dual Galois connection with isotone maps \( f \) and \( g \) iff \((e_{LX}, I_2, J_1, e_{LY})\) is a residuated connection with isotone maps \( f \) and \( g \).

5. If \((e_X, f, g, e_Y)\) is a dual Galois connection iff \((e_{LX}, F_2, G_2, e_{LY})\) is a Galois connection with antitone maps \( f \) and \( g \) iff \((e_{LX}, K_2, H_2, e_{LY})\) is a dual residuated connection with antitone maps \( f \) and \( g \) iff \((e_{LX}, M_2, L_2, e_{LY})\) is a dual Galois connection with antitone maps \( f \) and \( g \) iff \((e_{LX}, I_2, J_2, e_{LY})\) is a residuated connection with antitone maps \( f \) and \( g \).

6. If \((e_X, f, g, e_Y)\) is a dual residuated connection iff \((e_{LX}, F_1, G_2, e_{LY})\) is a Galois connection with isotone maps \( f \) and \( g \) iff \((e_{LX}, K_1, H_2, e_{LY})\) is a dual residuated connection with isotone maps \( f \) and \( g \) iff \((e_{LX}, M_1, L_2, e_{LY})\) is a dual Galois connection with isotone maps \( f \) and \( g \) iff \((e_{LX}, I_1, J_2, e_{LY})\) is a residuated connection with isotone maps \( f \) and \( g \).
(7) If \( e_X(x, y) \leq e_Y(f(x), f(y)) \), then

\[
F_1((e_X)_z) = (e_Y)_{f(z)}^{-1}, \quad F_2((e_X)_z) = (e_Y)_{f(z)}, \quad K_1((e_X)_z^{-1}) = ((e_Y)_{f(z)}^{-1})^*, \quad K_2((e_X)_z) = (e_Y)_{f(z)}^*, \\
I_1((e_X)_z) = (e_Y)_{f(z)}^{-1}, \quad I_2((e_X)_z) = (e_Y)_{f(z)}, \quad M_1((e_X)_z) = (e_Y)_{f(z)}^{-1}, \quad M_2((e_X)_z) = (e_Y)_{f(z)}.
\]

(8) If \( e_X(x, y) \leq e_Y(f(y), f(x)) \), then

\[
F_1((e_X)_z) = (e_Y)_{f(z)}^{-1}, \quad F_2((e_X)_z) = (e_Y)_{f(z)}, \quad K_1((e_X)_z) = ((e_Y)_{f(z)}^{-1})^*, \quad K_2((e_X)_z) = (e_Y)_{f(z)}^*, \\
I_1((e_X)_z) = (e_Y)_{f(z)}^{-1}, \quad I_2((e_X)_z) = (e_Y)_{f(z)}, \quad M_1((e_X)_z) = (e_Y)_{f(z)}^{-1}, \quad M_2((e_X)_z) = (e_Y)_{f(z)}.
\]

(9) If \( e_Y(x, y) \leq e_X(g(x), g(y)) \), then

\[
G_1((e_Y)_y) = (e_X)_{g(y)}^{-1}, \quad G_2((e_Y)_y) = (e_X)_{g(y)}, \quad H_1((e_Y)_y) = (e_X)_{g(y)}^{-1}, \quad H_2((e_Y)_y) = (e_X)_{g(y)}, \\
J_1((e_Y)_y^*) = ((e_X)_{g(y)}^{-1})^*, \quad J_2((e_Y)_y^*) = (e_X)_{g(y)}^*, \quad L_1((e_Y)_y^*) = (e_X)_{g(y)}^{-1}, \quad L_2((e_Y)_y^*) = (e_X)_{g(y)}.
\]

(10) If \( e_Y(x, y) \leq e_X(g(y), g(x)) \), then

\[
G_1((e_Y)_y) = (e_X)_{g(y)}^{-1}, \quad G_2((e_Y)_y) = (e_X)_{g(y)}, \quad H_1((e_Y)_y) = (e_X)_{g(y)}^{-1}, \quad H_2((e_Y)_y) = (e_X)_{g(y)}, \\
J_1((e_Y)_y^*) = ((e_X)_{g(y)}^{-1})^*, \quad J_2((e_Y)_y^*) = (e_X)_{g(y)}^*, \quad L_1((e_Y)_y^*) = (e_X)_{g(y)}^{-1}, \quad L_2((e_Y)_y^*) = (e_X)_{g(y)}.
\]

**Proof.** (1) and (2) follow from their definitions.

(3) Let \( e_X(x, g(y)) = e_Y(y, f(x)) \) be given. Since \( e_X(g(y), g(y)) = e_Y(y, f(g(y)) = 1 \), then \( g \) is an antitone map from:

\[
e_Y(y_1, y_2) = e_Y(y_1, y_2) \circ e_Y(y_2, f(g(y_2))) \\
\leq e_Y(y_1, f(g(y_2))) = e_X(g(y_2), g(y_1)).
\]

Similarly, \( f \) is an antitone map.

First, we will show that \( e_X(x, g(y)) = e_Y(y, f(x)) \) iff \( e_{LX}(A, G_1(B)) = e_{LY}(B, F_1(A)) \).
Let $e_X(x, g(y)) = e_Y(y, f(x))$ be given. By Lemma 1.2 (2,5), we have

$$e_{LY}(B, F_1(A)) = \bigwedge_{y \in Y} (B(y) \rightarrow F_1(A)(y))$$

$$= \bigwedge_{y \in Y} \left( B(y) \rightarrow \bigwedge_{x \in X} (A(x) \rightarrow e_Y(y, f(x))) \right)$$

$$= \bigwedge_{y \in Y} \bigwedge_{x \in X} \left( A(x) \rightarrow (B(y) \rightarrow e_X(x, g(y))) \right)$$

$$= \bigwedge_{x \in X} \left( A(x) \rightarrow G_1(B)(x) \right)$$

$$= e_{LX}(A, G_1(B)).$$

Conversely, put $A = 1_x$ and $B = 1_y$. By (1) and (2), we have

$$e_Y(y, f(x)) = F_1(1_x)(y) = e_{LY}(1_y, F_1(1_x))$$

$$= e_{LX}(1_x, G_1(1_y)) = G_1(1_y)(x) = e_X(x, g(y)).$$

Second, we will show that $e_X(x, g(y)) = e_Y(y, f(x))$ iff $e_{LX}(H_1(B), A) = e_{LY}(B, K_1(A))$.

Let $e_X(x, g(y)) = e_Y(y, f(x))$ be given. By Lemma 1.2 (3,5), we have

$$e_{LX}(H_1(B), A) = \bigwedge_{x \in X} (H_1(B)(x) \rightarrow A(x))$$

$$= \bigwedge_{x \in X} \left( \bigvee_{y \in Y} (e_X(x, g(y)) \circ B(y)) \rightarrow A(x) \right)$$

$$= \bigwedge_{x \in X} \bigwedge_{y \in Y} \left( B(y) \rightarrow (e_X(x, g(y)) \rightarrow A(x)) \right)$$

$$= \bigwedge_{y \in Y} \left( B(y) \rightarrow \bigwedge_{x \in X} (e_Y(y, f(x)) \rightarrow A(x)) \right)$$

$$= \bigwedge_{y \in Y} \left( B(y) \rightarrow K_1(A)(y) \right)$$

$$= e_{LY}(B, K_1(A))$$

Conversely, put $A = 1^*_x$ and $B = 1_y$. By (1) and (2), we have

$$e_X^*(x, g(y)) = H_1(1_y)^*(x) = e_{LX}(H_1(1_y), 1^*_x)$$

$$= e_{LY}(1_y, K_1(1^*_x)) = K_1(1^*_x)(y) = e_Y^*(y, f(x)).$$

Third, we will show that $e_X(x, g(y)) = e_Y(y, f(x))$ iff $e_{LX}(L_1(B), A) = e_{LY}(M_1(A), B)$. 

Let $e_X(x, g(y)) = e_Y(y, f(x))$ be given. By Lemma 1.2 (3,5,6), we have

$$e_{LY}(M_1(A), B) = \bigwedge_{y \in Y} (M_1(A)(y) \rightarrow B(y))$$

Conversely, put $A = 1^*_x$ and $B = 1^*_y$. Since $M_1(1^*_x)(y) = e_Y(y, f(x))$ and $L_1(1^*_y)(x) = e_X(x, g(y))$ from (1) and (2). Hence we have

$$e_{LY}(M_1(1^*_x), 1^*_y) = e_{LY}(1^*_x, 1^*_y) = e_{LY}(L_1(1^*_x), L_1(1^*_y)) = e_{LY}(1^*_x, 1^*_y).$$

Finally, we will show that $e_X(x, g(y)) = e_Y(y, f(x))$ iff $e_{LY}(A, J_1(B)) = e_{LY}(I_1(A), B)$.

Let $e_X(x, g(y)) = e_Y(y, f(x))$. Then

$$e_{LY}(I_1(A), B) = \bigwedge_{y \in Y} (I_1(A)(y) \rightarrow B(y))$$

Conversely, put $A = 1_x$ and $B = 1_y$. Since $I_1((e_X)_x)(y) = e_Y(y, f(x))$ and $J_1((e_Y)_y)(x) = e_X(x, g(y))$ from (1) and (2),

$$e_{LY}(1_x, 1_y) = e_{LY}(I_1(A), I_1(B)) = e_{LY}(1_x, 1_y).$$

(4) Let $e_X(x, g(y)) = e_Y(f(x), y)$ be given. Since $e_X(g(y), g(y)) = e_Y(f(g(y), y) = 1$, then $g$ is an isotone map from:

$$e_Y(y_1, y_2) = e_Y(y_1, y_2) \circ e_Y(f(g(y_1)), y_1)$$

$$\leq e_Y(f(g(y_1)), y_2) = e_X(g(y_1), g(y_2)).$$
Similarly, $f$ is an isotone map.

First, we will show that $e_X(x, g(y)) = e_Y(f(x), y)$ iff $e_{LX}(A, G_1(B)) = e_{LY}(B, F_2(A))$.

Let $e_X(x, g(y)) = e_Y(f(x), y)$ be given. By Lemma 1.2(2,5), we have

$$e_{LY}(B, F_2(A)) = \wedge_{y \in Y} (B(y) \to F_2(A)(y))$$

$$= \wedge_{y \in Y} \left( B(y) \to \wedge_{x \in X} (A(x) \to e_Y(f(x), y)) \right)$$

$$= \wedge_{y \in Y} \wedge_{x \in X} \left( A(x) \to (B(y) \to e_X(x, g(y))) \right)$$

$$= \wedge_{x \in X} \left( A(x) \to \wedge_{y \in Y} (B(y) \to e_X(x, g(y))) \right)$$

$$= \wedge_{x \in X} \left( A(x) \to G_1(B)(x) \right)$$

$$= e_{LX}(A, G_1(B)).$$

Conversely, put $A = 1_x$ and $B = 1_y$. By (1) and (2), $F_2(1_x) = (e_Y)_{f(x)}$ and $G_1(1_y) = (e_X)^{-1}_{g(y)}$.

$$e_Y(f(x), y) = F_2(1_x)(y) = e_{LY}(1_y, F_2(1_x))$$

$$= e_{LX}(1_x, G_1(1_y)) = G_1(1_y)(x) = e_X(x, g(y)).$$

Second, we will show that $e_X(x, g(y)) = e_Y(y, f(x))$ iff $e_{LX}(H_1(B), A) = e_{LY}(B, K_2(A))$.

If $e_X(x, g(y)) = e_Y(f(x), y)$, then

$$e_{LX}(H_1(B), A) = \wedge_{x \in X} (H_1(B)(x) \to A(x))$$

$$= \wedge_{x \in X} \left( (\forall y \in Y (e_X(x, g(y)) \circ B(y))) \to A(x) \right)$$

$$= \wedge_{x \in X} \wedge_{y \in Y} \left( B(y) \to (e_X(x, g(y)) \to A(x)) \right)$$

$$= \wedge_{y \in Y} \left( B(y) \to \wedge_{x \in X} (e_Y(f(x), y) \to A(x)) \right)$$

$$= \wedge_{y \in Y} \left( B(y) \to K_2(A)(y) \right)$$

$$= e_{LY}(B, K_2(A)).$$

Put $A = 1_x^*$ and $B = 1_y$. By (1) and (2), $K_2(1_x^*) = (e_Y)^{*}_{f(x)}$ and $H_1(1_w) = (e_X)^{-1}_{g(w)}$.

Hence

$$e_X^*(x, g(y)) = K_2(1_x^*)(y) = e_{LX}(H_1(1_y, 1_x^*)$$

$$= e_{LY}(1_y, K_2(1_x^*) = H_1(1_y)^*(x) = e_X^*(x, g(y)).$$

Other cases, (5) and (6) are similarly proved in (3).

(7) We have $F_2((e_X)^{-1}_{z}) = (e_Y)_{f(z)}$ from:
\[ F_2((e_X)^{-1}_z)(y) = \bigwedge_{x \in X} ((e_X)^{-1}_z(x) \to e_Y(f(x), y)) \]
\[ \leq (e_X)^{-1}_z(z) \to e_Y(f(z), y) = e_Y(f(z), y). \]

Since \( f \) is an isotone map,
\[ e_Y(f(z), y) \odot e_X(x, z) \leq e_Y((f(z), y) \odot e_Y(f(x), f(z)) \leq e_Y(f(x), y), \]
\[ e_Y(f(z), y) \leq \bigwedge_{z \in X} ((e_X)^{-1}_z(x) \to e_Y(f(x), y)) = F_2((e_X)^{-1}_z)(y). \]
\[ K_2((e_X)^*_y)(y) = \bigwedge_{z \in X} (e_Y(f(z), y) \to (e_X)^*_y(z)) \]
\[ \leq (e_Y(f(x), y) \to \bot) = e_Y(f(x), y)^*. \]

Thus, \( K_2((e_X)^*_y) \leq (e_Y)^*_y(x) \). Furthermore, \( K_2((e_X)^*_y) \geq (e_Y)^*_y(x) \) from:
\[ e_Y(f(z), y) \odot e_X(x, z) \leq e_Y(f(z), y) \odot e_Y(f(x), f(z)) \leq e_Y(f(x), y) \]
\[ \text{iff } (e_Y(f(x), y))^* \leq e_Y(f(z), y) \to (e_X(x, z))^*. \]

(9) We have \( G_1((e_Y)_y) \leq (e_X)^{-1}_y \) from:
\[ G_1((e_Y)_y)(x) = \bigwedge_{w \in Y} ((e_Y)_y(w) \to e_X(x, g(w))) \leq e_X(x, g(y)). \]

Moreover, \( G_1((e_Y)_y) \geq (e_X)^{-1}_y \) from:
\[ e_X(x, g(y)) \odot e_Y(y, w) \leq e_X(x, g(y)) \odot e_X(g(y), g(w)) \leq e_X(x, g(w)) \]
\[ e_X(x, g(y)) \leq e_Y(y, w) \to e_X(x, g(w)). \]

We have \( H_1((e_Y)^{-1}_w) = (e_X)^{-1}_w \) from:
\[ H_1((e_Y)^{-1}_w)(x) = \bigwedge_{y \in Y} ((e_Y)^{-1}_w(y) \odot e_X(x, g(y))) \geq (e_X)_w(g(w))(x). \]
\[ e_X(x, g(y)) \odot e_Y(y, w) \leq e_X(x, g(y)) \odot e_X(g(y), g(w)) \leq e_X(x, g(w)). \]

Since \( J_1(((e_Y)_y)^{-1})(x) = \bigwedge_{w \in Y} (e_X(x, g(w)) \to ((e_Y)_y)^{-1}(w) \leq (e_X(x, g(y)))^*, \) then
\[ J_1(((e_Y)_y)^{-1}) \leq ((e_X)^{-1}_y)^*. \]

Since \( e_X(x, g(w)) \odot e_Y(y, w) \leq e_X(x, g(w)) \odot e_X(g(w), g(y)) \leq e_X(x, g(y)), \) then
\[ e_X(x, g(w)) \to e_Y(w, y)^* \geq (e_X(x, g(y)))^*. \]

Thus, \( J_1(((e_Y)_y)^{-1}) \geq ((e_X)^{-1}_y)^*. \) Hence \( J_1(((e_Y)_y)^{-1}) = ((e_X)^{-1}_y)^*. \)
Example 2.2. Define a binary operation $\odot$ (called Lukasiewicz conjunction) on $L = [0,1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \quad x \to y = \min\{1 - x + y, 1\}.$$ 

Let $(X \times Y, e)$ satisfy Theorem 2.1(7). For example, $(1)$ we define $f$:

$$f(X) = (1) \cdot \min\{x, y\}, \quad f(Y) = (1) \cdot \min\{x, y\}.$$ 

Then $e$ is a Galois connection, $(e, e^0)$ is a dual Galois connection with antitone maps $f$ and $g$.

(2) We define $h : X \to Y$ with $h(a) = x, h(b) = f(c) = y$. Then $f$ is an isotone map. It satisfies Theorem 2.1(7). For examples,

$$F_2((e_X)^{-1}_a) = F_2(1,0.3,0.5) = (1,0.8,0.6) = (e_Y)_{f(a)} = (e_Y)_x,$$

$$F_2((e_X)^{-1}_b) = F_2(0.7,1,0.5) = (0.6,1,0.5) = (e_Y)_{f(b)} = (e_Y)_y,$$

$$F_2((e_X)^{-1}_c) = F_2(0.4,0.3,1) = (0.6,1,0.5) = (e_Y)_{f(c)} = (e_Y)_y.$$ 

(2) We define $h : X \to Y$ with $h(a) = x, h(b) = h(c) = z$. Then $f$ is an antitone map. It satisfies Theorem 2.1(8). For examples,

$$K_2((e_X)_a) = K_2(0,0.7,0.5) = (0,0.2,0.4) = (e_Y)_{h(a)}^*,$$

$$K_2((e_X)_b) = K_2(0.3,0,0.5) = (0.3,0.4,0) = (e_Y)_{h(b)}^*,$$

$$K_2((e_X)_c) = K_2(0.6,0.4,0) = (0.3,0.4,0) = (e_Y)_{h(c)}^*.$$ 

(3) We define $f$ and $g$ as $f(a) = x, f(b) = y, f(c) = z$ and $g(x) = c, g(y) = a, g(z) = b$. Then $e_Y^0(x, f(a)) = e_X(a, g(x))$ for all $a \in X, x \in Y$. By Theorem 2.1, $(e_X, f, g, e_Y^0)$ is a Galois connection, $(e_{LX}, F_1, G_1, e_{LY})$ is a Galois connection with antitone maps $f$ and $g$, $(e_{LX}, H_1, e_{LY})$ is a dual residuated connection with antitone maps $f$ and $g$, $(e_{LX}, M_1, L_1, e_{LY})$ is a dual Galois connection with antitone maps $f$ and $g$ and $(e_{LX}, J_1, e_{LY})$ is a dual residuated connection with antitone maps $f$ and $g$.
is a residuated connection with antitone maps \( f \) and \( g \). It satisfies Theorem 2.1(8) and (10). For examples,

\[
F_1((e_X)^{-1}(z)) = F_1(1, 0.3, 0.5)(z) = 0.7 = e_Y^0(z, x) \\
F_2((e_X)^{-1}) = F_2(0.7, 1, 0.5) = (0.7, 0.6, 1) = (e_Y)^{f(b)} \\
F_2((e_X)^{-1}) = F_2(0.4, 0.3, 1) = (0.7, 0.6, 1) = (e_Y)^{f(c)}
\]

Example 2.3. Let \( X = \{a, b, c\} \) be a set and \( f : X \rightarrow X \) a function as \( f(a) = b, f(b) = a, f(c) = c \). Define a binary operation \( \odot \) (called Lukasiewicz conjunction) on \( L = [0, 1] \) as Example 2.2.

(1) Let \( (X = \{a, b, c\}, e_1 = (e_X(a, b))) \) be a fuzzy poset as follows:

\[
e_1 = \begin{pmatrix}
1.0 & 0.6 & 0.5 \\
0.6 & 1.0 & 0.5 \\
0.7 & 0.7 & 1.0
\end{pmatrix}
\]

Since \( e_1(f(x), y) = e_1(x, f(y)) \), then \( (e_1, f, f, e_1) \) are both residuated and dual residuated connections. It satisfies Theorem 2.1 (4) and (6). Since \( f \) is an isotone map, it satisfies Theorem 2.1 (7) and (9). For examples,

\[
e_1(f(a), c) = 0.5 = F_2((e_1)^{-1}(c))(1 \rightarrow 0.5) \land (0.6 \rightarrow 0.5) \land (0.7 \rightarrow 1) \\
e_L((e_1)c, F_1((e_1)^{-1})) = (0.7 \rightarrow 0.6) \land (0.7 \rightarrow 1) \land (1 \rightarrow 0.5) \\
e_L((e_1)^{-1}, G_1((e_1)c)) = (1 \rightarrow 0.5) \land (0.6 \rightarrow 0.5) \land (0.7 \rightarrow 0.8) \\
e_1((e_1)\gamma(a) = (0.7 \rightarrow 0.6) \land (0.7 \rightarrow 1) \land (1 \rightarrow 0.5) \\
e_1(a, f(c)) = (e_1)^{-1}(f(c))(a)
\]

\[
e_1^*(f(c), a) = 0.3 = K_2((e_1)^*)(a) = (0.6 \rightarrow 0.3) \land (1 \rightarrow 0.3) \land (0.7 \rightarrow 0) \\
e_L((e_1)^{-1}, K_2((e_1)^*)) = (1 \rightarrow 0.3) \land (0.6 \rightarrow 0.3) \land (0.7 \rightarrow 0) \\
e_L((H_1((e_1)^{-1}), (e_1)^* = (0.6 \rightarrow 0.3) \land (1 \rightarrow 0.3) \land (0.7 \rightarrow 0) \\
e_1((e_1)^{-1}^*)(c) = e_1(c, f(a)))
\]
(2) Let \( (X = \{a, b, c\}, e_2 = (e_2(a, b))) \) be a fuzzy poset as follows:

\[
e_2 = \begin{pmatrix}
1.0 & 0.6 & 0.5 \\
0.6 & 1.0 & 0.7 \\
0.7 & 0.5 & 1.0
\end{pmatrix}
\]

Since \( e_1(y, f(x)) = e_1(x, f(y)) \), then \((e_1, f, f, e_1)\) are both both Galois and dual Galois connections. It satisfies Theorem 2.1 (3) and (5). Since \( f \) is an antitone map, it satisfies Theorem 2.1 (8) and (10). For examples,

\[
e_2(b, f(c)) = 0.7 = F_1((e_2)_c^{-1})(b) = (0.5 \to 1) \land (0.7 \to 0.6) \land (1 \to 0.7)
\]

\[
e_2(b, f(c)) = e_{LY}(e_Y y^{-1}, F_1(e_X x^{-1})) = (0.6 \to 0.5) \land (1 \to 0.7) \land (0.5 \to 1)
\]

\[
e_2(b, f(c)) = e_{LX}(e_X x^{-1}, G_1((e_Y y^{-1})) = (0.5 \to 1) \land (0.7 \to 0.6) \land (1 \to 0.7)
\]

\[
e_2(c, f(b)) = G_1((e_2)_b^{-1})(c) = e_2(c, f(b)).
\]

References