Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 4, 1144-1160
ISSN: 1927-5307

A SURVEY ON ECCENTRIC DIGRAPH OF A (DI)GRAPH

MEDHA ITAGI HUILGOL*

Department of Mathematics, Bangalore University, Central College Campus, Bangalore-560001

Abstract. The eccentricity e(u) of a vertex u is the maximum distance of u to any other vertex of G. A vertex v is an eccentric vertex of vertex u if the distance from u to v is equal to e(u). The eccentric digraph ED(G) of a graph(digraph) G is the digraph that has the same vertex as G and an arc from u to v exists in ED(G) if and only if v is an eccentric vertex of u in G. In this survey we take a look on the progress made till date in the theory of Eccentric digraphs of graphs and digraphs, in general. And list the open problems in the area.

Keywords: Eccentricity, Radius, Diameter, Antipodal Graphs, Eccentric digraphs.

2000 AMS Subject Classification: 05C12; 05C20

1. Introduction and Definitions

Its over a decade since Buckley [9] defined eccentric digraph $ED(G)$ of a graph G. Later Boland and Miller [6] introduced the concept of the *eccentric digraph* of a digraph. The eccentric digraph of a graph or digraph, is a distance based mapping, which assigns a binary relation induced by distances in a graph that can also be represented by a graph. Many distance based relations can be found in literature in the study of antipodal graphs [3], antipodal digraphs [17], eccentric graphs [1, 12] , etc.

*Corresponding author

E-mail address: medha@bub.ernet.in (M. I. Huilgol)

Research work of the author is supported by Bangalore University, Bangalore under the project U. O. No.: DEV: D2: YRB-BUIRF:2010-11.

Received March 9, 2012

All of these serve as the second level of abstraction, for all properties of graphs such as distance, diameter, radius, etc.

In the following we shall consider both directed graphs and symmetric digraphs (or, the undirected graphs). A *directed graph or digraph* G consists of a finite nonempty set $V(G)$ called *vertex set* with vertices and *edge set* $E(G)$ of ordered pairs of vertices called *arcs*; that is $E(G)$ represents a binary relation on $V(G)$. A graph G is a *symmetric digraph* if in G for any edge $(u, v) \in E(G)$ implies$(v, u) \in E(G)$. If (u, v) is an arc, it is said that u is adjacent to v and also that v is adjacent from u. The set of vertices which are adjacent from (to) a given vertex v is denoted by $N^+(u)[N^-(u)]$and its cardinality is the *out-degree* of v [*in-degree* of v]. A *walk of length* k from a vertex u to a vertex v in G is a sequence of vertices $u = u_0, u_1, u_2, ..., u_{k-1}, u_k = v$ such that each pair (u_{i-1}, u_i) is an arc of G. A digraph G is *strongly connected* if there is a u to v walk for any pair of vertices u and v of G. A digraph is *connected* if its underlying graph is connected. A digraph is *unilateral* if for every two distinct vertices u and v, there is either a $u - v$ path or a $v - u$ path. The *distance* $d(u, v)$ from u to v is the length of a shortest u to v walk. The *eccentricity,* $e(v)$ of v is the distance to a farthest vertex from v. If $dist(u, v) = e(u)(v \neq u)$ we say that v is an *eccentric vertex* of u. We define $dist(u, v) = \infty$ whenever there is no path joining the vertices u and v. The *radius* $rad(G)$ and *diameter* $diam(G)$ are minimum and maximum eccentricities, respectively. A graph is *self-centered* if $rad(G) = diam(G)$.An *eccentric path* of vertex v is a geodesic from v to an eccentric vertex of v. A vertex v is said to be an *antipodal vertex* of u if $d(u, v) = diam(G)$. The reader is referred to Buckley and Harary [10] and Chartrand and Lesniak [11] for additional, undefined terms.

Buckley [9] defines the *eccentric digraph* $ED(G)$ of a graph G as having the same vertex set as G and there is an arc from u to v if v is an eccentric vertex of u. It is worth noting that in the same paper Buckley has acknowledged Prof. Alan Goldman of John Hopkins University for indicating to consider a digraph where arcs are determined based on eccentric vertices in a graph.

The *antipodal digraph* of a digraph G denoted by $A(G)$, has the vertex set as G with an arc from vertex v in $A(G)$ if and only if v is an antipodal vertex of u in G; that is

$dist(u, v) = diam(G)$. This notion of antipodal digraph of a digraph was introduced by Johns and Sleno [17] as an extension of the definition of the *antipodal graph* of a graph given by Aravamudhan and Rajendran [3]. It is clear that $A(G)$ is a subgraph of $ED(G)$, and $A(G) = ED(G)$ if and only if G is self centered. A graph G is said to be a *self-antipodal* graph if $A(G) = G$.And the self-antipodal graphs were studied by Acharya and Acharya [2].

In [1] Akiyama et.al have defined *eccentric graph* of a graph G, denoted by G_e, having the same set of vertices as G with two vertices u and v being adjacent in G_e if and only if either v is an eccentric vertex of u in G or u is an eccentric vertex of v in G, that is $dist_G(u, v) = min\{e_G(u), e_G(v)\}$. Note that G_e is the underlying graph of $ED(G)$.

In [6] Boland and Miller introduced the concept of the *eccentric digraph of a digraph*. In [15] Gimbert et.al. have proved that $G_e = ED(G)$if and only if G is self-centered. In the same paper, the authors have characterized eccentric digraphs in terms of complement of the reduction of G, denoted by $\overline{G^-}$. Given a digraph G of order p a reduction of G, is derived from G by removing all its arcs incident from vertices with *out-degree* $p-1$. Note that $ED(G)$ is a subdigraph of $\overline{G^-}$.

In [16],Gimbert and Lopez have studied the behavior of sequences of iterated eccentric digraphs. Given a positive integer $k \geq 2$, the k^{th} iterated eccentric digraph of G is written as $ED^k(G) = ED(ED^{k-1}(G))$, where $ED^0(G) = G$ and $ED^1(G) = ED(G)$. One of these open problems was discussed in Medha Itagi Huilgol et.al [19]. We have characterized graphs with specified maximum degree such that $ED(G) = G$ and in [21] the higher powers of eccentric digraphs such that $ED^n(G) = G$ are considered.

In [20] we have considered the relations between the highly symmetric graphs viz. *distance degree regular (DDR) graphs* and eccentric digraphs. The *DDR* graphs being self centered, yield symmetric eccentric digraphs. We showed that the *unique eccentric node DDR graphs* have their all iterated eccentric digraphs as *DDR* graphs.

2. Fundamental results

In this section we start with the basic structural results while the definition was introduced by Buckley [9]. Following Chartrand and Ollermann [13], we use G^* to denote the symmetric digraph whose underlying graph is G. Thus G^* is the digraph that is obtained from G by replacing each edge of G by a symmetric pair of arcs. If D_1 and D_2 are digraphs, then Buckley [9] defines $D_1 \to D_2$ to be the digraph $D_1 U D_2$ with additional arcs from each vertex of D_1 to each vertex of D_2. Several results which give properties of eccentric digraphs of graphs are given below.

Remark 1. [9]*: If G is disconnected with components of order $n_1, n_2, \ldots\ldots., n_k$, then $ED(G) = (K_{n_1, n_2}, \ldots\ldots.,_{n_k})^*$.*

The eccentric digraphs of many familiar graphs can easily be determined.

Theorem 1. [9]*: For the complete graphs K_n, complete bipartite graphs $K_{m,n}$ and cycles C_n ,we have the following:*

$$ED(K_n) = (K_n)^*$$
$$ED(K_{1,n}) = K_1 \to (K_n)^* for\ n > 1$$
$$ED(K_{m,n}) = (K_m)^* U (K_n)^* for\ m, n \geq 2$$
$$ED(C_{2t}) = (tK_2)^*$$
$$ED(C_{2t+1}) = (C_{2t+1})^*$$

Remark 2. [6]*: If G is disconnected with k strongly connected components of order $n_1, n_2, \ldots\ldots., n_k$ then $ED(G) = (K_{n_1, n_2}, \ldots\ldots.,_{n_k})$, a (directed) complete multipartite graph.*

Remark 3. [19]*: $ED(G) = (K_{n_1, n_2}, \ldots\ldots.,_{n_k}) = K_{n_1}$.*

Remark 4. [6]*:An eccentric digraph has no vertex of* out-degree 0*.*

Remark 5. [6]*: If G is a complete digraph then $ED(G) = G$.*

Remark 6. [6]*: If G is a multipartite digraph then $ED^2(G) = G$.*

Remark 7. [6]*: The eccentric digraph of a directed cycle is a directed cycle, with directions reversed.*

Remark 8. [19]*: There exists no directed cycle in an eccentric digraph.*

The above two conditions are not sufficient for a graph to be an eccentric digraph.

Example 1. *Consider a symmetric cycle having a pendant vertex adjacent to one of the vertices on the cycle.*

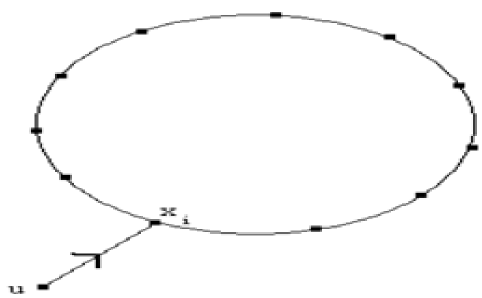

Figure 1.

The above graph cannot be an eccentric digraph.

Next we will discuss some results on trees.

Lemma 1. [19]*: For every vertex v in a tree T, each eccentric path of v contains a central vertex of T.*

The well-known result of Jordan [10] states that the center of a tree consists either a single vertex or two adjacent vertices. If $C(T) = \{u, v\}$, then let T_u denote the set of all vertices closer to u than to v, and let T_v be the set of all vertices closer to v than to u.

Lemma 2. [9]*: If $C(T) = \{u, v\}$, then the eccentric vertices for each vertex in T_u are within T_v , and the eccentric vertices for each vertex in T_v are within T_u.*

Lemma 3. [9]*: For each vertex x in a tree T, the set of eccentric vertices for x consists of all the eccentric vertices of the nearest central vertex c to x in T that are at least as far from x as from c.*

Remark 9. *: A central vertex c is nearest to itself.*

Theorem 2. [9]*: For any tree T, the eccentric digraph $ED(T)$ is connected. $ED(T)$ is unilateral if and only if $d(T) \leq 2$,and $ED(T)$ is strong if and only if $T = K_1 or K_2$.*

Lemma 4. [9]*: If $r(G) = 1$, then $ED(G) = (K_{|C(G)|})^* \to \left(\left\langle \overline{V(G) - C(G)} \right\rangle\right)^*$.*

3. Characterizations on Eccentric digraphs

Many open problems were posed by Boland, Buckley and Miller [5] concerning eccentric digraphs and characterizations. Mainly three of the previously posed problems were answered by Gimbert et.al. [15] viz.,

1. Characterize those graphs and digraphs that are antipodal.
2. Find hereditary properties of graphs and digraphs and their antipodal (eccentric).
3. Investigate relationships between antipodal and eccentric graphs(digraphs).

In case of undirected graphs, Buckley's [9] result is the most fundamental one which is in terms of complement of graph. Later on many results, most of the characterizations are based on some kind of complement of graph. We now look into these results.

Theorem 3. [9]*: Let G be a non trivial graph. Then G is a self-centered graph with radius 2 if and only if $ED(G) = (G)^*$.*

As noted in Bollabas' [3], almost all graphs are self-centered with radius 2, we have the following result.

Lemma 5. [9]*: For almost every graph G, its eccentric digraph is $ED(G) = (G)^*$.*

This result was extended by Gimbert et. al. [15] to the directed case by considering a modification of the complement operation, viz., the *complement of the reduction of G.*

Definition 1. [15]*: Given a digraph G of order n, a reduction of G, denoted by G^-, is derived from G by removing all its arcs incident from vertices with* out-degree $n-1$. *The digraph $\left(\overline{G^-}\right)$, is known as the* complement of the reduction of G.

Remark 10. *: $ED(G)$ is a subdigraph of $\left(\overline{G^-}\right)$ and moreover, their corresponding sets of vertices with out-degree $n-1$ are the same.*

Next result characterizes when the eccentric digraph of a graph (digraph) is equal to the complement of its reduction.

Proposition 1. [15]*: Let G be a digraph. Then $ED(G) = \left(\overline{G^-}\right)$, if and only if, for any vertex $u \in V(G)$ with eccentricity > 2 the following (local) transitive condition holds:*

$(u,v),(v,w) \in E(G) \Rightarrow (u,w) \in E(G), \forall v,w \in V(G)$ and $u \neq v$.

The next result is the generalization of Buckley's result for the undirected graphs.

Proposition 2. [15]*: Let G be a graph of order $n > 1$. Then $ED(G) = \left(\overline{G^-}\right)$, if and only if G satisfies one of the following conditions:*

(i). $rad(G) = 1$,

(ii). G is a self-centered of radius 2;

(iii). G is the union of complete graphs.

A digraph (graph) G is defined to be *eccentric* if there exists a digraph H such that $ED(H) \cong G$ as defined by Gimbert et.al.[15]. But in case of the term *"eccentric graph"* has been used by Chartrand and Gu[12] to denote a graph G such that all its vertices are *eccentric*(that is, $ED(G)$ has *minimum in-degree* ≥ 1). Next result gives a characterization for eccentric digraphs.

Theorem 4. [15]*: A digraph G is eccentric if and only if $ED(G) = \left(\overline{G^-}\right)$.*

Remark 11. *: The same cannot be assured in terms of undirected graphs, that is, if $ED(G) = \left(\overline{G^-}\right)$ then it is not necessary for the existence of an undirected graph H such that $ED(H) = G$.*

Example 2. *:*

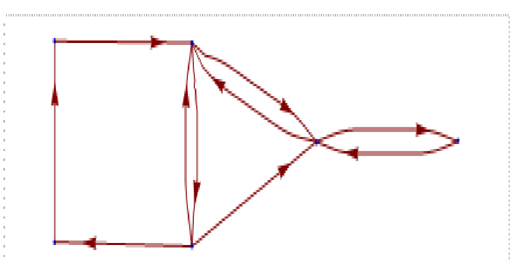

Figure 2

For the graph of *Figure 2*, $ED(G) = \left(\overline{G^-}\right)$ holds but, there does not exist an undirected graph H such that $ED(H) = G$.

In light of the above theorem and the note, a question remains unanswered, which we pose as an open problem as follows:

Problem 1. *: When there exists a symmetric digraph or undirected graph G for a digraph H such that $ED(G) = H$?*

For the undirected case the following result is due to Gimbert et.al.[15].

Theorem 5. [15]*: Let G be a graph of order $n > 1$. Then G is eccentric if and only if G is self-centered with radius two or G is the union of complete graphs.*

Corollary 1. [15]*:(i). Every non-connected graph with minimum degree > 0 is eccentric.*

(ii). The eccentric graphs of radius 1 are the complete multipartite graphs with at least one partite set of cardinality 1.

(iii). Every connected graph with radius ≥ 3 or diameter ≥ 4 is eccentric.

Corollary 2. [15]*: A tree is eccentric if and only if its diameter is not equal to* 3.

Remark 12. [6]*: If a digraph G is not eccentric then there exists an eccentric digraph H such that H contains G as an induced subdigraph and $|H| - |G| + 1$.*

In case of undirected case, the condition of being self-centered guarantees that the corresponding eccentric digraph is actually a graph since the distance is symmetric. But, this cannot be extended to the directed graphs, as the *distance is not a metric.* Hence the characterization of digraphs G such that $ED(G)$ is symmetric remains open. But, there is a complete solution in case of non-strongly connected graphs due to Gimbert. Et.al.[15].

Proposition 3. [15]*: Let G be a non-strongly connected digraph. Then $ED(G)$ is a symmetric digraph if and only if $G = C_1 \cup C_2 \cup ... \cup C_k$; $k \geq 2$,or $G = K_n \to C_1 \cup C_2 \cup ... \cup C_k$; $k \geq 1$,where $C_1, C_2, ..., C_k$ are strongly connected digraphs.*

Proposition 4. [15] *: Let G be a strongly connected digraph such that $ED(G)$ is symmetric. Then the following conditions hold:*

(i). $rad(G) > 1$, unless G is a complete digraph,

(ii). If $diam(G) = 2$ then G is self-centered graph.

4. Graphs with isomorphic Eccentric digraph

In this section we consider a special class of graphs for which the eccentric digraphs are isomorphic to the graph. In case of undirected graphs, Buckley [9] proved that the eccentric digraph of a graph G is equal to its complement, if and only if G is either a self-centered graph of radius two or G is the union of $k \geq 2$ complete graphs. In [15], Gimbert et. al. have proved that the eccentric digraph $ED(G)$ is symmetric if and only if G is self centered.

Also, we just mention about the period and tail of a graph, which will be elaborated in the next two sections. For every digraph G there exist smallest integers $p > 0$ and $t \geq 0$ such that $ED^t(G) = ED^{p+t}(G)$. In case of labeled graphs p is called the *period* of G and t the *tail* of G where as for unlabelled graphs these quantities are referred as *iso-period* ,donoted by $p\prime(G)$ and *iso-tail* , $t\prime(G)$ respectively.

Here we are looking at graphs which have their eccentric digraphs isomorphic to themselves. So by Gimbert's result these graphs are self-centered graphs. In this section we consider self-centered, undirected graphs. The following observations are easily justified.

Remark 13. [5]*: Odd cycles is a class of graphs for which* $ED(G) = G$.

Remark 14. [19]*: Odd cycles are graphs with minimum number of edges and maximum eccentricity on given number of vertices such that* $ED(G) = G$.

Remark 15. [19] *: For a self-centered graph* G *with radius* ≥ 3*, the complement* G *is self-centered with radius equal to two. Hence* $G \subset \overline{G}$*, and* $G \ncong \overline{G}$*, and* $ED(G)$ *is isomorphic to a subgraph of* G*. Further, by using Buckley's result [9], we can say that* $ED(\overline{G}) = \overline{\overline{G}} = G$*. That is if* $ED(\overline{G}) = ED(G)$*, then* $G = ED(G)$.

Remark 16. [5]*: Complete graphs is another class of graphs for which* $ED(G) = G$.

Remark 17. [19]*: It is easy to see that for graphs up to order* 7*, the only graphs for which* $ED(G) = G$*, are* $K_2, K_3, K_4, K_5, C_5, K_6, K_7, C_7$.

Remark 18. [19]*: Two isomorphic graphs have their eccentric digraphs isomorphic, but the converse need not be true always.*

As an example we give a pair of non-isomorphic, self-centered graphs with same eccentricity having one eccentric digraph.

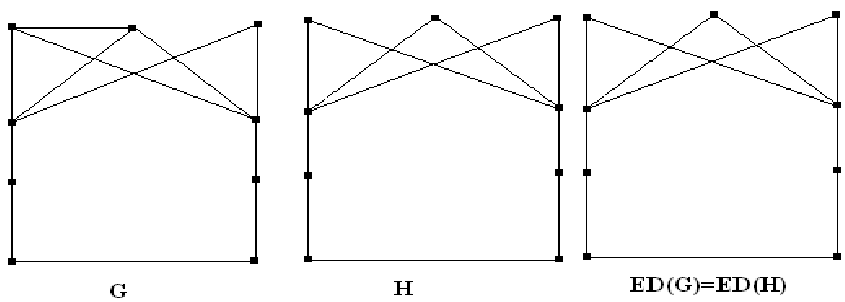

Figure 3

Lemma 6. [19] *: Let G be a self-centered graph with radius 2, then $ED(G) = G$ if and only if G is self-complementary.*

Lemma 7. [19]*: All self-centered graphs G with eccentricity greater than or equal to 3 with G having $period = 1, tail = 1$, satisfy the condition $ED(G) = G$.*

For G to be isomorphic to $ED(G)$, the necessary condition is that the for every vertex of degree, say k, there must exist another vertex with k number of eccentric vertices. This can be defined as eccentric degree of a vertex.

Definition 2. [19]*: For a vertex v of a graph G eccentric degree of v, denoted as $ecc.deg(v)$ is defined to be the number of vertices at eccentric distance from v. Also, the eccentric degree sequence of a graph is defined as a listing of eccentric degrees of vertices written in non-increasing order.*

So for $ED(G) = G$, the eccentric degree sequence of G should be equal to the degree sequence of G. But this condition is not sufficient as seen in the example below. Here both G and $ED(G)$ have their degree sequence and eccentric degree sequences as $(3^4, 2^9)$, but $ED(G) \ncong G$.

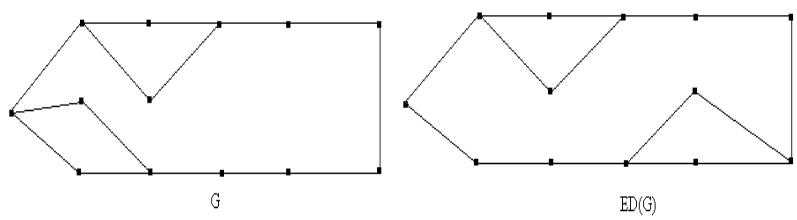

Figure 4

Next, we consider self-centered graphs with given maximum degree $\Delta(G)$. By [10],$\Delta(G) \leq p - 2r + 2$, for a self-centered graph G with radius r. The next result shows that there is no possibility of constructing a graph satisfying $ED(G) = G$, with $\Delta(G) = p - 2r + 2$.

Proposition 5. [19]*: There does not exist a graph G with $\Delta(G) = p - 2r + 2$,such that $ED(G) = G$.*

Theorem 6. [19]*: A connected self-centered graph G with $\Delta(G) = p-2r+1$, is isomorphic to its eccentric digraph if and only if its degree sequence is of the form $(p - 2r + 1)^2, 2^{p-2}$ with structure $K_1 + \overline{(K^{p-2r+1})} \cdot F \cdot \overline{K_2} + (\overline{K_2} - H) + (\overline{K_2} - H) + \cdots + (\overline{K_2} - H)$ $\{r - times\}$, where F is the graph obtained by joining one vertex of $\overline{(K^{p-2r+1})}$ to one vertex of $\overline{K_2}$ and remaining $p - 2r$ vertices to one vertex of $\overline{K_2}$, and H is the $1 - factor$ removed from successive $\overline{K_2} + \overline{K_2}$.*

In the next result we consider a particular case of graphs with $ED(G) = G$, that is, odd cycles.

Lemma 8. [19]*: In a labeled odd cycle $C_{2n+1}, n \geq 1$, two vertices v_i, v_j are at eccentric distance in $ED(C_{2n+1})$, if and only if $d_G(v_i, v_j) = (n/2)$ or $((n + 1)/2)$.*

Remark 19. [19]*: For unlabelled odd cycles, iterations of $ED(C_{2n+1})$ can be packed into K_n , since there are $((n - 1)/2) - 1 = ((n - 3)/2), ED(G)s$, whereas, $rad(C_{2n+1}) = n$.*

Remark 20. [19]*: In case of labeled odd cycles the sequence of* $ED(G)'s$ *can be packed into* K_n*, if the permutation on* n *number of vertices defined by* $f(1) = 1; f(2i) = r + 2 - i, i = 1, 2, 3, ..., r; f(2i+1) = 2r+2-i, i = 1, 2, 3, ..., r;$ *is a product of three cyclic permutations of length* $1, r, r,$ *respectively.*

Next, we construct a class of self-centered graphs which have $ED(G) = G$, which contain an odd cycle as a base.

Proposition 6. [19]*: There exists a self-centered graph* G*, such that* $ED(G) = G$*, containing an odd cycle.*

Following is an example of a self-centered graph with radius 4, satisfying $ED(G) \cong G$,with C_9, as base. So $S = \{1, 2, 3, \ldots 9\}$ and $V(C_n) = \{S_1, S_2, S_3, S_4, S_5\}$,where $S_1 = \{1\}, S_2 = \{2, 5, 8\}, S_3 = \{4\}, S_4 = \{7\}, S_5 = \{3, 6, 9\}$. For each $S_i, 1 \leq i \leq 4$ we have $S_{1_1} = \{1_1\}$, $S_{1_2} = \{1_2\}, S_{1_3} = \{1_3\}, S_{2_1} = \{2_1, 5_1, 8_1\}, S_{2_2} = \{2_2, 5_2, 8_2\}$, $S_{2_3} = \{2_3, 5_3, 8_3\}; S_{3_1} = \{4_1\}, S_{3_2} = \{4_2\}, S_{3_3} = \{4_3\}, S_{4_1} = \{7_1\}, S_{4_2} = \{7_2\}, S_{4_3} = \{7_3\}$.

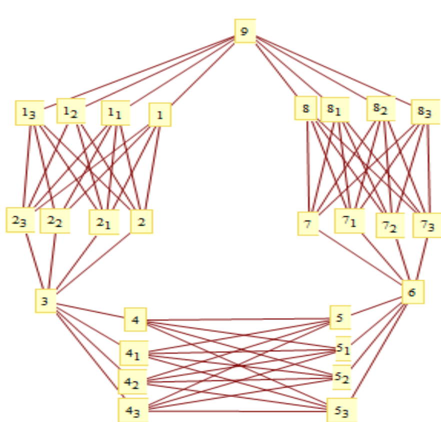

Figure 5

5. ECCENTRIC DIGRAPHS OF DDR GRAPHS

In this section we consider a special class of graphs, viz., the eccentric digraphs of *distance degree regular (DDR) graphs.* The *distance degree sequence (dds)* of a vertex v in a graph $G = (V, E)$ is a list of the number of vertices at distance $1, 2, \ldots, e(v)$ in that order, where $e(v)$ denotes the eccentricity of v in G. Thus, the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the *dds* of the vertex v_i in G where, d_{i_j}denotes number of vertices at distance j from v_i. The concept of *distance degree regular (DDR) graphs* was introduced by G. S. Bloom et.al.[4], as the graphs for which all vertices have the same *dds*.

A graph is said to be a *unique eccentric node(u.e.n.) graph* if for every vertex there exists a unique eccentric vertex in G.

Proposition 7. [20]*: There exists a* DDR *graph G such that $ED(G) \cong G$.*

Proposition 8. [20]*:There exists a* DDR *graph G such that, $ED(G)$ is a disconnected graph each of whose components are complete bipartite graphs.*

Proposition 9. [20]*:For a* unique eccentric node(u. e. n.) DDR *graph, if $p > 4$ then $tail = 1$, $period = 2$, and if $p = 4$, $tail = 0, period = 2$.*

Proposition 10. [20]*:For a given diameter $d(\geq 3)$, there exist $2d - 6$* DDR *graphs with $tail = 1$ and $period = 1$.*

Remark 21. [20]*: The unique eccentric node DDR graphs have their all iterated eccentric digraphs as DDR graphs.*

For non unique eccentric node DDR graphs the problem remains still open.

Problem 2. *: When do non-unique eccentric node DDR graphs have their all iterated digraphs as DDR graphs?*

6. ITERATED ECCENTRIC DIGRAPHS

Another line of investigation concerns the iterated sequence of eccentric digraphs. It is obvious that for any digraph G with $|V(G)| = n$,its eccentric digraph sequence $ED^k(G)$ cannot cycle through all possible digraphs. For every digraph G there exist smallest integers $p > 0$ and $t \geq 0$ such that $ED^t(G) = ED^{p+t}(G)$. In case of labeled graphs p is called the *period* of G and t the *tail* of G where as for unlabelled graphs these quantities are referred as *iso-period* $p\prime(G)$ and *iso-tail* $t\prime(G)$ respectively. A graph G is said to be *periodic* if $t(G) = 0$. Also, the iso-period of G divides period of G. We list some results on iterated eccentric digraphs.

Remark 22. [6]*: If a digraph G is the union of $k > 1$ vertex disjoint strongly connected digraphs of orders $n_1, n_2, ..., n_k$, for $m > 0$,*

$$ED^m(G) = \left\{ \begin{array}{c} K_{n_1,n_2,...,n_k}, \text{if m is odd;} \\ K_{n_1} \cup K_{n_2} \cup .... \cup K_{n_k}, \text{if m is even.} \end{array} \right\}.$$

Theorem 7. [21] *There exist self centered graphs with radius $r = (p-1)/2$ with odd cycle C_p, as a base, satisfying $ED^n(C_p) = C_p$,where n is the least positive integer such that for a labeled cycle C_p, whenever (i) p is an odd integer and $p \geq 7, p \neq 3(2m+1), m = 1, 2, 3 \ldots$ and $p \neq x^2, x(\geq 5)$is a prime integer (ii)$p \geq 7, p = 3(2m+1), m = 1, 2, 3 \ldots$ (iii)$p = x^2, x(\geq 5)$ is a prime integer.*

Theorem 8. [21]*:In a labeled odd cycle on $3(2m+1), m = 1, 2, 3 \ldots$vertices, the distance between any pair of vertices is $2m+1$ if and only if the distance between these pair of vertices is $2m+1$ in its eccentric digraph.*

Theorem 9. [21]*: For a given integer $n \geq 3$, there exist two odd cycles with period n.*

Theorem 10. [21]*: If G is a graph having a chordless cycle of length $2d+1$, where d is a diameter of G and each of whose vertices having eccentricity equal to d, then $ED(G)$ also has a chord less cycle of length $2d+1$.*

Next result proves that the *tail* and *iso-tail* of any digraph always coincide, which answers a question posed in [14].

Proposition 11. [16]*: For every digraph $G, t(G) = t\prime(G)$.*

Period and tail of random graphs were considered by Gimbert and Lopez [16] in detail. Some terminology about random graphs can be obtained from Bollabas [2,3].

Proposition 12. [3]*: Let be fixed natural numbers and let $0 < p < 1$ be also fixed. Then in $G(n, p)$ almost every digraph $G = (V, E)$ is such that for every sequence of d distinct vertices $x_1, x_2, ..., x_d$ there exists a vertex $x \in \{x_1, x_2, ..., x_d\}$ such that*

$$\left\{ \begin{array}{l} (x_i, x) \in ED(G), \ \ \textit{if } 1 \leq i \leq a, \\ (x, x_i) \in ED(G), \ \ \textit{if } a < i \leq b, \\ (x_i, x) \notin ED(G), \ \ \textit{if } b < i \leq c, \\ (x, x_i) \notin ED(G), \ \ \textit{if } c < i \leq d \end{array} \right\}$$

As a corollary to this result Gimbert and Lopez have proved the following result.

Corollary 3. [16]*: Let* $0 < p < 1$ *be fixed. Then in* $G(n, p)$ *almost every digraph* G *is such that* G *and its complement are both self-centered with radius* 2*.*

In Gimbert et.al[14] it was pointed out that if you pick a digraph G at random on a computer then it has its period equal to two. This had led to the conjecture that almost every digraph has period two and was proved by Gimbert and Lopez[16] as follows:

Proposition 13. [16]*: Let* $0 < p < 1$ *be fixed. Then in* $G(n, p)$ *almost every digraph has period two and tail zero.*

The odd cycle plays an important in the construction of eccentric digraphs. Gimbert and Lopez[16] had constructed digraphs whose period is equal to the period of the odd cycle by taking an odd cycle as a basis and using the generalized lexicographic product. As a consequence, the iso-period takes any value. Following results [21] we have shown the existence of non-isomorphic graphs with same order with tail zero and iso-period n.

Theorem 11. [21]*: For a given integer* $n \geq 3$*, there exist at least two non-isomorphic self- centered graphs having same order with tail* 0 *and iso-period* n*.*

But, the complete characterization is still awaited. Here is the original problem as listed in [5].

Problem 3. [5]*: Find graphs or digraphs for which* m *is the smallest integer such that* $ED^m(G) = G$*, for* $m = 1, 2, 3, ..., f(n)$*.*

As noted above we already have partial results. There is another related problem.

Problem 4. [5]*: Given the order of a graph or a digraph, what is the maximum possible value of* $f(n)$ *in Open Problem 3?*

7. PLANARITY

There exists another very important aspect of eccentric digraphs which needs to be answered is with respect to the planarity of $ED(G)$. In the first paper of Buckley[9] the following Problem is stated. But a complete characterization exists for undirected *trees* due to Buckley[9].

Problem 5. [5]*: Find a characterization for $ED(G)$ to be planar for digraphs.*

Theorem 12. [9]*: For a tree T, its eccentric digraph $ED(T)$ is planar if and only if none of the following conditions hold:*

(i) $d(T) = 2$ and T has order at least 5,

(ii) $d(T) = 3$ and the double star $S_{2,3} \subseteq T$,

(iii) $d(T) \geq 4, d(T)$ is odd and one of the two central vertices has at least three eccentric vertices, or

(iv) $d(T) \geq 4, d(T)$ is even and either the unique central vertex c has at least four eccentric vertices or c has three eccentric vertices x, y, and z and the set of branches at c containing none of x, y, or z contains at least two vertices other than c.

In [22] we have proved that a planar or non-planar graph G can have a planar eccentric digraph. But we are still in search of a characterization.

8. CONCLUSION

We conclude this survey with a comment that even though there are many results, characterizations in terms of complements of graphs exist in literature, there are open problems, structural results, constructions, graph properties in relation to the eccentric digraphs remain untouched.

References

[1] J. Akiyama, K. Ando, D. Avis, "Eccentric graphs", *Discrete Mathematics*, 56(1985)1-6.

[2] B. D. Acharya and M.Acharya, "On self-antipodal graphs", *National Academy Science Letters*, Vol. 8, No.5 (1985).

[3] R. Aravamudhan, B. Rajendran, "On antipodal graphs", *Discrete Mathematics*, 49(1984)193-195.

[4] G.S. Bloom, L.V. Quintas and J.W. Kennedy, "Distance Degree Regular Graphs", The Theory and Applications of Graphs, 4th International conference, Western Michigan University, Kalamazoo, MI, May (1980), John Wiley and Sons, New York, (1981), 95-108.

[5] J. Boland, F. Buckley, M. Miller, “Eccentric digraphs”, *Discrete Mathematics*,286 (1-2)(2004) 25-29.

[6] J. Boland, M. Miller, “The Eccentric digraph of a digraph”, in *Proceedings of the12th Australasian Workshop of Combinatorial Algorithms* (AWOCA 2001) 2001, pp.66-70.

[7] B. Bollobas, Graph Theory, An Introductory Course, Springer-Verlag, New York, 1979.

[8] B. Bollobas, Random Graphs,2nd ed.,Cambridge University Press, 2001.

[9] F. Buckley, “The Eccentric digraph of a graph”, *Congr. Numer.*, 149,(2001)65-76.

[10] F. Buckley, F. Harary, Distance in Graphs, Addison-Wesley, Redwood City, CA, 1990.

[11] G. Chartrand, L. Lesniak, Graphs and Digraphs, 3rd Ed., Chapman &Hall, London, 1996.

[12] G. Chartrand , W Gu, M. Schultz, S. Winters, "Eccentric Graphs“, *Networks* 34(2) (1999) 115-121.

[13] G. Chartrand, O. R. Ollermann, Applied and Algorithmic Graph Theory, McGraw-Hill, NewYork(1993).

[14] J. Gimbert, M. Miller, F. Ruskey, J. Ryan, “Iterations of eccentric digraphs”, *Bull. Inst.Combin. Appl.* 45(2005) 41-50.

[15] J. Gimbert, N.Lopez, M. Miller, J. Ryan, “Characterization of eccentric digraphs”, *Discrete Mathematics*, 306(2006) 210-219.

[16] J. Gimbert, N.Lopez, “On the behavior of sequences of iterated eccentric digraphs”, (Preprint).

[17] G. Johns, K. Sleno, “Antipodal graphs and digraphs”, *Internat. J. Math. Sci.* 16(3) (1993) 579- 586.

[18] Medha Itagi Huilgol, Syed Asif Ulla S, Sunilchandra A.R, "On Eccentric Digraphs of graphs", Proceedings of International Conference on Emerging Trends in Mathematics and Computer Applications (ICETMCA) held at Mepco Schlenk Engineering College, Sivakasi, Allied Publishers Pvt. Ltd., 41-44, 2010.

[19] Medha Itagi Huilgol, Syed Asif Ulla S, Sunilchandra A.R, "On Eccentric Digraphs of graphs", *Applied Mathematics*, Vol 2 (2011) 705-710.

[20] Medha Itagi Huilgol, Rajeshwari M, Syed AsifUlla, “Distance Degree Regular graphs and their eccentric digraphs”, *International J. of Math. Sci. & Engg. Appls.*Vol. 5, No. VI, (2011) 405-416.

[21] Medha Itagi Huilgol, Syed AsifUlla, “On powers of eccentric digraphs of graphs”,(Preprint).

[22] Medha Itagi Huilgol, “Planarity of eccentric digraphs”, (Preprint).