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EXPRESSIONS FOR THE FIRST EIGENVALUES AND REGULARIZED TRACE FORMULAE FOR A SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH A TURNING POINT

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Abstract : Consider the system of second order differential equation

$y''(x)+(\lambda^2 R(x) + Q(x))y(x) = 0,\ 0 \le x \le \pi$

where $y(x) = (y_1(x), y_2(x))^T$, $Q(x)= \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}$, $R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix}$,

p(x),q(x),r(x),s(x),t(x) being real–valued continuously differentiable functions of x on $[0, \pi]$.

In the present paper we determine the expressions for the first eigenvalues for the system in different cases for s(x), t(x) satisfying on$[0,\pi]$, the conditions

a) $s(x) = xs_1(x)$, $t(x) = xt_1(x)$, $s_1(x) > 0$, $t_1(x) > 0$, or,

b) $s(x) = s_1(x) / x$, $t(x) = t_1(x) / x$, $s_1(x) > 0$, $t_1(x) > 0$, or,

c) $s(x) > 0$, $t(x) > 0$

by using the asymptotic expressions for the nth eigenvalue (λ_n) and those of the corresponding normalized eigenvector $\psi(x, \lambda_n) = (\psi_1(x, \lambda_n), \psi_2(x, \lambda_n))^T$ under the Dirichlet and Neumann boundary conditions. Further, we determine the expressions for the regularized trace matrix for the system with $s(x) = t(x) = 1$, $0 \le x \le \pi$ under the Neumann and general boundary conditions by employing the corresponding asymptotic expressions for the nth eigenvalue (λ_n).

Keywords: asymptotic solutions, turning points, Dirichlet and Neumann boundary conditions, general boundary conditions, eigenvalues, normalized eigenvectors, first eigenvalues, regularized trace.

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1. Introduction

Consider the second order boundary value problem (Sturm – Liouville type)

$$y''(x)+\{\lambda + q(x)\}y(x) = 0, \quad 0\leq x\leq\pi \qquad \text{. . . . (1)}$$

$$y'(0) – hy(0) = 0,\ y'(\pi) + Hy(\pi) = 0 \qquad \text{. . . .(2)}$$

where $q(x) \in C_2(0,\pi)$, h, H are real finite numbers. Let $\{\lambda_n\}_0^\infty$ be the sequence of eigenvalues of the boundary value problem (1) – (2). It is well-known (see Levitan and Sargsjan [12], pp 77 – 81) that the series

$$S_\lambda=\sum_{n=0}^{\infty}(\lambda_n - n^2 - c), c = \frac{2}{\pi}(h + H + \frac{1}{2}\int_0^\pi q(x)dx) \qquad \text{. . . . (3)}$$

is convergent and

$$S_\lambda= \frac{1}{4}[q(0) + q(\pi)] - \frac{H}{\pi} - \frac{H^2}{2} \qquad \text{. . . . (4)}$$

S_λ being called the 'Regularized trace' and the formula was first obtained by Gelfand and Levitan [7, 8]. Works of Levitan [10, 11], Gelfand [6] Dikii [4] on regularized trace are also worth notice. Further references may be made to the works of Bogachev [2], Lyubiskhin [17], Mitrokhin [18], Pikula and Martinovic [19] among others.

A general discussion on the calculation of the regularized trace formula for the two-term fourth order operator.

$$l(y) = y^{IV} + q(x)y = \lambda y,\ 0 \leq x \leq\pi,$$

$$y(0) = y'(0) = y(\pi) = y'(\pi) = 0 \qquad \text{…. (5)}$$

occurs in the monograph by Levitan and Sargsjan [12].

Defining the 'trace' of an operator as the sum of its eigenvalues, Levitan and Sargsjan [12] obtained for a Dirac system

$$y_2' – \{ \lambda + p(x)\}y_1 = 0,\ y_1' + \{ \lambda + r(x)\}y_2 = 0,$$

$y_2(0)\, \mathrm{Cos}\alpha + y_1(0)\mathrm{Sin}\alpha = 0,\ y_2(\pi)\mathrm{Cos}\beta + y_1(\pi)\mathrm{Sin}\beta = 0$ . . . . (6)

the trace formula given by

$$\lambda_0 + \sum_{n=1}^{\infty}(\lambda_n + \lambda_{-n}) = \frac{r(0) - p(0)}{4}\mathrm{Cos}2\alpha + \frac{r(\pi) - p(\pi)}{4}\mathrm{Cos}2\beta \qquad . . . . (7)$$

where $\{\lambda_n\}_0^{\infty}$ are the sequence of eigenvalues of the boundary value problem (6) .

The trace formulae for the Dirac system are also obtained by Abdukadyrov [1] and Guseinov [9].

For the Sturm-Liouville equation of type

$$y''(x) + (\lambda^2 r(x) + q(x))y(x) = 0, \text{ over } [0, \pi] \qquad . . . . (8)$$

where $r(x) = x^{\alpha} r_1(x)$, α, being some positive or negative number and $r_1(x) > 0$, Dorodnicyn [5], using the asymptotic expressions for the eigenvalues, determined the first K-eigenvalues $\lambda_0^2, \lambda_1^2, \ldots. \lambda_{k-1}^2$ by forming K-equations

$$\sum_{n=0}^{k-1} \lambda_n^{-2m} = g_m - \sum_{n=k}^{\infty} \lambda_n^{-2m}, \ (m = 1, 2, \ldots\ldots., k) \qquad . . . . (9)$$

where $g_m = \int_0^{\pi} r(x) G_m(x, x)\, dx$, ... (10)

$$G_m(x, x) = \sum_{n=0}^{\infty} \lambda_n^{-2m} (\bar{y}(x, \lambda_n))^2 \qquad . . . . (11)$$

$\bar{y}(x, \lambda_n)$ being the normalized eigenfunctions.

Introducing the generalized Zeta function

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s} \qquad . . . . (12)$$

Dorodnicyn [5] obtained expressions for the series $\sum_{n=0}^{k-1} \lambda_n^{-2m}$, m = 1, 2, …., k, expressed in terms of the generalized Zeta function, λ_n being the nth eigenvalue of the system (8) under certain boundary conditions. He went deep into the subject by considering among others the Mathieu equations and the Kelvin and Darwin equations viz.

$\frac{d}{d\mu}\left(\frac{1 - \mu^2}{f^2 - \mu^2} \cdot \frac{d\rho}{d\mu}\right) + \beta^2 \rho = 0$[see Dorodnicyn [5] Pp. 58 – 66].

Sevcenko [22] obtained trace formula for more general cases. A procedure for calculating trace formulae for general problems involving ordinary differential equations over a finite interval is given in Lidskii and Sadovnicii [13, 14, 15].

In the present paper we consider the system of second order differential equations

$$y''(x) + (\lambda^2 R(x) + Q(x))y(x) = 0, \quad 0 \le x \le \pi \qquad \ldots . (13)$$

where , $y(x) = (y_1(x), y_2(x))^T$,

$$Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}, \qquad R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix},$$

p(x), q(x), r(x), s(x), t(x) being real-valued continuously differentiable functions of x on $[0, \pi]$ and further s(x), t(x) are specified in the following different ways :

a) $s(x) = xs_1(x)$, $t(x) = xt_1(x)$, $s_1(x), t_1(x) > 0$ for $0 \le x \le \pi$ . . . . (14)

b) $s(x) = s_1(x) / x$, $t(x) = t_1(x) / x$, $s_1(x), t_1(x) > 0$ for $0 \le x \le \pi$ . . . . (15)

c) $s(x) > 0$, $t(x) > 0$ for $0 \le x \le \pi$ . . . . (16)

In this paper we obtain the expressions for the first eigenvalues for the system (13) with s(x), t(x) satisfying relations (14) or (15) or (16) under (i) the Dirichlet boundary conditions i.e.,

$$y_1(0) = y_2(0) = y_1(\pi) = y_2(\pi) = 0 \qquad \ldots . (17)$$

or, (ii) the Neumann boundary conditions i.e.,

$$y_1'(0) = y_2'(0) = y_1'(\pi) = y_2'(\pi) = 0 \qquad \ldots . (18)$$

satisfied by the solution $y(x) = (y_1(x), y_2(x))^T$ of the system (13) at $x = 0$, $x = \pi$ by using the asymptotic expressions for the nth eigenvalue (λ_n) and those of the corresponding normalized eigenvector $\psi(x, \lambda_n) = (\psi_1(x, \lambda_n), \psi_2(x, \lambda_n))^T$ which are determined by Sengupta [23, 24, 25].

Further, we determine in what follows the expressions for the regularized trace matrix connected with the system (13) with $s(x) = t(x) = 1$ under the Neumann boundary conditions (18) and the general boundary conditions

$a_{i1}y_{j1}(0) + a_{i2}y_{j1}'(0) + a_{i3}y_{j2}(0) + a_{i4}y_{j2}'(0) = 0,$

$b_{i1}y_{j1}(\pi) + b_{i2}y_{j1}'(\pi) + b_{i3}y_{j2}(\pi) + b_{i4}y_{j2}'(\pi) = 0, i,j = 1,2$ . . . . (19)

satisfied by the solution $y(x) = (y_1(x), y_2(x))^T$ of the system (13) at $x = 0$, $x = \pi$ where

$y_i(x) = (y_{i1}(x), y_{i2}(x))^T$, i = 1, 2

and a_{ij}, b_{ij}, i = 1, 2; j = 1,2,3,4 are real-valued constants independent of λ satisfying

(i) $\text{rank}(a_{ij}) = \text{rank}(b_{ij}) = 2$, i = 1, 2 ; j =1, 2, 3, 4 where at least one of

$\begin{vmatrix} a_{j1} & a_{j3} \\ a_{k2} & a_{k4} \end{vmatrix}$, j, k = 1, 2 ; $\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$, $\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \neq 0$

(ii) $a_{j1}a_{k2} + a_{j3}a_{k4} = 0$, j, k = 1, 2

(iii) $b_{11}b_{22} - b_{12}b_{21} + b_{13}b_{24} - b_{14}b_{23} = 0,$

by using the asymptotic formula for the nth eigenvalue (λ_n) as obtained in the monograph of Sengupta [25].

2. Certain results for the normalized eigenvector $\psi(x, \lambda_n)$

Let $G(x, z) = (G_{ij}(x, z))$, i, j = 1, 2 be the 2 x 2 symmetric matrix where $G_{ij}(x, z)$, i, j = 1, 2 are continuous and posses continuous differential coefficients upto the order two with respect to x for $0 \le x \le \pi$.

Let G(x, z) satisfy

(i) $\frac{\delta^2 G(x,z)}{\delta x^2} + Q(x)\, G(x, z) = 0,$ . . . . (20)

(ii) the discontinuity conditions

$G'_x(z, z + 0) - G'_x(z, z - 0) = I$, I being 2 x 2 unit matrix, and

(iii) the vectors $G_k(x, z) = (G_{k1}(x, z), G_{k2}(x, z))^T$, k= 1,2 satisfy the boundary conditions (17) or (18), It may be noted that the matrix G(x, z), so defined, may be called the special Greens' matrix associated with the system (13). Evidently the matrix G(x, z) is independent of λ.

Let $f(x) = (f_1(x), f_2(x))^T$ be a real-valued continuous functions of x over $[0, \pi]$ and $f^T(x)R(x)f(x) \in L[0, \pi]$. Let us define the vector $\Phi(x) = (\Phi_1(x), \Phi_2(x))^T$ given by

$$\Phi(x) = \int_0^{\pi} G(x,z)R(z)f(z)dz \qquad \text{. . . . (21)}$$

Differentiating $\Phi(x)$ w.r. to x twice we obtain

$$\Phi'(x) = \int_0^{\pi} G'(x,z)R(z)f(z)dz \text{ and}$$

$$\Phi''(x) = \int_0^{x} G''(x,z)R(z)f(z)dz + \int_x^{\pi} G''(x,z)R(z)f(z)dz$$

$$+(G'(x, x - 0) - G'(x, x + 0))R(x)f(x)$$

$$= \int_0^{\pi} G''(x,z)R(z)f(z)dz + (G'(x, x - 0) - G'(x, x + 0))R(x)f(x)$$

$$= -Q(x)\,\Phi(x) - R(x)f(x).$$

$$\text{i.e.; } \Phi''(x) + Q(x)\,\Phi(x) = -R(x)f(x) \qquad \text{. . . . (22)}$$

Thus $\Phi(x)$ satisfies the differential equation

$$y''(x) + Q(x)y(x) = -R(x)f(x) \qquad \text{. . . . (23)}$$

Expressing equation (13) in the form

$$y''(x) + Q(x)y(x) = -R(x)[\lambda^2 y(x) + f(x)]$$

it follows that the solution $y(x) = (y_1(X), y_2(x))^T$ of (13) can be obtained in the form

$$y(x) = \int_0^{\pi} G(x,z)R(z)\left[\lambda^2 y(z) + f(z)\right]dz$$

$$= \lambda^2\int_0^{\pi} G(x,z)R(z)y(z)dz + g(x), \text{ say} \qquad .... (24)$$

where $g(x) = \int_0^{\pi} G(x,z)R(z)f(z)dz$ .... (25)

Therefore the system (13) is equivalent to the integral equation

$$y(x) - \lambda^2\int_0^{\pi} G(x,z)R(z)y(z)dz = g(x) \qquad .... (26)$$

where g(x) is given by (25).

By making f(x) = 0, it follows from (25) and (26) that the homogeneous boundary value problems (13) – (17) or (13) – (18) with s(x), t(x) satisfying (14) or (15) or (16) is equivalent to the integral equation

$$y(x) = \lambda^2\int_0^{\pi} G(x,z)R(z)y(z)dz \qquad .... (27)$$

where G(x, z) is the special Green's matrix associated with the system (13) and $\lambda \neq 0$ is an eigenvalue,

Let $\{\psi(x, \lambda_n)\} = \{(\psi_1(x, \lambda_n), \psi_2(x, \lambda_n))^T\}$ be a sequence of normalized eigenvectors corresponding to the eigenvalues $\{\lambda_n\}_0^{\infty}$ of the boundary value problem (13) – (17) or (13) – (18).

Let $\{\psi(x, z; \lambda_n)\} = \begin{pmatrix} \psi_1(x,\lambda_n)\psi_1(z,\lambda_n) & \psi_1(x,\lambda_n)\psi_2(z,\lambda_n) \\ \psi_2(x,\lambda_n)\psi_1(z,\lambda_n) & \psi_2(x,\lambda_n)\psi_2(z,\lambda_n) \end{pmatrix}$ .... (28)

and $H(x, z) = \sum_{n=0}^{\infty} \lambda_n^{-2}\ \psi(x,z;\ \lambda_n)$. .... (29)

For the boundary value problems (13) – (17) or (13) – (18) with s(x), t(x) satisfying (14) or (15) or (16) by using the asymptotic expressions for the $\psi(x, \lambda_n)$, as determined by Sengupta [23, 24, 25], it follows that

$$\psi_i(x, \lambda_n)\psi_j(z, \lambda_n) = K.g_i(x).g_j(z) + 0(\lambda_n^{-1}), \quad .... (30)$$

for n > 0 and i,j = 1, 2, k is a certain constant and $g_i(x)$ are the functions involving s(x), t(x) but independent of λ_n. Hence using the asymptotic expressions for the corresponding λ_n, as given in Sengupta [23, 24, 25], we obtain from (29) and (30) that the series H(x, z) converges uniformly for all x, z in every finite interval.

We prove the following theorem :

Theorem : 1 *A necessary and sufficient condition for λ_n to be an eigenvalue of the boundary value problem (13) – (17) or (13) – (18) with s(x), t(x) satisfying (14) or (15) or (16) is that*

$$P(x, z) \equiv H(x, z) - G(x, z) = \sum_{n=0}^{\infty} \lambda_n^{-2}\, \psi(x, z; \lambda_n) - G(x, z) \quad .... (31)$$

is zero identically, where H(x, z), $\psi(x, z; \lambda_n)$ are given in (29), (28) respectively and G(x, z) is the special Green's matrix associated with the system (13).

Proof : From the definitions of H(x, z), G(x, z) it follows that P(x, z) is continuous and symmetric i.e.; $P^T(x, z) = P(z, x)$. Hence by the well-known theorem of the theory of integral equations [see Courant and Hilbert [3], Pp – 122] that it P(x, z) is not identically zero, there exists at least one eigenvector. Hence there exists a number λ_0 and a vector $u(x) \neq 0$ satisfying

$$u(x) + \lambda_0^2 \int_0^{\pi} P(x,z)R(z)u(z)dz = 0 \quad .... (32)$$

where R(z) is defined in (13).

Let $\psi(x, \lambda_k) = (\psi_1(x, \lambda_k), \psi_2(x, \lambda_k))^T$ be the normalized eigenvector of the given problem corresponding to the eigenvalue $\lambda_k \neq 0$. Then from (27) we obtain

$$\int_0^{\pi} G(x,z)R(z)\,\psi(z,\lambda_k)dz = \lambda_k^{-2}\,\psi(x, \lambda_k) \quad .... (33)$$

Put $M(x) \equiv (M_1(x), M_2(X))^T = \int_0^\pi P(x,z)R(z)\,\psi(z,\lambda_k)dz$ . . . . (34)

$N(x) \equiv (N_1(x), N_2(x))^T = \int_0^\pi H(x,z)R(z)\,\psi(z,\lambda_k)dz$ . . . . (35)

Thus by using the expressions for H(x, z) given in (29) it follows that

$N(x) = \lambda_k^{-2}\psi(x, \lambda_k)$ . . . . (36)

Then by using (33), (36) it follows from (34) that M(x) = 0 . . . . (37)

Hence from (32) we obtain u(x) = 0, which contradicts our hypothesis. Therefore P(x, z) is identically zero.

Conversely, let $u(x) = (u_1(x), u_2(x))^T$ be a solution of the integral equation (32). Then

$$\int_0^\pi u^T(x)R(x)\,\psi(x,\lambda_n)dx + \lambda_0^2 \int_0^\pi u^T(z)R(z)\left(\int_0^\pi P^T(x,z)R(x)\,\psi(x,\lambda_n)dx\right)dz = 0$$

. . . . (38)

As $P^T(x, z) = P(z, x)$, using (37) we obtain $\int_0^\pi u^T(x)R(x)\,\psi(x,\lambda_n)dx = 0$

. . . . (39)

Therefore u(x) is orthogonal to the eigenvector $\psi(x, \lambda_n)$. Thus from (31), it follows that

$$\int_0^\pi P(z,x)R(x)u(x)dx = - \int_0^\pi G(z,x)R(x)u(x)dx \quad . . . . (40)$$

From (32), we now obtain

$$u(x) - \lambda_0^2 \int_0^\pi G(x,z)R(z)u(z)dz = 0 \quad . . . . (41)$$

i.e.; u(x) is an eigenvector of the given boundary value problem. Since u(x) is orthogonal to all the eigenvectors, $u(x) \equiv 0$ and therefore P(x, z) = 0. This completes the proof.

Thus if $\lambda_n \neq 0$ be an eigenvalue of the boundary value problem (13) – (17) or (13) – (18) we obtain

$$G(x, z) = \sum_{n=0}^{\infty} \lambda_n^{-2} \psi(x, z; \lambda_n) \quad \text{. . . . (42)}$$

where $\psi(x, z; \lambda_n)$ is given by (28) and G(x, z) is the special Green's matrix associated with the system (13).

3. Evaluation of the first eigenvalues

Let $\{\lambda_n\}_0^{\infty}$ be the sequence of eigenvalues λ_n for the boundary value problem (13) – (17) or (13) – (18) with s(x), t(x) satisfying (14) or (15) or (16). We now form the special iterated Green's matrices in the following way :

Put $G_2(x, z) = \int_0^{\pi} G(x,y)R(y)G(y,z)dy$ . . . . (43)

and for m > 1, $G_{m+1}(x, z) = \int_0^{\pi} G_m(x,y)R(y)G(y,z)dy$ . . . . (44)

where G(x, y) is the special Green's matrix as defined before.

By using (42) it now follows that

$$G_2(x, z) = \sum_{n=0}^{\infty} \lambda_n^{-4} \psi(x, z; \lambda_n) \quad \text{. . . . (45)}$$

and $G_m(x, z) = \sum_{n=0}^{\infty} \lambda_n^{-2m} \psi(x, z; \lambda_n)$ for m > 1 . . . . (46)

Put $g_m = \sum_{i,j=1}^{2} T_{ij}$ . . . . (47)

where T_{ij}, i, j = 1, 2 are the elements of the matrix

$$T = (T_{ij}) = \int_0^{\pi} G_m(x,x)R(x)dx \quad \text{.......(48)}$$

$G_m(x, z)$ being given in (46).

For $m \geq 1$, from (47), (48) we obtain

$$g_m = \sum_{n=0}^{\infty} \lambda_n^{-2m} \qquad \text{. . . . (49)}$$

Now starting with some suffix, say n = k, we apply the asymptotic representation of the eigen-values and the first 'k' of the equations (49) give a system of k-equations which determine the first k-eigenvalues $\lambda_0^2, \lambda_1^2, \ldots\ldots\lambda_{k-1}^2$, namely

$$\sum_{n=0}^{k-1} \lambda_n^{-2m} = g_m - \sum_{n=k}^{\infty} \lambda_n^{-2m}, \; m = 1, 2, \ldots, k \qquad \text{. . . . (50)}$$

We prove the following theorems :

Theorem : 2 *Let $\{\lambda_n\}$ be the sequence of eigenvalues of the boundary value problem (13) – (17) where s(x), t(x) satisfy the conditions (14) then*

$$\sum_{n=k}^{\infty} \lambda_n^{-2m} = (\pi)^{-2m} \left[\int_0^{\pi} \sqrt{z(x)}dx \right]^{2m} . \; \zeta\left(2m, k- \frac{1}{12}\right) + 0 \left(\sum_{n=k}^{\infty} n^{-2m-2}\right) \qquad \text{. . . . (51)}$$

where $\zeta(s, a)$ is the generalized Zeta function defined in (12) and z(x) is either s(x) or t(x) according as $\int_0^x \sqrt{s(x)}\, dx >$ or $< \int_0^x \sqrt{t(x)}dx$.

Proof *:* From the asymptotic representation for λ_n of the given boundary value problem (13) – (17), as given in (65)(66) of Sengupta [23], it follows that

$$\sum_{n=k}^{\infty} \lambda_n^{-2m} = (\pi)^{-2m} \left[\int_0^{\pi} \sqrt{z(x)}dx \right]^{2m} \sum_{n=k}^{\infty} \left(n - \frac{1}{12}\right)^{-2m} + 0 \left(\sum_{n=k}^{\infty} n^{-2m-2}\right) \text{. . . .(52)}$$

where z(x) is either s(x) or t(x) according as $\int_0^x \sqrt{s(x)}dx >$ or $< \int_0^x \sqrt{t(x)}dx$.

Introducing the zeta function $\zeta(s, a)$ as given in (12), from (52) the theorem is proved.

Similarly using (69) or (70) of Sengupta [23], we obtain

Theorem : 3 *Let {λ_n} be the sequence of eigenvalues of the boundary value problem (13) – (18) where s(x), t(x) satisfy the conditions (14), then*

$$\sum_{n=k}^{\infty} \lambda_n^{-2m} = (\pi)^{-2m}. \left[\int_0^{\pi} \sqrt{z(x)}dx \right]^{2m} \zeta(2m, k + \tfrac{1}{12}) + 0 \left(\sum_{n=k}^{\infty} n^{-2m-5/3}\right) \quad \ldots . (53)$$

where z(x) is either s(x) or t(x) according as $\int_0^X \sqrt{s(x)}dx$ *> or <* $\int_0^x \sqrt{t(x)}dx$*.*

Also by making use of the relations (31), (32) of Sengupta [24], we obtain

Theorem : 4 *Let {λ_n} be the sequence of eigenvalues of the boundary value problem (13) – (17) where s(x), t(x) satisfy the conditions (15), then*

$$\sum_{n=k}^{\infty} \lambda_n^{-2m} = (2\pi)^{-2m} \left[\int_0^{\pi} \sqrt{z(x)}dx \right]^{2m} \zeta(2m, k + \tfrac{1}{4}) + 0 \left(\sum_{n=k}^{\infty} n^{-2m-3}\right) \quad \ldots . (54)$$

where z(x) is either s(x) or t(x) according as $\int_0^x \sqrt{s(x)}dx$ *> or <* $\int_0^x \sqrt{t(x)}dx$*.*

Further, using (24) of Sengupta [25] we obtain

Theorem : 5 *Let {λ_n} be the sequence of eigenvalues of the boundary value problem (13) – (17) or (13) – (18) where s(x), t(x) satisfy the conditions (16) then*

$$\sum_{n=k}^{\infty} \lambda_n^{-2m} = \sum_{n=k}^{\infty} (2\pi)^{-2m} \left[\int_0^{\pi} \sqrt{z(x)}dx \right]^{2m} + 0\left(\sum_{n=k}^{\infty} n^{-2m-2}\right) \quad \ldots . (55)$$

where z(x) is either s(x) or t(x) according as $\int_0^x \sqrt{s(x)}dx$ *> or <* $\int_0^x \sqrt{t(x)}dx$*.*

4. Evaluation of regularized trace matrix in the cases where s(x) = t(x) = 1

In this case the system (13) reduces to

$$y''(x) + (\lambda^2 I + Q(x))y(x) = 0 \ , \ 0 \le x \le \pi \qquad \text{. . . . (56)}$$

where I is the 2 x 2 unit matrix and Q(x) is defined in (13)

Now, by using (39) and (40) of Sengupta [25] it follows that for the boundary- value problem (56) – (19), for sufficiently large n, the asymptotic expressions for the nth eigenvalue λ_n satisfy

$$\lambda_n^2 = n^2 + \overline{P}_{ij} + 0(n^{-2}),\ i, j = 1, 2 \qquad \text{. . . . (57)}$$

where $P_{ij} = 2\overline{P}_{ij} / \pi$, P_{ij}'s being those given in (38) of Sengupta [25].

$$\text{Hence, } S_{ij} = \sum_{n=0}^{\infty}(\lambda_n^2 - n^2 - \overline{P}_{ij}) < \infty,\ i, j = 1, 2 \qquad \text{. . . . (58)}$$

The matrix S = (S_{ij}), after Levitan and Sargsjan [12], is defined as the regularized trace matrix for the system (13) when s(x) = t(x) =1 .

Let $y_j(x, \lambda) = (y_{j1}(x, \lambda), y_{j2}(x, \lambda))^T$, j = 1, 2 be two linearly independent solutions of the system (56) which with their first derivatives take some prescribed values a_{ij} and b_{ij}, i = 1,2 , j = 1, 2, 3, 4 at x = 0 and x = π respectively, as explicitly stated in (31) of Sengupta [3]. The eigenvalues λ_n are the roots of the entire analytic function

$$b_{i1}y_{j1}(\pi, \lambda) + b_{i2}y'_{j1}(\pi, \lambda) + b_{i3}y_{j2}(\pi, \lambda) + b_{i4}y'_{j2}(\pi, \lambda),\ i, j = 1, 2.$$

Hence for fixed i, j = 1, 2, we have (the eigenvalues being represented by λ_n)

$$b_{i1}y_{j1}(\pi, \lambda) + b_{i2}y'_{j1}(\pi, \lambda) + b_{i3}y_{j2}(\pi, \lambda) + b_{i4}y'_{j2}(\pi, \lambda) = A_{ij}B_{ij}(\lambda) \qquad \text{. . . . (59)}$$

where $Bij(\lambda) = \prod_{n=0}^{\infty}(1 - \lambda^2 / \lambda_n^2)$, $(\lambda, \lambda_n \neq 0)$ and A_{ij} are some constants will be determined later.

Put $B_{ij}(\lambda) = C_{ij}(\lambda_0^2 - \lambda^2).\ D_{ij}(\bar{\lambda})\text{Sin } \pi\lambda / \pi\lambda$, i, j = 1, 2 . . . . (60)

where $C_{ij} = \lambda_0^2 \prod_{n=1}^{\infty} n^2 / \lambda_n^2$,

$$D_{ij}(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda^2 - \lambda_n^2}{n^2 - \lambda^2}\right),\ i, j = 1, 2 \qquad \text{. . . . (61)}$$

Now, $\log D_{ij}(\lambda) = -\sum_{k=1}^{\infty} \quad \sum_{n=1}^{\infty} \frac{1}{k}\left(\frac{n^2 - \lambda_n^2}{n^2 - \lambda^2}\right)^k$ . . . . (62)

Let $|\lambda_n^2 - n^2| < a$, then

$$\left| \sum_{n=1}^{\infty}\left(\frac{n^2 - \lambda_n^2}{n^2 - \lambda^2}\right)^k \right| < \frac{a^k}{|\lambda|^k}\left|\sum_{n=1}^{\infty} \frac{1}{(n - \lambda)^k}\right|,\ \lambda \neq n .$$

Also, $\left|\sum_{n=1}^{\infty} \frac{1}{(n-\lambda)^k}\right| < \int_0^{\infty} \frac{dx}{(x-\lambda)^k}$.

Hence, $\sum_{k=2}^{\infty}\left|\sum_{n=1}^{\infty} \frac{1}{k}\left(\frac{n^2 - \lambda_n^2}{n^2 - \lambda^2}\right)^k\right| = 0(|\lambda|^{-3})$ . . . . (63)

$$\text{Again, } -\sum_{n=1}^{\infty} \frac{n^2 - \lambda_n^2}{n^2 - \lambda^2} = \frac{1}{\lambda^2}\sum_{n=1}^{\infty} \frac{(\lambda_n^2 - n^2 - \bar{P}_{ij})n^2}{n^2 - \lambda^2} - \frac{1}{\lambda^2}\sum_{n=1}^{\infty}(\lambda_n^2 - n^2 - \bar{P}_{ij}) + \sum_{n=1}^{\infty} \frac{\bar{P}_{ij}}{n^2 - \lambda^2} \qquad \text{. . . . (64)}$$

It also follows that

Sup $\left|(\lambda_n^2 - n^2 - \bar{P}_{ij})n^2\right| < \infty$ and

$\sum_{n=1}^{\infty} \frac{1}{n^2 - \lambda^2} = \frac{3}{4\lambda^2}$. $(n \neq \lambda)$ [See Titchmarsh [26], Pp. 34].

Hence, $\frac{1}{\lambda^2}\sum_{n=1}^{\infty} \frac{(\lambda_n^2 - n^2 - \bar{P}_{ij})n^2}{n^2 - \lambda^2} = 0\ (|\lambda|^{-4})$ . . . . (65)

From (62), by using (63) – (65), it follows that

$D_{ij}(\lambda) = 1 + \lambda^{-2} (\lambda_0^2 - S_{ij} - \bar{P}_{ij}/4) + 0(|\lambda|^{-3})$ . . . . (66)

where $S_{ij} = \sum_{n=0}^{\infty}(\lambda_n^2 - n^2 - \bar{P}_{ij})$ . . . . (67)

From (60), we therefore obtain

$B_{ij}(\lambda) = \frac{C_{ij}}{\pi}.\ \text{Sin}\ \pi\lambda[- \lambda + \lambda^{-1} . (S_{ij} + P_{ij} / (2\pi))+ 0(|\lambda|^{-2})],$ . . . . (68)

where P_{ij} is given in (38) of Sengupta [25].

Making use of the expressions (34) – (35) of Sengupta [25], we obtain

$$b_{i1}y_{j1}(\pi, \lambda) + b_{i2}y'_{j2}(\pi, \lambda) + b_{i3}y_{j2}(\pi, \lambda) + b_{i4}y'_{j2}(\pi, \lambda)$$

$$= \text{Cos}\lambda\pi[b_{i1}a_{j2} - b_{i2}a_{j1} + b_{i3}a_{j4} - b_{i4}a_{j3}$$

$$- \frac{b_{i2}a_{j2}}{2}\int_0^{\pi} p(z)dz - \frac{b_{i2}a_{j4}}{2}\int_0^{\pi} r(z)dz$$

$$- \frac{b_{i4}a_{j2}}{2}\int_0^{\pi} r(z)dz - \frac{b_{i4}a_{j4}}{2}\int_0^{\pi} q(z)dz]$$

$$- \lambda\text{Sin}\lambda\pi[b_{i1}a_{j1} + b_{i3}a_{j3} + \frac{b_{i2}\ a_{j2}}{4} (p(\pi) + p(0))$$

$$+ \frac{b_{i2}a_{j4}}{4}(r(\pi) + r(0)) + \frac{b_{i4}a_{j2}}{4} (r(\pi) + r(0)) + \frac{b_{i4}a_{j4}}{4}(q(\pi) + q(0))$$

$$+ \frac{b_{i1}a_{j2}}{2}\int_0^{\pi} p(z)dz + \frac{b_{i1}a_{j4}}{2}\int_0^{\pi} r(z)dz + \frac{b_{i3}a_{j2}}{2}\int_0^{\pi} r(z)dz$$

$$+ \frac{b_{i3}a_{j4}}{2}\int_0^{\pi} q(z)dz] + 0(|\lambda|^{-2}), \text{ for i, j = 1, 2} \quad . . . . (69)$$

We choose $A_{ij} = \pi / C_{ij}$, then making use of the expressions (68) – (69) and comparing the coefficients of λ^{-1} from both sides of (59) we obtain the expressions for S_{ij} and consequently the matrix S.

In particular, if the boundary conditions be Neumann (as given in (18)) we evaluate the value of S_{ij} by specializing a_{ji}, b_{ij}'s of the general boundary conditions (19) in different ways (explicitly given in article – 7 of Sengupta [25]).

For example, when

$a_{12} = a_{14} = a_{22} = 1, \quad a_{11} = a_{13} = a_{21} = a_{24} = 0$;

$b_{12} = b_{14} = b_{22} = 1, \quad b_{11} = b_{13} = b_{21} = b_{24} = 0$,

[case – I of article – 7 of Sengupta[25]),

the values of S_{ij}, i, j = 1, 2 are determined and we obtain

$$S_{11} = \frac{1}{4}\Big[p(\pi) + 2r(\pi) + q(\pi) + p(0) + 2r(0) + q(0)$$

$$+ \frac{1}{\pi}\int_0^{\pi}(p(z) + 2r(z) + q(z))dz \Big],$$

$$S_{12} = S_{21} = \frac{1}{4}\Big[p(\pi) + p(0) + r(\pi) + r(0) + \frac{1}{\pi}\int_0^{\pi}(p(z) + r(z))dz \Big],$$

$$S_{22} = \frac{1}{4}\Big[p(\pi) + p(0) + \frac{1}{\pi}\int_0^{\pi}p(z)dz \Big] \qquad \ldots . (70)$$

The values of S_{ij}, i, j = 1, 2 in the other three cases, given explicitly in article – 7 of Sengupta [25], are determined similarly.

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