MINIMIZING TOTAL COMPLETION TIME IN A TWO-MACHINE FLOW-SHOP SCHEDULING PROBLEMS WITH A SINGLE SERVER

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Abstract. We consider the problem of two-machine flow-shop scheduling with a single server and equal processing times, we show that this problem is \(NP\)-hard in the strong sense and present a busy schedule for it with worst-case bound 7/6

Keywords: flow-shop scheduling problem, total completion time worst-case, single server.

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1. Introduction

We consider the two-machine flow-shop scheduling problem, which is described as follows. We are given \(n\) jobs \(J_1, J_2, \cdots, J_n\), and two machines \(M_1\) and \(M_2\). Each job \(J_j\) consists of a chain \((O_{1,j}, O_{2,j}, \cdots, O_{n,j})\) of operations, and \(O_{i,j}\) is to be processed on machine \(M_i\) for \(p_{i,j}\) time units. Each machine can only process one operation at a time, and each job can be processed on at most one machine at a time. No preemption is allowed, i.e., once start, any operation can not be interrupted before it is completed. Immediately before processing an operation \(O_{i,j}\) the corresponding machine, which takes a setup time of \(s_{i,j}\) time units.

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During such a setup the machine is also occupied for \( s_{i,j} \) time united, i.e., No other job can be processed on it. The setup times are assumed to be separable from the processing times, i.e., a setup on a subsequent machine may be performed while the job is still processed on the preceding machine. All setups have to be done by a single server \( M_S \), which can perform at most one setup at a time. The problem we consider is to find a schedule \( S \) which minimizes the total completion times, that is \( \sum_{i=1}^{n} C_j \). Following the three-field notation schedule introduced by Lentra et al [1], we denote this problem as \( F2, S1||\sum_{i=1}^{n} C_j \). If all processing are equal to \( p \), that is \( p_{i,j} = p (i = 1, 2; j = 1, 2, \cdots , n) \), we have the \( F2, S1|p_{i,j} = p|\sum_{i=1}^{n} C_j \) problem.

Complexity results for flow-shop problems obtained by Garey, et al [2], who studied two-machine flow-shop problem with minimizing total completion times, that is \( F2||\sum_{i=1}^{n} C_j \). J.A.Hoogereen, et al [3] studied some special cases for two-machine flow-shop problems with minimizing total completion times, and proved that the problem with equal processing on first machine, that is \( F2, S1|p_{1,j} = p|\sum_{i=1}^{n} C_j \), is \( NP \)-hard in the strong sense, and present an \( O(n \log(n)) \) approximation algorithm for it with worst-case bound \( 4/3 \). Complexity results for flow-shop problems with a single server was obtained by Bruchher, et al [4]. The complexity of parallel dedicated machine with a single server was obtained by Glass, et al [5].

In this paper, we derive some new complexity results for special cases of two-machine problem with a single server. The remainder of the paper is organized as follows. In section 2 we show that flow-shop problem with a single server, equal processing times, and minimizing total completion times is \( NP \)-hard in the strong sense. In section 3 we introduce a improved algorithm, and prove that its worst case is \( 7/6 \), the bound is tight.

2. Complexity of the \( F2,S1|p_{1,j} = p|\sum_{i=1}^{n} C_j \) problem

Let \( C_{i,j} \) denote the completion times of job \( J_j \) on machine \( M_i \). If there are no idle times on machine and machine, we have
\[
C_{1,1} = s_{1,1} + p_{1,1}, C_{2,1} = s_{1,1} + p_{1,1} + s_{2,1} + p_{2,1},
\]
\[
C_{i,j} = C_{1,j-1} + s_{1,j} + p_{1,j},
\]
\[C_{2,j} = \max(C_{2,j-1}, C_{1,j}) + s_{2,j} + p_{2,j}\), for \(j = 2, 3, \cdots, n\)

**Theorem 2.** The problem of \(F2,S1|p_{1,i} = p|\sum_{i=1}^{n} C_j\) is NP-hard in the strong sense.

**Proof.** Our proof is based upon a reduction from the problem Numerical Matching with Target Sums or, in short, Target Sum, which is known to be NP-hard in the strong sense[6].

Target Sum. Given two multisets \(X = x_1, x_2, \cdots, x_n\) and \(Y = y_1, y_2, \cdots, y_n\) of positive integers and an target vector \(z_1, z_2, \cdots, z_n\), where \(\sum_{i=1}^{n} (x_j + y_j) = \sum_{i=1}^{n} z_j\), is there a position of the set \(X \cup Y\) into \(n\) disjoint set \(Z_1, Z_2, \cdots, Z_n\), each containing exactly one element from each of \(X\) and \(Y\), such that the sum of the numbers in \(Z_j\) equal \(z_j\), for \(i = 1, 2, \cdots, n\)?

1. **P-jobs:** \(s_{1,i} = b, p_{1,i} = b, s_{2,i} = b + x_ip_{2,i} = b(i = 1, 2, \cdots, n)\)
2. **Q-jobs:** \(s_{1,i} = 0, p_{1,i} = b, s_{2,i} = b + y_i, p_{2,i} = b(i = 1, 2, \cdots, n)\)
3. **R-jobs:** \(s_{1,i} = 0, p_{1,i} = b, s_{2,i} = b - z_i, p_{2,i} = b(i = 1, 2, \cdots, n)\)
4. **U-jobs:** \(s_{1,i} = 0, p_{1,i} = b, s_{2,i} = 0, p_{2,i} = b(i = 1, 2, \cdots, n)\)
5. **L-jobs:** \(s_{1,i} = 4b, p_{1,i} = b, s_{2,i} = b, p_{2,i} = b(i = 1, 2, \cdots, n)\)

Observe that all processing times are equal to \(y\). To prove the theorem we show that in this constructed if the \(F2,S1|p_{1,i} = p|\sum_{i=1}^{n} C_j\) problem a schedule \(S_0\) satisfying \(\sum_{i=1}^{n} C_j(S_0) \leq y = \sum_{i=1}^{n} x_j + \sum_{i=1}^{n} (x_j + y_j) + (77n^2 - 13n - 4)b/2\) exists if and only if Target Num has a solution.

Suppose that Target Num has a solution. The desired schedule \(S_0\) exists and can be described as follows. No machine has intermediate idle time. Machine \(M_i\) process the jobs in order of the sequence \(\sigma\), i.e., in the sequence

\[
\sigma = (\sigma_{P_{1,1}}, \sigma_{Q_{1,1}}, \sigma_{R_{1,1}}, \sigma_{U_{1,1}}, \sigma_{V_{1,1}}, \sigma_{L_{1,1}}, \cdots, \sigma_{P_{1,n}}, \sigma_{Q_{1,n}}, \sigma_{R_{1,n}}, \sigma_{U_{1,n}}, \sigma_{V_{1,n}}, \sigma_{L_{1,n}})
\]

While machine \(M_2\) process the jobs in the sequence

\[
\tau = (\tau_{P_{2,1}}, \tau_{Q_{2,1}}, \tau_{R_{2,1}}, \tau_{U_{2,1}}, \tau_{V_{2,1}}, \tau_{L_{2,1}}, \cdots, \tau_{P_{2,n}}, \tau_{Q_{2,n}}, \tau_{R_{2,n}}, \tau_{U_{2,n}}, \tau_{V_{2,n}}, \tau_{L_{2,n}})
\]

as indicated in Figure 1.

![Fig.1 Gantt chart for the \(F2, S1|p_{1,i} = p|\sum_{i=1}^{n} C_j\) problem](image)
Then we define the sequence and shown in Figure 1. Obviously, these sequence $\sigma$ and $\tau$ fulfills $C(S) = C(\sigma, \tau) \leq y$. Conversely, assume that the flow-shop scheduling problem has a solution $\sigma$ and $\tau$ with $C(S) \leq y$.

Considering the path composed of machine $M_1$ operations of jobs $(P_{1,1}, Q_{1,1}, R_{1,1}, U_{1,1}, V_{1,1}, W_{1,1})$. Machine $M_2$ operations of jobs $(R_{1,1}, U_{2,1}, V_{2,1}, W_{2,1}, L_{2,1}, \cdots, R_{2,n}, U_{2,n}, V_{2,n}, W_{2,1}, L_{2,n})$ ; we obtain that

$$C(S) \geq 3b + x_1 + 5b + x_1 + y_1 + 7b + x_1 + y_1 - z_1 + 8b + 9b + 10b + \cdots (3 + (n - 1)11)b + x_n + (5 + (n - 1)11)b + x_n = y_n + (7 + (n - 1)11)b + \cdots + (11n + 1)b = \sum_{i=1}^{n} x_j + \sum_{i=1}^{n} (x_j + y_j) + \frac{77n^2 - 13n - 4}{2}b/2 = y.$$ So we have $C(S) = y$.

(a) If $S$ has a partition $\mu$, then there is a schedule with finish times $y$. One such schedule is shown in Figure 1.

(b) If $S$ has no partition, then all schedule must have a finish times $> y$. Since $S$ has no partition, then $x_i + y_i \neq z_i (i = 1, 2, \cdots, n)$. Let $\xi_i = x_i + y_i - z_i$, we have

$$\sum_{i=1}^{n} C_j(S) = \sum_{i=1}^{n} x_j + \sum_{i=1}^{n} (x_j + y_j) + \frac{77n^2 - 13n - 4}{2}b/2 + 5 \sum_{i=1}^{n} \xi_i + 10 \sum_{i=1}^{n-1} \xi_i + \cdots + 5n \xi_1 > y.$$ 

### 3. Worst-case for the $F2,S1|p_{1,j} = p| \sum_{i=1}^{n} C_j$ problem

In examining "worst" schedule, we restrict ourselves to busy schedule. A busy schedule is a schedule in which at all times from start to finish at least one server is processing a task.

**Theorem 3.** The problem of $F2,S1|p_{1,j} = p| \sum_{i=1}^{n} C_j$ problem, let $S_0$ be a busy schedule for this problem, $S^*$ be the optimal solution for the $F2,S1|p_{1,j} = 1| \sum_{i=1}^{n} C_j$ problem, then $\sum_{i=1}^{n} C_j(S_0)/\sum_{i=1}^{n} C_j(S^*) \leq 7/6$. The bound is tight.

**Proof.** For a schedule $S$, let $I_{i,j}(S)(i = 1, 2, j = 1, 2, \cdots, n)$ denote the total idle times of job $J_j$ on machine $M_i$.

Considering the path composed of machine $M_1$ operations of jobs $1, 2, \cdots, l$ , machine $M_2$ operation of job $j$, we obtain that $C_j = \sum_{i=1}^{j} (s_{1,i} + p_{1,i}) + I_{1,j} + s_{2,j} + p_{2,j}$ (1)

Considering the path composed of machine $M_1$ operations of job $1$ , machine $M_2$ operation of jobs $1, 2, \cdots, j$ , we obtain that $C_j = s_{1,1} + p_{1,1} + \sum_{i=1}^{j} (s_{2,i} + p_{2,i}) + I_{2,j}$ (2)
Considering the path composed of machine $M_1$ operations of jobs $1, 2, \cdots, l$, machine $M_2$ operation of jobs $l, l+1, \cdots, j$, we obtain that $C_j = \sum_{i=1}^{l}(s_{1,i} + p_{1,i}) + I_{1,j} + \sum_{i=1}^{j}(s_{2,i} + p_{2,i}) + I_{2,j}$ (3)

So we have

$$6 \sum_{j=1}^{n} C_j(S_0) = 2(\sum_{i=1}^{l}(s_{1,i} + p_{1,i}) + I_{1,j} + s_{2,j} + p_{2,j}) + 2(s_{1,1} + p_{1,1} + \sum_{i=1}^{l}(s_{2,i} + p_{2,i}) + I_{2,j})$$

$$+ 2(\sum_{i=1}^{l}(s_{1,i} + p_{1,i}) + I_{1,j} + \sum_{i=1}^{j}(s_{2,i} + p_{2,j}) + I_{2,j}) \leq 7 \sum_{j=1}^{n} C_j(S^*)$$

$$\sum_{i=1}^{n} C_j(S_0) / \sum_{i=1}^{n} C_j(S^*) \leq 7/6.$$ 

To prove the bound is tight, introduce the following example as shown in Fig. 2 and Fig. 3.

(1) $s_{1,i} = 2b, p_{1,i} = b, s_{2,i} = 2b, p_{2,i} = b (i = 1, 2)$

(2) $s_{1,i} = 0, p_{1,i} = b, s_{2,i} = 0, p_{2,i} = b (i = 3, 4)$

So we have $\sum_{i=1}^{n} C_j(S_0) / \sum_{i=1}^{n} C_j(S^*) = 35b/30b = 7/6$, the bound is tight.

References


