ON SEMISIMPLE AND LEFT QUASI-REGULAR ELEMENTS OF ORDERED SEMIGROUPS

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Abstract. We prove, among others, that an ordered semigroup contains a left (resp. right) quasi-regular element if and only it contains a left (resp. right) regular element. It has a semisimple element if and only if it has an intra-regular element. An element \( a \) of an ordered semigroup \( S \) is a semisimple element of \( S \) if and only if there exists an intra-regular element \( b \) of \( S \) such that \( I(a) = I(b) \). The element \( a \) is a left (resp. right) quasi-regular element of \( S \) if and only if there exists a left (resp. right) regular element \( b \) of \( S \) such that \( L(a) = L(b) \) (resp. \( R(a) = R(b) \)). As a consequence, if the ideal \( I(a) \) generated by an element \( a \) of \( S \) has an intra-regular generator, then \( a \) is semisimple. If the principal left (resp. right) ideal \( L(a) \) (resp. \( R(a) \)) of an element \( a \) of \( S \) has a left (resp. right) regular generator, then \( a \) is a left (resp. right) quasi-regular element of \( S \).

Keywords: ordered semigroup; semisimple element; left (right) quasi-regular element; intra-regular, left (right) regular element; \( \pi \)-semisimple, left (right) quasi \( \pi \)-regular element.

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1. Introduction and prerequisites

Let \( S \) be an ordered semigroup. A nonempty subset \( T \) of \( S \) is called a subsemigroup of \( S \) if \( T^2 \subseteq T \). For a subset \( H \) of \( S \), we denote by \( (H) \) the subset of \( S \) defined by

\[ \{ t \in S \mid t \leq h \text{ for some } h \in H \}. \]

A nonempty subset \( A \) of \( S \) is called a left (resp.
right) ideal of \( S \) if (1) \( SA \subseteq A \) (resp. \( AS \subseteq A \)) and (2) if \( a \in A \) and \( b \in S \) such that \( b \leq a \), then \( b \in A \). A nonempty subset \( A \) of \( S \) is called an ideal of \( S \) if it is both a left and a right ideal of \( S \). The left (resp. right) ideals, and so the ideals of \( S \) are clearly subsemigroups of \( S \). For an element \( a \) of \( S \), \( I(a) \), \( L(a) \), \( R(a) \), denotes the ideal, left ideal, and the right ideal of \( S \), respectively, generated by \( a \) \((a \in S)\), and we have \( I(a) = (a \cup Sa \cup aS \cup SaS) \), \( L(a) = (a \cup Sa) \), \( R(a) = (a \cup aS) \) [2]. As usual, for an element \( b \) of \( S \), denote by \(< b >\) the subsemigroup of \( S \) generated by \( b \), that is, the smallest (with respect to the inclusion relation) subsemigroup of \( S \) containing \( b \). We have \(< b > = \{ b, b^2, b^3, ...., b^n \mid n \in N \}\) \((N = \{1,2,......\} \) is the set of natural numbers). An element \( a \) of \( S \) is called intra-regular if \( a \in (Sa^2S) \), equivalently, if \( a \leq xa^2y \) for some \( x,y \in S \) (cf., for example, [4]). It is called left (resp. right) regular if \( a \in (Sa^2) \) (resp. \( a \in (a^2S) \)), that is, if \( a \leq xa^2 \) (resp. \( a \leq a^2x \)) for some \( x \in S \) [3]. An element \( a \) of an ordered semigroup \( S \) is called semisimple if \( a \in (SaSaS) \) [5]; it is called left (resp. right) quasi-regular if \( a \in (SaSa) \) (resp. \( a \in (aSaS) \)) [5,6]. So the element \( a \) is semisimple, left quasi-regular or right quasi-regular if \( a \leq xayaz, a \leq xaya, a \leq axay \) for some \( x,y,z \in S \), respectively. An ordered semigroup \( S \) is called intra-regular, left (right) regular, semisimple or left (right) quasi-regular if every element of \( S \) is so.

Semisimple and left (right) quasi-regular elements of semigroups (without order) as well as \( \pi \)-semisimple and left quasi \( \pi \)-regular semigroups have been studied in [1]. An ordered semigroup \( S \) is a semilattice of left strongly simple semigroups if and only if every left ideal of \( S \) is a semisimple subsemigroup of \( S \) (cf. [7; Theorem 9]). It is a semilattice of simple and regular semigroups if and only if every bi-ideal of \( S \) is a semisimple subsemigroup of \( S \), equivalently, if every left ideal of \( S \) is a right quasi-regular subsemigroup of \( S \) [8]. Characterization of left (right) quasi-regular and semisimple ordered semigroups in terms of fuzzy sets has been given in [5]. In the present paper we prove, among others, that an ordered semigroup contains a left (resp. right) quasi-regular element if and only if it contains a left (resp. right) regular element. It contains a semisimple element if and only if it contains an intra-regular element. Moreover we prove that an element \( a \) of an
ordered semigroup $S$ is a semisimple element of $S$ if and only if there exists an intra-regular element $b$ of $S$ such that $I(a) = I(b)$. An element $a$ of an ordered semigroup $S$ is a left (resp. right) quasi-regular element of $S$ if and only if there exists a left (resp. right) regular element $b$ of $S$ such that $L(a) = L(b)$ (resp. $R(a) = R(b)$). As a consequence, if $S$ is an ordered semigroup, $a \in S$ and $b$ an intra-regular element of $S$ such that $I(a) = \langle b \rangle$, then $a$ is semisimple. If $b$ is a left (resp. right) regular element of $S$ and $L(a) = \langle b \rangle$ (resp. $R(a) = \langle b \rangle$), then $a$ is left (resp. right) quasi-regular. We use the terminology semisimple, $\pi$-semisimple instead of the terminology intra quasi-regular, intra quasi $\pi$-regular given for semigroups (without order) in [1]. For an ordered semigroup $S$, we denote by $IQReg(S)$ (or $Sems(S)$), $LQReg(S)$, $RQReg(S)$ the set of semisimple (intra quasi-regular according to [1]), left quasi-regular and right quasi-regular elements of $S$, respectively. It has been proved in [1] that a semigroup $S$ is left quasi $\pi$-regular if and only if it is $\pi$-semisimple and $IQReg(S) = LQReg(S)$. If an ordered semigroup $S$ is $\pi$-semisimple and $IQReg(S) = LQReg(S)$, then it is left quasi $\pi$-regular, but the converse statement does not seem to be true, in general. However, for an ordered semigroup $S$ we have $LQReg(S) \subseteq IQReg(S)$, and if $S$ is left quasi $\pi$-regular, then it is $\pi$-semisimple.

2. Main results

**Proposition 1.** Let $S$ be an ordered semigroup. If $a$ is an intra-regular element of $S$, then $a$ is a semisimple element of $S$. If $a$ is a left (resp. right) regular element of $S$, then $a$ is a left (resp. right) quasi-regular element of $S$.

**Proof.** Let $a$ be an intra-regular element of $S$ and let $x, y \in S$ such that $a \leq xa^2y$. Then we have $a \leq x(xa^2y)ay = (x^2a)(ay)(ay) \in SaSaS$, so $a \in (SaSaS]$, and $a$ is semisimple. Let $a$ be a left regular element of $S$ and $x \in S$ such that $a \leq xa^2$. Then we have $a \leq x(xa^2)a = x^2aaa \in SaSa$, so $a \in (SaSa]$, and $a$ is a left quasi-regular element of $S$. Finally, if $a$ is a right regular element of $S$ and $y \in S$ such that $a \leq a^2y$, then $a \leq a(a^2y)y = aaay^2 \in aSaS$, so $a \in (aSaS]$, and $a$ is a right quasi-regular element of $S$. □
Proposition 2. Let $S$ be an ordered semigroup and $a \in S$. If $a \leq xayaz$ for some $x,y,z \in S$, then the element $yaz$ is an intra-regular element of $S$. If $a \leq xaya$ for some $x,y \in S$, then the element $ya$ is a left regular element of $S$. If $a \leq axay$ for some $x,y \in S$, then the element $ax$ is a right regular element of $S$.

Proof. Let $a \leq xayaz$ for some $x,y,z \in S$. Then we have
\[
yaz \leq y(xayaz)z = (yx)a(yaz^2) \leq (yx)(xayaz)(yaz^2)
= (yx^2a)(yaz)^2, \quad \text{where } yx^2a, z \in S,
\]
so the element $yaz$ is an intra-regular element of $S$.

Let $a \leq xaya$ for some $x,y \in S$. Then we have
\[
ya \leq y(xaya) = (yx)a(ya) \leq (yx)(xaya)(ya)
= (yx^2a)(ya)^2, \quad \text{where } yx^2a \in S,
\]
so the element $ya$ is a left regular element of $S$.

Similarly, if $a \leq axay$ for some $x,y \in S$, then
\[
ax \leq (axay)x = (ax)a(yx) \leq (ax)(axay)(yx)
= (ax)^2(ay^2x), \quad \text{where } ay^2x \in S,
\]
so the element $ax$ is a right regular element of $S$. \hfill \square

By Propositions 1 and 2 we have the following

Theorem 3. (cf. also [5]) An ordered semigroup has a semisimple element if and only if it has an intra-regular element. An ordered semigroup contains a left (resp. right) quasi-regular element if and only if it contains a left (resp. right) regular element.

Remark 4. If $a$ is an intra-regular element of an ordered semigroup $S$, then $I(a) = (SaS]$. In fact: First of all, $(SaS] \subseteq (a \cup Sa \cup aS \cup SaS] = I(a)$. Let now $t \in I(a)$. Then $t \leq a$ or $t \leq pa$ or $t \leq aq$ or $t \leq paq$ for some $p,q \in S$. On the other hand, since $a$ is intra-regular, there exist $x,y \in S$ such that $a \leq xa^2y$. If $t \leq a$, then $t \leq xa^2y \in SaS$, and $t \in (SaS]$. If $t \leq pa$ for some $p \in S$, then $t \leq pxa^2y \in SaS$, so $t \in (SaS]$. If $t \leq aq$ for some
q \in S$, then $t \leq xa^2yq \in SaS$, so $t \in (SaS]$. Finally, if $t \leq paq$ for some $p,q \in S$, then $t \leq pxa^2yq \in SaS$, and $t \in (SaS]$. 

**Theorem 5.** Let $S$ be an ordered semigroup. An element $a$ of $S$ is a semisimple element of $S$ if and only if there exists an intra-regular element $b$ of $S$ such that $I(a) = I(b)$.

**Proof.** $\Longrightarrow$. Let $a$ be a semisimple element of $S$. Then $a \leq xayaz$ for some $x, y, z \in S$. By Proposition 2, the element $yaz$ is an intra-regular element of $S$. In addition, we have $I(a) = I(yaz)$. In fact: First of all,

$$I(yaz) = (yaz \cup Syaz \cup yazS \cup SyazS) \subseteq (SaS) \subseteq I(a).$$

Let now $t \in I(a)$. Then $t \leq a$ or $t \leq pa$ or $t \leq aq$ or $t \leq paq$ for some $p,q \in S$. If $t \leq a$, then $t \leq xayaz \in Syaz$. Then $t \in (Syaz) \subseteq I(yaz)$, so $t \in I(yaz)$. If $t \leq pa$ for some $p \in S$, then $t \leq p(xayaz) \in Syaz$, and $t \in (Syaz) \subseteq I(yaz)$, so $t \in I(yaz)$. If $t \leq aq$ for some $q \in S$, then $t \leq (xayaz)q \in S(yaz)S$, and $t \in (S(yaz)S) \subseteq I(yaz)$, so $t \in I(yaz)$. If $t \leq paq$ for some $p,q \in S$, then $t \leq p(xayaz)q \in S(yaz)S$, so $t \in (S(yaz)S) \subseteq I(yaz)$, and $t \in I(yaz)$.

$\Longleftarrow$. Let $a \in S$ and $b$ an intra-regular element of $S$ such that $I(a) = I(b)$. Then $I(a) = I(b)$ and $b \leq xb^2y$ for some $x, y \in S$. Then we have

$$a \in I(b) \subseteq I(xb^2y) = (xb^2y \cup Sxb^2y \cup xb^2yS \cup Sxb^2yS) \subseteq (Sb^2S).$$

Since $b \in I(a)$, we have $b \leq a$ or $b \leq pa$ or $b \leq qa$ or $b \leq paq$ for some $p,q \in S$. If $b \leq a$, then $b^2 \leq a^2$, $a \in (Sb^2S) \subseteq (Sa^2S)$, so $a \in (Sa^2S)$, and $a$ is an intra-regular element of $S$. Then, by Proposition 1, $a$ is a semisimple element of $S$. If $b \leq pa$ for some $p \in S$, then $b^2 \leq papa$, then $a \in (Sb^2S) \subseteq (SpapaS) \subseteq (SaSaS)$, so $a \in (SaSaS)$, and $a$ is semisimple. If $b \leq aq$ for some $q \in S$, then $b^2 \leq aqaq$, $a \in (Sb^2S) \subseteq (SaqaqS) \subseteq (SaSaS)$, $a \in (SaSaS)$, and $a$ is semisimple. If $b \leq paq$ for some $p,q \in S$, then $a \in (Sb^2S) \subseteq (SpaqpaqS) \subseteq (SaSaS)$, and $a$ is semisimple. $\square$

**Theorem 6.** Let $S$ be an ordered semigroup. An element $a$ of $S$ is a left quasi-regular element of $S$ if and only if there exists a left regular element $b$ of $S$ such that $L(a) = L(b)$. 


**Proof.** $\implies$. Let $a$ be a left quasi-regular element of $S$. Then $a \leq xaya$ for some $x, y \in S$. By Proposition 2, the element $ya$ is a left regular element of $S$. Moreover we have $L(a) = L(ya)$. In fact: First of all,

$$L(ya) = (ya \cup Sya] \subseteq (Sa] \subseteq (a \cup Sa] = L(a).$$

Let now $t \in L(a)$. Then $t \leq a$ or $t \leq pa$ for some $p \in S$. If $t \leq a$, then $t \leq xaya \in Sya$. Then $t \in (Sy[a] \subseteq (ya \cup Sya] = L(ya)$, and $t \in L(ya)$. If $t \leq pa$ for some $p \in S$, then $t \leq p(xaya) \in Sya$, and $t \in L(ya)$.

$\impliedby$. Let $a \in S$, $b$ a left regular element of $S$ such that $L(a) = L(b)$, and $x \in S$ such that $b \leq xb^2$. Then we have $a \in L(b) \subseteq L(xb^2) = (xb^2 \cup Sxb^2] \subseteq (Sb^2]$. Since $b \in L(a)$, we have $b \leq a$ or $b \leq pa$ for some $p \in S$. If $b \leq a$, then $a \in (Sb^2] \subseteq (Sa^2]$, $a \in (Sa^2]$, and $a$ is a left regular element of $S$. By Proposition 1, $a$ is a left quasi-regular element of $S$. If $b \leq pa$ for some $p \in S$, then $a \in (Sb^2] \subseteq (Spapa] \subseteq (SaSa]$, and $a$ is again a left quasi-regular element of $S$.

The right analogue of Theorem 6 also holds, and we have

**Theorem 7.** Let $S$ be an ordered semigroup. An element $a$ of $S$ is a right quasi-regular element of $S$ if and only if there exists a right regular element $b$ of $S$ such that $R(a) = R(b)$.

**Theorem 8.** Let $S$ be an ordered semigroup and $a, b \in S$. If $I(a) =\lhd b >$, then $I(a) = I(b)$. If $L(a) =\lhd b >$ (resp. $R(a) =\lhd b >$), then $L(a) = L(b)$ (resp. $R(a) = R(b)$).

**Proof.** Let $I(a) =\lhd b >$. As $I(b)$ is an ideal of $S$ containing $b$, it is a subsemigroup of $S$ containing $b$. On the other hand, $\lhd b >$ is the smallest subsemigroup of $S$ containing $b$, so $\lhd b > \subseteq I(b)$, and $I(a) \subseteq I(b)$. Let now $t \in I(b)$. Then $t \leq b$ or $t \leq xb$ or $t \leq by$ or $t \leq xby$ for some $x, y \in S$. If $t \leq b$, then $t \leq b \in \lhd b > = I(a)$, and $t \in I(a)$. If $t \leq xb$ for some $x \in S$, then $t \leq xb \in S < b >= SI(a) \subseteq I(a)$, so $t \in I(a)$. If $t \leq by$ for some $y \in S$, then $t \leq by \in \lhd b > = I(a)S \subseteq I(a)$, and $t \in I(a)$. If $t \leq xby$ for some $x, y \in S$, then $t \leq xby \in S < b > = SI(a)S \subseteq I(a)$, and $t \in I(a)$.

Let now $L(a) =\lhd b >$. $L(b)$ as a left ideal, it is a subsemigroup of $S$, thus

$$L(a) =\lhd b > \subseteq L(b).$$
Let now \( t \in L(b) \). Then \( t \leq b \) or \( t \leq xb \) for some \( x \in S \). If \( t \leq b \), then \( t \leq b \in L(a) \), and \( t \in L(a) \). If \( t \leq xb \) for some \( x \in S \), then \( t \leq xb \in S < b > = SL(a) \subseteq L(a) \), and \( t \in L(a) \). In a similar way we prove that \( R(a) =< b > \) implies \( R(a) = R(b) \). \( \square \)

By Theorems 5, 6, 7 and 8 we have the following

**Theorem 9.** Let \( S \) be an ordered semigroup and \( a \in S \). If \( b \) is an intra-regular element of \( S \) such that \( I(a) =< b > \), then \( a \) is a semisimple. If \( b \) is a left (resp. right) regular element of \( S \) and \( L(a) =< b > \) (resp. \( R(a) =< b > \)), then \( a \) is left (resp. right) quasi-regular.

For the sake of completeness, we keep the notation given for semigroups (without order) in [1] and, for an ordered semigroup \( S \), we denote by \( IQReg(S) \), \( LQReg(S) \), \( RQReg(S) \) the set of semisimple, left quasi-regular and right quasi-regular elements of \( S \), respectively.

**Remark 10.** We have \( LQReg(S) \subseteq IQReg(S) \) and \( RQReg(S) \subseteq IQReg(S) \). In fact, if \( a \in LQReg(S) \), then \( a \in (SaSa) \) i.e. \( a \leq xaya \) for some \( x, y \in S \), then \( a \leq xay(xaya) \in SaSaS \), so \( a \in (SaSaS) \), and \( a \in IQReg(S) \). If \( a \in RQReg(S) \), then \( a \leq axay \leq (axay)xay \in SaSaS \), and \( a \in IQReg(S) \).

**Proposition 11.** Let \( S \) be an ordered semigroup. If \( a \in IQReg(S) \), then there exist \( x, y, z \in S \) such that \( a \leq (xay)^naz^n \) for every \( n \in N \).

**Proof.** Let \( a \in IQReg(S) \). Then there exist \( x, y, z \in S \) such that \( a \leq xayaz \). Then we have

\[
a \leq (xay)az = (xay)(xayaz)z = (xay)^2az^2 \\
\leq (xay)^2(xayaz)z^2 = (xay)^3az^3 \\
\leq \ldots \leq (xay)^naz^n
\]

for every \( n \in N \). \( \square \)

**Definition 12.** An element \( a \) of an ordered semigroup \( S \) is called \( \pi \)-semisimple if there exists \( n \in N \) such that the element \( a^n \) is a semisimple element of \( S \), that is, if a power of \( a \) is semisimple. An element \( a \) of an ordered semigroup \( S \) is called left (resp. right) quasi-\( \pi \)-regular if there exists \( n \in N \) such that the element \( a^n \) is a left (resp. right) quasi-regular element of \( S \) i.e. if a power of \( a \) is left (resp. right) quasi-regular. An ordered semigroup
$S$ is called $\pi$-semisimple, left quasi $\pi$-regular or right quasi $\pi$-regular, respectively, if every element of $S$ is so.

**Theorem 13.** Let $S$ be an ordered semigroup and $a \in S$. If $a$ is left quasi $\pi$-regular (or right quasi $\pi$-regular), then it is $\pi$-semisimple as well. "Conversely", if $a$ is $\pi$-semisimple and $IQ\text{Reg}(S) = LQ\text{Reg}(S)$, then $a$ is a left quasi $\pi$-regular element of $S$. If $a$ is $\pi$-semisimple and $IQ\text{Reg}(S) = RQ\text{Reg}(S)$, then $a$ is a right quasi $\pi$-regular element of $S$.

**Proof.** Let $a$ be left quasi $\pi$-regular element of $S$. Then there exists $n \in \mathbb{N}$ such that $a^n \in (Sa^nSa^n]$. Then we have

$$a^n \in (Sa^nS(Sa^nSa^n]) = (Sa^nS(Sa^nSa^n)] \subseteq (Sa^nSa^nS],$$

so $a$ is a $\pi$-semisimple element of $S$. Let now $a$ be a $\pi$-semisimple element of $S$ and $IQ\text{Reg}(S) = LQ\text{Reg}(S)$. Since $a$ is $\pi$-semisimple, there exists $n \in \mathbb{N}$ such that $a^n \in (Sa^nSa^nS]$. That is $a^n \in IQ\text{Reg}(S) (= LQ\text{Reg}(S))$. Then $a^n \in LQ\text{Reg}(S)$ which means that the element $a$ is a left quasi $\pi$-regular element of $S$. Finally, let $a$ be $\pi$-semisimple element of $S$, $IQ\text{Reg}(S) = RQ\text{Reg}(S)$, and $a^n \in (Sa^nSa^nS]$ for some $n \in \mathbb{N}$. Then $a^n \in IQ\text{Reg}(S) (= RQ\text{Reg}(S))$, and $a$ is a right quasi $\pi$-regular element of $S$. \qed

**Theorem 14.** If an ordered semigroup $S$ is left quasi $\pi$-regular (or right quasi $\pi$-regular), then it is $\pi$-semisimple. "Conversely", if $S$ is $\pi$-semisimple and $IQ\text{Reg}(S) = LQ\text{Reg}(S)$ (resp. $IQ\text{Reg}(S) = RQ\text{Reg}(S)$), then it is left (resp. right) quasi $\pi$-regular.

**Problem.** Find an example of a left quasi $\pi$-regular ordered semigroup having a semisimple element which is not left quasi-regular.

**References**


