POTENTIALLY RIGHT SIMPLE $\pi$-REGULAR SEMIGROUPS ARE RIGHT GROUPS
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Abstract. We prove that every $\pi$-regular semigroup such that all its elements are divided by one another on the left in potential sense, is a right group.

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1. Introduction

The simple universal algebras (the simple groups, rings, the simple and congruence-free semigroups) attracted the attention of algebraists always. The “simple” objects may be simple in usual sense. For example, the semigroups without non-trivial right (or left) congruences are exactly either semigroups of no more than two elements or cyclic groups of prime order. However some “simple” objects may have a complicated structure. L.A. Bokut proved that any semigroup may be isomorphically embedded into a congruence-free semigroup [2]. Earlier R.H. Bruck proved that every semigroup has an embedding into a simple semigroup with an identity element ([3], Theorem 8.3).

The notion “simplicity” may be generalized to “potential simplicity”. It is well-known that any right simple semigroup (a semigroup having only one right ideal) containing an idempotent, is a right group, i.e. it is isomorphic to the direct product of a group and a right zero semigroup (see [4], 1.27). It was proved in [6] that the finite potentially right simple semigroups are also right groups. The aim of this work is to generalize this result to $\pi$-regular semigroups.

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Let \( \mathcal{P} \) be a property (of elements, subsets, subsemigroups, congruences…) of a semigroup \( S \). We may consider \( \mathcal{P} \) as a condition. We say that the condition \( \mathcal{P} \) is fulfilled potentially if there exists an oversemigroup \( T \supseteq S \) such that \( \mathcal{P} \) is fulfilled in \( T \) (see [4], Chap. 10). In the papers [11, 12], the potential divisibility and the potential invertibility of elements of a semigroup were investigated. The potential divisibility of Boolean matrices was considered in [8]. The paper [7] contains some results on potential one-sided congruences of a semigroup and the potential solubility of systems of equations.

Now we provide some definitions and notations. As usual we put for a semigroup \( S \)

\[
S^1 = \begin{cases} S & \text{if } S \text{ has a unity}, \\ S \cup \{1\} & \text{if } S \text{ has no unity}. \end{cases}
\]

The symbol \( \mathbb{N} \) will denote the set of positive integers. Let \( X \) be a set. Denote by \( T(X) \) the semigroup of all transformations \( \alpha : X \to X \) with the multiplication \( x(\alpha \beta) = (x\alpha)\beta \). The kernel of the element \( \alpha \in T(S^1) \) is the set \( \ker \alpha = \{(x, y) \in X \times X \mid x\alpha = y\alpha\} \). It is well known that any semigroup \( S \) may be isomorphically embedded into the semigroup \( T(S^1) \), the embedding is so: \( a \mapsto \varphi_a \), where \( x\varphi_a = xa \) for \( a \in S \), \( x \in S^1 \). Let \( a, b \) be elements of a semigroup \( S \). We say that \( a \) is divided by \( b \) on the left if \( a \in bS^1 \) (see [9], 2.1.2), and \( a \) is divided potentially by \( b \) on the left if \( a \in bT^1 \) in some oversemigroup \( T \supseteq S \). Recall that \( \mathcal{R} = \{(a,b) \in S \times S \mid aS^1 = bS^1\} \) is the Green’s relation on the semigroup \( S \). Clearly, \((a,b) \in \mathcal{R}\) if and only if the elements \( a, b \) are divided by one another on the left. The generalized Green’s relation \( \mathcal{R}^* \) (see [5]) is a “potential analogue” of \( \mathcal{R} \), i.e. \((a,b) \in \mathcal{R}\) if and only if the elements \( a, b \) are divided potentially by one another on the left.

A semigroup \( S \) is called right simple if \( aS = S \) for any \( a \in S \). This is equivalent to the requirement that any elements \( a, b \in S \) are divided by one another on the left. A semigroup \( S \) is potentially right simple if any elements \( a, b \in S \) are divided potentially by one another on the left.

Note that if \( a \in bT^1 \) for \( T_1 \supseteq S \) and \( b \in aT^1 \) for \( T_2 \supseteq S \) then there exists an oversemigroup \( T_3 \supseteq S \) such that \( a \in bT_3^1 \) and \( b \in aT_3^1 \) (one may take \( T_3 = T(S^1) \)). Therefore \((a,b) \in \mathcal{R}^*\) if and only if \((a,b) \in \mathcal{R}\) in some oversemigroup \( T \supseteq S \).
Clearly, \( \mathcal{R} \subseteq \mathcal{R}^* \). This inclusion may be strong. For example, in the semigroup \( (\mathbb{N}, \cdot) \) we have \( \mathcal{R}^* = \mathbb{N} \times \mathbb{N} \) but \( \mathcal{R} = \{(a, a) \mid a \in \mathbb{N}\} \). The relation \( \mathcal{R}^* \) on arbitrary semigroup was characterized by E.G. Shutov in [11] as follows.

**Lemma 1** ([11], see also [9], 10.1.6, 10.1.7). The following conditions are equivalent for any elements \( a, b \) of a semigroup \( S \):

(i) \( (a, b) \in \mathcal{R}^* \);

(ii) \( \forall x, y \in S^1 \ (xa = ya \iff xb = yb) \).

### 2. Preliminaries

For the right simple semigroups we have the following assertion which may be proved easily.

**Lemma 2.** The following conditions are equivalent for any semigroup \( S \):

(i) \( S \) is right simple;

(ii) \( \mathcal{R} = S \times S \);

(iii) \( \forall a \in S \ aS = S \).

The next statement is an analogue of Lemma 2 for potentially right semigroups.

**Proposition 3.** The following conditions are equivalent for any semigroup \( S \):

(i) \( S \) is potentially right simple;

(ii) \( \mathcal{R}^* = S \times S \);

(iii) \( \forall a \in S \ \exists T \supseteq S \ aT \supseteq S \).

**Proof.** (i) \( \Rightarrow \) (ii). If \( a, b \in S \) then \( a \in bT_1^1 \) and \( b \in aT_2^1 \) for some oversemigroups \( T_1, T_2 \supseteq S \). We seen earlier that \( a \in bT_1^1 \), \( b \in aT_2^1 \) for \( T_3 \supseteq S \). This means that \( (a, b) \in \mathcal{R}^* \).

(ii) \( \Rightarrow \) (iii). Let \( a, b \in S \). Then \( (a, b) \in \mathcal{R}^* \). By Lemma 1, \( xa = ya \iff xb = yb \). This implies \( \ker a = \ker b \) (we consider \( a, b \) as elements of \( T(S^1) \)). Then \( b \in a \cdot T(S^1) \). Since \( b \) is arbitrary, we have \( S \subseteq a \cdot T(S^1) \).

(iii) \( \Rightarrow \) (i). Let \( a, b \in S \). In view of (iii) we have \( b \in aT \) for some \( T \supseteq S \). It means that \( b \) is divided potentially by \( a \). Therefore \( S \) is potentially right simple.
Lemma 1 shows that every right cancellative semigroup with the unity is potentially right simple. The assumption of the existence of the unity is essential as shows the example of a non-trivial left zero semigroup.

Recall a number of definitions in addition. An element $a$ of a semigroup $S$ is called regular if $aba = a$ for some $b \in S$. An element $a$ is a group element if it lies in a subgroup of the semigroup $S$. Let $\text{Reg} S$ be the set of all regular elements of the semigroup $S$, and $\text{Gr} S$ be the set of all group elements. A semigroup $S$ is called an epigroup (a quasiperiodic semigroup, in another terminology) if
\[
\forall a \in S \quad \exists n \in \mathbb{N} \quad a^n \in \text{Gr} S
\]
(see [10]). A semigroup $S$ is called $\pi$-regular if
\[
\forall a \in S \quad \exists n \in \mathbb{N} \quad a^n \in \text{Reg} S
\]
(see [1]). Obviously, the following containments of classes of semigroups take place:
\[
\text{finite} \subset \text{periodic} \subset \text{epigroups} \subset \pi\text{-regular}
\]
It is seen from the statement (iii) of Lemma 2 that the class of right simple semigroups is closed under taking homomorphic images. At the same time, the example of the Baer–Levi semigroup (see [4], 8.1) shows that this class is not closed under taking subsemigroups, since the Baer–Levi semigroup contains the infinite cyclic semigroup which is not right simple. It is converse for the class of all potentially right simple semigroups: this class is closed under taking of subsemigroups (it follows from the definition) but it is not closed under homomorphic images as shows the following example. Let $S = \{1, a, a^2, \ldots\}$ is an infinite cyclic monoid. As $S$ is cancellative semigroup with unity then $S$ is potentially right simple. However, for the ideal $I = \{a, a^2, \ldots\}$ the Rees quotient semigroup $S/I = \{1, 0\}$ is not potentially right simple as shows Proposition 3 (here $(1, 0) \notin \mathcal{R}^*$ by Lemma 1).

The following statement is known. We give the proof for the sake of completeness.

**Lemma 4.** $\mathcal{R}^* = \mathcal{R}$ in any regular semigroup.

**Proof.** Let $S$ be a regular semigroup and $a, b \in S$ are such that $(a, b) \in \mathcal{R}^*$. As $S$ is regular then $aca = a$ for some $c \in S$. We have $ac \cdot a = 1 \cdot a$. As $(a, b) \in \mathcal{R}^*$ then, by Lemma 1, $ac \cdot b = 1 \cdot b$. Thus $b \in aS$. It can be prove similarly that $a \in bS$. Therefore $(a, b) \in \mathcal{R}$.
Lemma 5. If a semigroup $S$ is potentially right simple then every its idempotent is a left unity.

Proof. Let $e^2 = e \in S$. Then $e \cdot e = 1 \cdot e$. As $\mathcal{R}^* = S \times S$ then, by Lemma 1, $e \cdot a = 1 \cdot a$ for all $a \in S$.

Lemma 6. Any potentially right simple $\pi$-regular semigroup is an epigroup.

Proof. Let $S$ be a semigroup which satisfies Lemma’s condition. Take $a \in S$. It follows from the $\pi$-regularity that $a^n \in \text{Reg} S$ for some $n \in \mathbb{N}$. This implies $a^n ba^n = a^n$. Clearly $e = ba^n$ is an idempotent. Moreover $a^n = a^n ba^n = a^n e \in eS$. By Lemma 5, $e$ is a left unity, therefore $Se = eSe$. Thus $a^n \in eSe$. Prove that $a^n$ is an invertible element of the semigroup $eSe$ relatively its unity $e$. Indeed, $ebe \cdot a^n = ba^n = e$, therefore $a^n$ is left invertible. As $a^n$, $ebe \in eSe$ and $ebe \cdot a^n = e$ (which is the unity of the semigroup $eSe$), then the element $a^n \cdot ebe$ is an idempotent. By Lemma 5, $a^n \cdot ebe$ is a left unity. Therefore $a^n \cdot ebe \cdot e = e$, hence $a^n$ is right invertible. Thus $a^n$ is invertible in $eSe$ which implies that $a^n$ belongs to the maximal subgroup $G_e$ with the unity $e$.

3. Main Results

Theorem 7. Any potentially right simple $\pi$-regular semigroup is a right group.

Proof. Let $S$ be a potentially right simple $\pi$-regular semigroup and $a \in S$. By Lemma 6, $S$ is an epigroup. Therefore $a^n \in G_e$ for some for some $n \in \mathbb{N}$ and a maximal subgroup $G_e$ with the unity $e$. We may think that $n \geq 2$. Put $g = a^n$ and denote by $g^{-1}$ the inverse element for $g$ in the group $G_e$. We have $a^n e = a^n$, $g^{-1} a^n = e$. By Lemma 5, $e$ is a left unity. Therefore $a = ea = g^{-1} a a = g^{-1} a \cdot a^n$. This implies $eae = e \cdot g^{-1} a \cdot a^n \cdot e = g^{-1} a^n \cdot a = ea = a$, i.e. $a \in eSe$.

Show that the element $a$ is invertible in the semigroup $eSe$. We have $g^{-1} a^{n-1} \cdot a = e$. As $g^{-1} a^{n-1}$, $a \in eSe$ and $e$ is the unity of $eSe$, then $a \cdot g^{-1} a^{n-1}$ is an idempotent. By Lemma 5, $ag^{-1} a^{n-1}$ is a left unity. Therefore $ag^{-1} a^{n-1} \cdot e = e$. This follows $ag^{-1} a^{n-1} = e$. Thus, $a$ has in $eSe$ a two-sided invertible element $g^{-1} a^{n-1}$. Hence $a \in G_e$. The elements of $G_e$ are regular,
therefore \(a \in \text{Reg} \, S\). As \(a\) is arbitrary element then \(S\) is a regular semigroup. Then, by Lemma 4, we obtain \(\mathcal{R} = \mathcal{R}^* = S \times S\). This implies that \(S\) is a right simple semigroup. As \(S\) is regular then \(S\) is a right group.

Conflict of Interests
The author declares that there is no conflict of interests.

REFERENCES