ORE EXTENSIONS OVER NOETHERIAN $\delta$-RINGS

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Abstract. Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ such that $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and $\delta$ a $\sigma$-derivation of $R$ such that $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of $R$. In this paper we show that if $R$ is Noetherian ring such that $\sigma(P) = P$ for all minimal prime ideal $P$ of $R$, then $R[x; \sigma, \delta]$ is 2-primal.

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1. Introduction

Throughout this paper $R$ will denote an associative ring with identity $1 \neq 0$. The field of complex numbers is denoted by $\mathbb{C}$, the set of real numbers is denoted by $\mathbb{R}$ and the field of rational numbers is denoted by $\mathbb{Q}$, the ring of integers is denoted by $\mathbb{Z}$, and the set of positive integers is denoted by $\mathbb{N}$. The set of prime ideals of $R$ is denoted by $Spec(R)$. The set of minimal prime ideals of $R$ is denoted by $MinSpec(R)$. The prime radical and the set of nilpotent elements of $R$ are denoted by $P(R)$ and $N(R)$ respectively. Let $I$ and $J$ be any two ideals of a ring $R$. Then $I \subset J$ means that $I$ is strictly contained in $J$.

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Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Recall that a $\sigma$-derivation of $R$ is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In case $\sigma$ is the identity map, $\delta$ is called just a derivation of $R$. For example for any endomorphism $\tau$ of a ring $R$ and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a $\tau$-derivation of $R$.

Let $\sigma$ be an automorphism of a ring $R$ and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a map defined by $\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}$, for all $r \in R$. Then $\delta$ is a $\sigma$-derivation of $R$ if and only if $\phi$ is a ring homomorphism. Also let $R = K[x]$, $K$ a field. Then the formal derivative $d/dx$ is a derivation of $R$.

Recall that $R[x; \sigma, \delta]$ is the usual polynomial ring with coefficients in $R$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x; \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. An ideal $I$ of a ring $R$ is called $\sigma$-invariant if $\sigma(I) = I$ and is called $\delta$-invariant if $\delta(I) \subseteq I$. If an ideal $I$ of $R$ is $\sigma$-invariant and $\delta$-invariant, then $I[x; \sigma, \delta]$ is an ideal of $O(R)$ and as usual we denote it by $O(I)$. In the case $\delta$ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$. In the case $\sigma$ is the identity map, we denote the differential operator ring $R[x; \delta]$ by $D(R)$.

**2-primal rings**

2-primal rings have been studied in recent years and are being treated by authors for different structures. In [10], Greg Marks discusses the 2-primal property of $O(R)$, where $R$ is a local ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. In Greg Marks [10], it has been shown that for a local ring $R$ with a nilpotent maximal ideal, the Ore extension $O(R)$ will or will not be 2-primal depending on the $\delta$-stability of the maximal ideal of $R$. In the case where $O(R)$ is 2-primal, it will satisfy an even stronger condition; in the case where $O(R)$ is not 2-primal, it will fail to satisfy an even weaker condition.
Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [8]. 2-primal near rings have been discussed by Argac and Groenewald in [1]. Recall that a ring \( R \) is 2-primal if and only if \( N(R) = P(R) \), i.e., if the prime radical is a completely semiprime ideal. An ideal \( I \) of a ring \( R \) is called completely semiprime if \( a^2 \in I \) implies \( a \in I \) for \( a \in R \). We also note that a reduced ring is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [1]- [3], [8], [10].

**Example 1.1**

1. Let \( R = \mathbb{F}[x] \) be the polynomial ring over the field \( \mathbb{F} \). Then \( R \) is 2-primal with \( P(R) = \{0\} \).
2. Let \( R = M_2(\mathbb{Q}) \), the set of \( 2 \times 2 \) matrices over \( \mathbb{Q} \). Then \( R[x] \) is a prime ring with non-zero nilpotent elements and, so cannot be 2-primal.

Now let \( R \) be a Noetherian ring, which is also an algebra over \( \mathbb{Q} \), \( \sigma \) be an automorphism of \( R \) such that \( a\sigma(a) \in P(R) \) implies \( a \in P(R) \) for \( a \in R \) and \( \delta \) be a \( \sigma \)-derivation of \( R \) such that \( a\delta(a) \in P(R) \) implies \( a \in P(R) \). Let \( P \in \text{MinSpecR} \). Then \( P(O(R) = O(P(R)) \) if and only if \( O(R) \) is 2-primal. This is proved in Theorem (3.3).

### 2. Ore extensions

**Definition 2.1 \( \sigma(*) \)-Ring:** In Kwak [9], a ring \( R \) is said to be \( \sigma(*) \)-ring if \( a\sigma(a) \in P(R) \) implies \( a \in P(R) \) for \( a \in R \).

**Example 2.2**

1. Let \( R = \mathbb{C} \) and \( \sigma : \mathbb{C} \to \mathbb{C} \) be the map defined by 
   \[ \sigma(a + ib) = a - ib; \quad a, b \in \mathbb{R}. \]
   Then \( R \) is \( \sigma(*) \)-ring.
2. Let \( R = \mathbb{F}[x] \) be the polynomial ring over the field \( \mathbb{F} \). Let \( \sigma : R \to R \) be an endomorphism defined by \( \sigma(f(x)) = f(0) \).
   Then \( R \) is not a \( \sigma(*) \)-ring.

**Definition 2.3 \( \delta \)-Ring:** Let \( R \) be a ring. Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) be a \( \sigma \)-derivation of \( R \). Then \( R \) is a \( \delta \)-ring if \( a\delta(a) \in P(R) \) implies \( a \in P(R) \).
Proposition 2.4 Let $R$ be a ring and $\sigma$ an automorphism of $R$. Then $R$ is a $\sigma(*)$-ring implies $R$ is 2-primal.

Proof.

Let $a \in R$ be such that $a^2 \in P(R)$. Then
\[ a\sigma(a)\sigma(a') = a\sigma(a')a^2(a) \]
\[ \in \sigma(P(R)) = P(R). \]
Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R).
\]
i.e., every $\sigma(*)$-ring is a 2-primal ring but converse need not be true.

Example 2.5 Let $R = F[x]$ be the polynomial ring over the field $F$. Then $R$ is 2-primal with $P(R) = \{0\}$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$.
Then $R$ is not a $\sigma(*)$-ring.

Proposition 2.6 Let $R$ be a 2-primal ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\delta(P(R)) \subseteq P(R)$. If $P \in \text{MinSpec}(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.

Proof.

Let $P \in \text{MinSpec}(R)$. Now $P$ is a completely prime ideal, therefore, for any $a \in P$, there exists $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of Shin [11]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \subseteq P(R)$; i.e., $\delta(a)\sigma(b) + a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies that $\delta(a)\sigma(b) \in P$. Now $\sigma(P) = P$ implies that $\sigma(b) \notin P$ and since $P$ is completely prime in $R$, we have $\delta(a) \in P$. Hence $\delta(P) \subseteq P$.

Theorem 2.7 Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $\delta$-ring and $\delta(P(R)) \subseteq P(R)$. Then $R$ is 2-primal.

Proof.

Define a map $\rho : R/P(R) \to R/P(R)$ by $\rho(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau : R/P(R) \to R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$, then it can be seen that $\tau$ is an automorphism of $R/P(R)$ and $\rho$ is a $\tau$-derivation of $R/P(R)$. Now $a\delta(a) \in P(R)$ if and only if $(a + P(R))\rho(a + P(R)) = P(R)$ in $R/P(R)$. Thus as
in Proposition (5) of Hong, Kim and Kwak [7], $R$ is a reduced ring and, therefore as mentioned in introduction, $R$ is 2-primal.

**Proposition 2.8** Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$. Then:

1. For any completely prime ideal $P$ of $R$ with $\sigma(P) = P$ and $\delta(P) \subseteq P$, $O(P)$ is a completely prime ideal of $O(R)$.

2. For any completely prime ideal $U$ of $O(R)$, $U \cap R$ is completely prime ideal of $R$.

**Proof.**

See Proposition (4) of [4].

**Corollary 2.9** Let $R$ be a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is moreover a $\delta$-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in \text{MinSpec}(R)$ be such that $\sigma(P) = P$. Then $O(P)$ is a completely prime ideal of $O(R)$.

**Proof.**

$R$ is 2-primal by Theorem (2.7), and so by Proposition (2.6) $\delta(P) \subseteq P$. Further more as mentioned in Proposition (2.6) above, $P$ is a completely prime ideal of $R$. Now use Proposition (2.8), and the proof is complete.

### 3. Main results

**Proposition 3.1** Let $R$ be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ be a $\sigma$-derivation of $R$ such that $R$ is a $\delta$-ring. If $P \in \text{MinSpec}(R)$ is such that $\sigma(P) = P$ then $\delta(P) \subseteq P$.

**Proof.**

Let $P \in \text{MinSpec}(R)$. Then by Proposition (1.1) of [3] $\delta(P(R)) \subseteq P(R)$ and by Theorem (2.7) $R$ is 2-primal. Since $\sigma(P) = P$ by Proposition (2.6) $\delta(P) \subseteq P$.

**Corollary 3.2** Let $R$ be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a $\sigma(*)$-ring and $\delta$ be a $\sigma$-derivation of $R$ such that $R$ is a $\delta$-ring. Let $P \in \text{MinSpec}(R)$. Then $O(P)$ is completely prime ideal of $O(R)$.

**Proof.**
Let $P \in \text{MinSpec}(R)$. Then $\sigma(P) = P$ by Theorem (2) of [6] and $\delta(P) \subseteq P$ by Proposition (3.1). Also $P$ is completely prime ideal of $R$ by Theorem (2) of [6]. Now use Proposition (2.8), and the proof is complete.

We now prove the following Theorem, which is crucial in proving Theorem (3.5).

**Theorem 3.3** Let $R$ be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a $\sigma(\ast)$-ring and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $\delta$-ring. Let $P \in \text{MinSpec}(R)$. Then $P(O(R)) = O(P(R))$ if and only if $O(R)$ is 2-primal.

**Proof.**

Suppose $P(O(R)) = O(P(R))$. We will show that $O(R)$ is 2-primal. Let $g(x) = \sum_{i=0}^{n} x^i b_i \in O(R)$, $b_n \neq 0$, be such that $(g(x))^2 \in P(O(R)) = O(P(R))$. We will show that $g(x) \in P(O(R))$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in \text{MinSpec}(R)$. Now $\sigma(P) = P$ and $P$ is completely prime by Theorem (2) of [6]. Also, $R$ is 2-primal by Proposition (2.4). Therefore we have $b_n \in P$, for all $P \in \text{MinSpec}(R)$; i.e., $b_n \in P(R)$. Now since $\delta(P) \subseteq P$ for all $P \in \text{MinSpec}(R)$ by Proposition (3.1), we get $(\sum_{i=0}^{n-1} x^i b_i)^2 \in P(O(R)) = O(P(R))$ and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all $i$, $0 \leq i \leq n$. Thus we have $g(x) \in O(P(R))$, i.e., $g(x) \in P(O(R))$. Therefore $P(O(R))$ is a completely semiprime ideal of $O(R)$. Hence $O(R)$ is 2-primal.

Conversely, let $O(R)$ be 2-primal. Now by Corollary (3.2) $P(O(R)) \subseteq O(P(R))$. Let $f(x) = \sum_{j=0}^{n} x^j a_j \in O(P(R))$. Now $R$ is a 2-primal subring of $O(R)$ by Proposition (2.4) which implies that $a_j$ is nilpotent and thus $a_j \in N(O(R)) = P(O(R))$, and so we have $x^j a_j \in P(O(R))$ for each $j$, $0 \leq j \leq n$, which implies that $f(x) \in P(O(R))$. Hence $P(O(R)) = O(P(R))$.

**Theorem 3.4** Let $R$ be a Noetherian $\mathbb{Q}$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then:

1. $P_1 \in \text{MinSpec}(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in \text{MinSpec}(O(R))$.
2. $P \in \text{MinSpec}(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in \text{MinSpec}(R)$. 
Proof.

See Theorem (2.3) of [5].

**Theorem 3.5** Let $R$ be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ such that $R$ is a $\sigma(\cdot)$-ring and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $\delta$-ring. If $\sigma(P) = P$ for all $P \in \text{MinSpec}(R)$. Then $O(R)$ is 2-primal.

**Proof.**

Let $P_1 \in \text{MinSpec}(R)$. Since $R$ is 2-primal, $\sigma(P_1) = P_1$, and therefore Theorem (3.4) implies $O(P_1) \in \text{MinSpec}(O(R))$. Similarly for any $P \in \text{MinSpec}(O(R))$ such that $\sigma(P \cap R) = P \cap R$ Theorem (3.4) implies that $P \cap R \in \text{MinSpec}(R)$. Therefore, $O(P(R)) = P(O(R))$, and now the result is obvious by using Theorem (3.3).

**References**


