REGULAR PROPER *-SEMIGROUP EMBEDDINGS AND INVOLUTIONS

ADEL A. ABDELKARIM

Department of Mathematics, Faculty of Science, Jerash University, Jerash, Jordan

Abstract. It is proved that if \((S, \star)\) is a proper *-semigroup and if \(D\) is 0-characteristic integral domain then \((D[S], \star)\) is nil-semisimple provided that \(S\) is finite or \(i \in D\). Let \((S, \star)\) be a finite proper *-semigroup and \(F\) be a finite field of characteristic \(p\) such that \((F[S], \star)\) is a proper *-ring. Then \(F[S]\) is a direct product of fields and \(2 \times 2\) matrix rings over fields. Furthermore, \(p \neq 2, p \neq 1 \mod 4\).

Keywords: proper *; maximal proper *; symmetric, alpha inner.

2010 AMS Subject Classification: 20M17, 20M19.

1. Introduction

A semigroup with involution \((S, \star)\) is called a *-semigroup. It is called a \(p^\ast\)-semigroup if the involution \(\ast\) is proper. Thus \(\forall a, b \in S, aa^\ast = ab^\ast = bb^\ast \Rightarrow a = b\). A ring with involution \((R, \star)\) is a *-ring. It is called a \(p^\ast\)-ring if the involution \(\ast\) is proper. Thus \(aa^\ast = 0 \Rightarrow a = 0\) for all \(a \in R\). Let \((S, \star), (T, \star)\) be two *-semigroups. An injective mapping \(f : (S, \star) \rightarrow (R, \star)\) from a *-semigroup \((S, \star)\) into a *-ring \((R, \star)\) such that for all \(a, b \in (S, \star)\), \(f(ab) = f(a)f(b), f(a^\ast) = (f(a))^\ast\) is called a *-embedding. Let \((S, \star)\) be a *-semigroup and consider the semigroup ring \(Z[S]\) of \(S\) over \(Z\). If \((S, \star)\) is a \(p^\ast\)-semigroup then \((Z[S], \star)\) need not be a \(p^\ast\)-ring as in ( [6]). Let \((S, \star)\) be a *-semigroup. The involution \(\ast\) is called a maximal proper involution if for every distinct elements \(s_1, \ldots, s_n \in S\), there exists an element \(s_i\) such that \(s_is_i^\ast \neq s_js_j^\ast, j \neq i\), and

Received June 4, 2014
A.A. ABDELKARIM

$s_is_i^* = s_is_i^* = s_is_i^*; k, l = 1, ..., n$. Such a *-semigroup is called an mp-semigroup. For example any inverse semigroup is an mp-semigroup under its inverse involution as in ([6]). If $(S, \ast)$ is an mp-semigroup then $(Z[S], \ast)$ is a p*-ring and $(S, \ast)$ is *-embeddable in $(Z[S], \ast)$, ([6]). Let $(R, \ast)$ be a *-ring and let $n$ be a fixed positive integer. If for every distinct elements $r_1, ..., r_n \in R$ it holds that $\sum r_ir_i^* = 0$ implies that $r_i = 0, i = 1, ..., n$ then we say that $(R, \ast)$ is $n$-formally complex. Let $F$ be a field, let $\alpha$ be an automorphism of order 1 or 2 and let $D \in M_n(F)$ be a diagonal matrix. Then $F$ is $D(\alpha) - formally complex$ if and only if $\sum d_i\alpha(a_i) = 0$ implies all $a_i = 0$. If $D$ is the identity matrix we say that $F$ is $n-$formally complex and if this true for all $n$ we say that $F$ is formally-complex. On the other hand, if $\alpha$ is the identity then we say that $F$ is $D(\alpha) - real$ and if $D$ is the identity we say that $F$ is $n-$formally real and if this is the case for all $n$ we say that $F$ is formally real. If $(S, \ast)$ is an mp-semigroup and $(R, \ast)$ is formally complex *-ring then $(R[S], \ast)$ is a p*-ring and $(S, \ast)$ is *-embeddable in $(R[S], \ast)$, as in [6]) where it is shown there is a finite p*-semigroup that cannot be *-embedded in any p*-ring. Let $(R, \ast)$ be a *-ring. An ideal $I$ in $R$ is called a *-ideal if $I^\ast = I$. In this case the ring $R/I$ is a *-ring under the involution $(r + I)^\ast = r^* + I$.

Let $F$ be a field and let $\alpha$ be an automorphism on $F$ of order 1 or 2. Let $R = M_n(R)$ and let $A \in R$. If we apply to every entry in $A$ the automorphism $\alpha$ we get $A^\alpha$. An involution $\ast$ on $R$ is called $\alpha-$inner if there is an invertible matrix $P$ such that for all $A$ in $R$ we have $A^\ast = P^{-1}A^{\alpha t}P$ and if $\alpha$ is the identity mapping then $\ast$ is called inner.

Let $F$ be a field and let $\alpha$ be an automorphism on $F$ and let two matrices $A, B \in M_n(F)$. We say that the matrices $A, B$ are $\alpha-$congruent if there is a matrix $C$ such that $A = CBC^{\alpha t}$. Also we say that a matrix $A \in M_n(F)$ is $\alpha-$symmetric if $A = A^{\alpha t}$ and it is called $\alpha-$antisymmetric if $A^{\alpha t} = -A$. Here $A^\alpha$ is got from the matrix $A$ by applying $\alpha$ to its entries. It is known that if $A$ is a symmetric matrix in $M_n(F), F$ is a field then it is congruent to a diagonal matrix and if $A$ is anti-symmetric invertible matrix then $A$ is congruent to a direct sum of 2 by 2 matrices each of which is of the form $\alpha \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \alpha \in F$. See [3] pp. 365-372.

Let $(S, \ast)$ be a proper *-semigroup of order 5 or less. It was noticed (through a computer program) that once the involution $\ast$ in the *-semigroup ring $(Z[S], \ast)$ is not proper then the
p*-semigroup \((S, *)\) is not *-embeddable in any ring p*-ring. Up to now there is no proof or disproof for this claim.

In the first part of this note we find a necessary and sufficient condition for a certain class of involutions on \(R = M_n(F)\), \(F\) is a field, to be proper involutions. In the second part we give a plan to decide if a given proper *-semigroup is *-embeddable in a p*-ring and if so we seek to find a p*-algebra of matrices that *-embeds \((S, *)\) and we look for all involutions *' on \(S\) that makes \((S, *')\) *-isomorphic with \((S, *)\). Incase \((S, *)\) is not *-embeddable in a p*-ring we locate the *-subsemigroup \((T, *)\) such that \((S/T, *)\) is *-embeddable in a p*-ring.

2. Preliminaries

We cite the following known facts.

**Theorem 1.** (A) Let \((S, *)\) be an mp semigroup and let \((R, *)\) be a formally complex ring. Then \((R[S], *)\) is a proper *-ring and hence it has a zero nil radical, ([6]).

We cite the following version of Wedderburn Theorem from [2] p. 435

**Theorem 2.** (B) If \(R\) is a non zero left Artinean nil-semisimple ring then it isomorphic with a finite direct sum of finite matrix rings over a division ring.

We Also cite the following from [5], p.63.

**Theorem 3.** (B): If \(A\) is a left Noetherian ring, then every nil ideal is nilpotent.

We also cite the following version of Skolem-Noether theorem; see[2], p.460.

**Theorem 4.** (C): Let \(R\) be a simple left-Artinian ring and let \(K\) be the center of \(R\) ( so that \(R\) is a \(K\)-algebra). Let \(A\) and \(B\) be finite dimensional simple \(K\)-algebras of \(R\) that contain \(K\). If \(\alpha : A \to B\) is a \(K\)-algebra isomorphism that leaves \(K\) fixed elementwise, then \(\alpha\) extends to an inner automorphism of \(R\).

We cite the following theorem from [1], p136.

**Theorem 5.** (D): Let \((R, *)\) be a semi-simple *-ring with involution * such that \(\forall x \in R, \exists n(x), (x + x^*)^n(x) = x + x^*\). Then \(R\) is a subdirect product of fields and 2 \(\times\) 2 matrix rings over fields.
Proposition 6. Let $F$ be a field and let $P \in M_n(F)$ be a symmetric matrix then there is a diagonal matrix $D$ congruent to $P$; i.e.,

$$\exists C \in M_n(F), CPC^t = D,$$ see [4], for example. If $P$ is antisymmetric then $P$ is congruent to a direct sum of matrices of the form $\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and 0-matrices where $\alpha \in F$.

As a generalization we state a similar proposition whose proof is similar to that of proposition [6] and its proof is omitted.

Proposition 7. Let $F$ be a field and let $\alpha$ be an automorphism of order 2 on $F$. Let $P \in M_n(F)$ be an invertible matrix such that $P^{\alpha t} = P$. Then there is a matrix $C$ and a diagonal matrix $D$ such that $CPC^{\alpha t} = D$.

3. Main results

Given a semigroup $S$ we can ask how to find all proper involutions on $S$. For example if $S$ is an inverse semigroup then the inverse operator is one of the proper involutions on $S$. Similarly given a ring $R$ there is a problem of finding all proper involutions on $R$. For example if we take a field $F$ and its corresponding matrix ring $R = M_n(F)$ the problem is to find all proper involutions on $R$. The transpose operator is an involution which need not be proper unless $F$ is $n$-real. For example the transpose involution is not proper on $R = M_2(Z_2)$.

Let $F$ be a field and let $R = M_n(F)$ be the matrix ring over $F$ and let $Z(R) = \{cI : c \in F\}$ be the center of $R$. Let $\ast$ be an involution on $R$. Let $A \in Z(R)$. Then for all $X \in R, AX = XA$ implies that $A^tX^t = X^tA^*$ and so $A^* \in Z$. Thus for all $c \in F, (cI)^* = c^*I$ and so $\ast$ induces an automorphism (called the corresponding automorphism) of order at most 2 on $F$. Conversely we will show that any automorphism $\alpha$ of order at most 2 on $F$ induces an involution $\ast$ on $R = M_n(F)$ given by $A^* = P^{-1}A^{\alpha t}P$ for all $A \in R$ as shown in the following proposition.

Proposition 8. (1) Let $\ast$ be an involution on $R = M_n(F)$ whose corresponding automorphism is the identity on $F$. Then there is an invertible matrix $P$ such that $A^* = P^{-1}A^{\alpha t}P$ for every matrix $A$ in $M_n(F)$. 
(2) Let \( * \) be an involution on \( M_n(F) \) whose corresponding automorphism \( \alpha \) on \( F \) has order 2. Then there is an invertible matrix \( P \) such that \( A^* = P^{-1}A^{\alpha}P \) for every matrix \( A \in M_n(F) \).

**Proof.** (1) The operator \( h : A \mapsto A^{*t} \) is an automorphism that fixes the center of \( M_n(F) \) elementwise. From Noether-Skolem Theorem it follows that there is an invertible matrix \( P \) such that for all \( A \in R, h(A) = A^{*t} = PAP^{-1} \). Thus \( A^* = Q^{-1}A^tQ, Q = P^t \) for every \( A \in M_n(F) \).

(2) The operator \( k : A \mapsto A^{*\alpha} \) is an automorphism on \( M_n(F) \) that fixes the center \( Z(R) = \{ cI : c \in F \} \) elementwise. From Noether-Skolem Theorem there is an invertible matrix \( P \) such that for every matrix \( A \) we have \( k(A) = A^{*\alpha} = P^{-1}AP \). Thus for every matrix \( A \in R \) we have \( A^* = P^\alpha A^{\alpha}P^{-1} = Q^{-1}A^{\alpha}Q, Q = P^{-1}A^{\alpha} \).

**Corollary 9.** Let \( * \) be an involution on \( R = M_n(F) \) whose corresponding automorphism \( \alpha \) is of order 1 or 2 on \( F \). Then there is an invertible matrix \( P \) such that \( A^* = P^{-1}A^{\alpha}P \) for every matrix \( A \) in \( M_n(F) \).

We can generalize the preceding propositions to division rings. The proof of the following proposition is similar to the proof of proposition 8 and it is omitted.

**Proposition 10.** Let \( R = M_n(D) \) be a matrix ring on a division ring \( D \). Let \( * \) be an involution on \( R \). Let \( Z(R) \) be the center of \( D \). Then there is an automorphism \( \alpha \) on the ring \( Z(R) \) of order 1 or 2 and there is an invertible matrix \( P \) such that for all \( A \in R \), \( A^* = P^{-1}A^{\alpha}P \).

We prove the following.

**Proposition 11.** Let \( \alpha \) be an automorphism of order 1 or 2 on the field \( F \). Let \( P \in R \) be an invertible matrix on \( F \). Define \( * \) on \( R \) as \( A^* = P^{-1}A^{\alpha}P \) for all \( A \in R \). Then \( * \) is an involution if and only if \( P^\alpha = cI, c = \pm 1, c^n = 1 \).

**Proof.** We have for all \( A, B \in R, (A + B)^* = A^* + B^*, (AB)^* = B^*A^* \). To make \( * \) as an involution we need \( A^{**} = A \) to hold on \( R \). Thus \( P^{-1}P^\alpha AP^{-1}A^{\alpha}P = A \) for all \( A \in R \). Thus \( P^{-1}P^\alpha = cI \) or \( P^\alpha = cP \) for some nonzero scalar \( c \). Also we notice that \( P^{**} = P \) and from \( P^* = P^{-1}P^\alpha P = P^{-1}cPP = cP \) we get \( P = P^{**} = (cP)^* = c^2P \) and so \( c^2 = 1 \) and \( c = \pm 1 \). From \( P^t = cP \) and upon taking determinants we get we get \( c^n = 1 \). If \( n \) is odd we must have \( c = 1 \) and if \( n \) is even we still have \( c = \pm 1 \).
Remark 1. If one of the diagonal elements of $P$ in proposition (11) is nonzero then $c = 1$ and $P^t = P$. Otherwise and if all diagonal elements are 0 we have only the condition $c = \pm 1$ and $n$ is even.

Next we discuss conditions on $P$ that guarantees that the involution $*$ is proper

Proposition 12. Let $F$ be a field and let $R = M_n(F)$.

(1) Let $*$ be an involution on $R$ defined by $A^* = P^{-1}A^tP$ for all $A \in R$. Let $P = P$. If $P^{-1} = QQ^t$ for some matrix $Q$ and if $F$ is formally real then $*$ is a proper involution.

(2) Let $*$ be an involution on $R$ defined by $A^* = P^{-1}A^\alpha tP$ for all $A \in R$ with $P^t = P$ and let the corresponding automorphism $\alpha$ on $F$ be of order 2. If $P^{-1} = QQ^\alpha$ for some matrix $Q$ and if $F$ is formally $\alpha$–complex then $*$ is a proper involution.

Proof. (1) For $*$ to be proper we need the condition $AA^* = 0$ to hold if and only if $A = 0$ for all $A \in R$. This is equivalent to require that $AP^{-1}A^tP = 0$ implies that $A = 0$. Or $AP^{-1}A^t = 0$ implies that $A = 0$. Or, $AQ^t = 0$ implies that $A = 0$. If $F$ is formally real this is equivalent to $AQ = 0$ implies that $A = 0$ which is the case since $Q$ is invertible.

(2) For $*$ to be proper we need the condition $AA^* = 0$ to hold if and only if $A = 0$ for all $A \in R$. This is equivalent to $AP^{-1}A^\alpha tP = 0$ if and only if $A = 0$. Or $AP^{-1}A^\alpha t = 0$ if and only if $A = 0$. But $P^{-1} = QQ^\alpha$ and so $AP^{-1}A^\alpha t = AQQ^\alpha A^\alpha t = 0$ implies that $AQ$ and hence $A = 0$ since $F$ is $\alpha$–formally complex. ■

Proposition 13. Let $R = M_n(F), F$ being a field. Let $*$ be an involution on $R$ with a corresponding automorphism $\alpha$ and a corresponding matrix $P, P^\alpha = P$. Let $D$ be the corresponding diagonal matrix that is congruent to $P$ as was mentioned in proposition 7. If $\alpha$ is the identity mapping then $*$ is proper if and only if $F$ is $D$–real. If $\alpha$ is of order 2 then $*$ is proper if and only if $F$ is $D$–complex.

Proof. We need to show, for $*$ to be proper, that $AP^{-1}A^\alpha t = 0$ if and only if $A = 0$. Since $P^{-1} = CDC^\alpha$, we see that we need $ACDC^\alpha A^\alpha t = 0$ if and only if $A = 0$ if and only if $AC = 0$ if and only if $A = 0$. It is clear that we need $F$ to be $D(\alpha)–$complex. ■
Proposition 14. Let $F$ be a $p$-characteristic field and let $*$ be a proper involution on $R = M_n(F)$ such that its corresponding automorphism is the identity. Let $P$ be the corresponding matrix for the involution $*$ as in the proof of proposition (11) and let $D$ be a diagonal matrix congruent to $P$ with diagonal entries set $D = \{d_1, ..., d_n\}$. Then $p \neq 2, P^* = P = P^t$, and $F$ is $D$-real. Conversely if $F$ is $D$-real then the involution is proper.

Proof. We have seen in the proof of proposition (11) that $P^t = \pm P$. Assume, to get a contradiction, that $P^* = -P$. Let $Q = P^t$. Define $f : F^n \times F^n \to F^n$ by $f(u, v) = u^t Q v$. Then $f$ is a bilinear form on $F^n$. In fact, $f$ is alternating because $f(u, v) = (f(u, v))^t \Rightarrow u^t Q v = v^t Q^t u = -v^t Q u = -f(v, u), \forall u, v \in F^n$. Thus $\forall v \neq 0, f(v, v) = 0$. Let us pick one such $v$ and let us form the matrix $A$ whose first row is $v^t$ and whose all other rows are zero rows. Straightforward calculations show that $A^t QA = 0$. Thus $A^t PA = 0$. Thus $A \neq 0, A^* A = P^{-1} A^t PA = 0$, a contradiction with properness of $*$ on $R$. It follows that $p \neq 2$, otherwise $P = -P$ and we saw that this contradicts properness of $*$. To complete the proof let $C$ be an invertible matrix such that $CP^{-1} C^t = D$, a diagonal invertible matrix. Now $\forall A \in R, \exists B \in R, A = BC, AA^* = 0 \Leftrightarrow BC(P^{-1} C^t B^t P) = BDB^t P = 0 \Leftrightarrow BDB^t = 0$. Thus $* \text{ is proper if and only if the only solution in } B \in M_n(F) \text{ for the equation } BDB^t = 0 \text{ is } B = 0$. If we take for $B$ a matrix which is everywhere 0 except possibly on its first row $\{x_1, ..., x_n\}$ we see that the condition implies the equation $\sum d_i x_i^2 = 0$ has only the trivial solution. Thus $F$ is $D$-real.

Let $*$ be an involution on $R = M_n(F), n \text{ is even, with a corresponding matrix } P \text{ with } P^* = -P$. We give an example that $*$ is not proper.

Example 1. Let $F$ be any field and let $R = M_2(F)$ and we take the invertible anti-symmetric matrix matrix $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $\alpha$ be an automorphism on $F$ of degree 1 or 2. We define an involution $*$ on $R$ defined by $A^* = P^{-1} A\alpha^t P$ for all $A \in R$. This involution is not proper if we take $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then a simple calculation reveals that $AA^* = 0$-matrix although $A$ is not zero.

Proposition 15. Let $F$ be a field and let $* \text{ be a proper involution on } M_n(F) \text{ with a corresponding matrix } P. \text{ Then } P^t = P \text{ and } \text{char}(F) \neq 2.$
Proof. If \( P' = -P \) then from the fact in the introduction and from the preceding example * is not proper. If the characteristic of the field is 2 then \( P' = -P \) and again the involution is not proper. \( \square \)

**Proposition 16.** Let \((S, \ast)\) be a finite proper \(*\)-semigroup and \( F \) be a finite field of characteristic \( p \neq 0 \) such that \((R, \ast) = (F[S], \ast)\) is a proper \(*\)-ring. Then \( R \) is a direct product of fields and \( 2 \times 2 \) matrix rings over fields. Furthermore, \( p \neq 2, p \neq 1 \mod 4. \) The converse is also true.

Proof. \( x \in R, y = x + x^* \). Then not all positive powers of \( y \) are distinct owing to the finiteness of \( R \). Let \( m > 1 \) be a positive power of \( y \) such that \( \exists n > m, y^n = y^n \) such that \( m = 2k, n = 2l \). Then, since \( y = y^*, y^n = (yy^*)^k = y^n = (yy^*)^l \). Using \(*\)-cancellation, we get \( y^k = y, k > 1 \). Thus \( \forall x \in R, \exists n(x), (x + x^*)^{n(x)} = x + x^* \) and Theorem D applies. The last part follows from the fact that any involution on \( M_2(Z_p) \) is transpose-inner and the transpose involution is proper if and only if \( p \neq 2, p \neq 1 \mod 4. \) \( \square \)

**Proposition 17.** Let \((R, \ast) = (M_m(Z_n), \ast)\) be a proper \(*\)-ring. Then \( m = 2, n = p_1\ldots p_k, p_i \neq p_j(i \neq j), p_i \neq 2, p_i \neq 1 \mod 4, \forall i = 1, \ldots, k. \)

Proof. That \( m = 2 \) follows from Theorem D. That \( p_i \neq p_j(i \neq j) \) follows from \(*\) being proper: \( p_1 = p_2 \Rightarrow \frac{n}{p_1}(\frac{n}{p_1})^* = 0 \neq \frac{n}{p_1} \). The proof of the other parts is similar to the proof in proposition 16. \( \square \)

**Proposition 18.** Let \((R, \ast) = (M_2(Z_p), \ast)\) be a proper \(*\)-ring. Then \(*\) is inner.

Proof. Let \( C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Then \( C, D \) generate the ring \( R \). This is easily seen. Let \( C^* = A, D^* = B \). We are looking for a matrix \( u = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \) such that \( C^* = A = u^{-1}Cu = u^{-1}C^*u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), \( D^* = B = u^{-1}Du = u^{-1}D^*u = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \). Thus \( uA = Cu, uB = Du \Rightarrow uA = CD^{-1}uB = GuB \Rightarrow \begin{pmatrix} z & t \\ -x & -y \end{pmatrix}. \) \( A = B. \) The last matrix equation gives rise to solutions in \( x, y, z \) and \( t \) since \( A \) and \( B \) are invertible. Furthermore the resulting
matrix \( \begin{pmatrix} z & t \\ -x & -y \end{pmatrix} \), which has the same determinant as that of \( u \), is invertible since \( A \) and \( B \) are. Thus \( u \) is invertible. Thus * is inner at least for the matrices \( C \) and \( D \). But \( C \) and \( D \) generate the whole matrix ring and, for example, \((CD)^* = D^*C^* = u^{-1}D'u u^{-1}C'u = u^{-1}(CD)^*u \). Thus * is inner in general.

3.1. *-Semigroup Embedding in a Proper *-Ring. We start this subsection with the following remarks:

Although the following remarks are almost routine we present them here for the sake of completeness.

**Remark 2.** Let \((R,\ast)\) be an \( m \)-characteristic proper *-ring without \( 1 \). Then either \( m = 0 \) or \( m \) is square-free. Also \((R,\ast)\) can be *-embedded in an \( m \)-characteristic proper *-ring \((R_1,\ast)\) with \( 1 \).

**Illustration 1.** Let \( r \) be a nonzero element of \( R \) such that there is a smallest positive integer \( m \) with \( mr = 0 \) and \( m = kp^2 \), \( k \) is not a unit and \( p \) is a prime. then \( kp \) is not zero. But then \((kpr)(kpr)^* = 0 \). From properness of * it follows that \( kpr = 0 \) which is a contradiction with \( kpr \) not zero. To prove the other part we have two cases to consider.

**Illustration 2.**

**Case 1.** : \( m = 0 \). In this case we take the Cartesian product \( Z \otimes R \) and define addition and multiplication as follows. \((m,r) + (m',r') = (m + m', r + r'), (m,r)(m',r') = (mm', mr' + m'r + rr')\) for every \( m, m' \in Z, r, r' \in R \). This makes of \( Z \otimes R \) a ring \( R_1 \). We define an operator * on \( R' \) by \((m,r)^* = (m, r^*)\). Then it is straightforward to see that * is an involution. In fact, it is proper. For, \((m,r)(m,r)^* = (0,0) = (m^2, mr + mr^* + rr^*) \Rightarrow m = 0, rr^* = 0 \Rightarrow r = 0, (m, r) = (0,0)\).

**Illustration 3.**

**Case 2.** : \( m \neq 0 \). In this case \( m \) is square-free. For if \( m = p^2k \), \( p \) is prime, then there exists \( 0 \neq r \in R, mr = 0, nr \neq 0 \) for all positive integers \( n < m \). But then \( 0 \neq pkr, (pkr)(pkr)^* = 0 \), a contradiction with the properness of the involution * . Now we form \( Z_m \otimes R \). We define addition and multiplication as in Case 1. It is straightforward to see that these operations are well-defined making of \( Z_m \otimes R \) a ring denoted by \( R_2 \). We define * on \( R_1 \) as in Case 1. Then * is an
involution and it is proper. For, 
\[(0,0) = (k, r)(k, r)^* = (k^2, kr^* + kr + rr^*) \Rightarrow k^2 = 0 \Rightarrow k = 0\]
for all \(k \in \mathbb{Z}_m, r \in R\). The last implication follows since \(m\) is square-free forcing \(\mathbb{Z}_m\) to have a 0-radical. It follows that \(rr^* = 0\) and so \(r = 0, (m, r) = (0, 0)\).

**Remark 3.** Let \((R, *)\) be an 0-characteristic proper *-ring. Then \((R, *)\) can be *-embedded in a 0-characteristic proper *-algebra \((R_1, *)\) over \(Q\).

**Illustration 4.** We may assume that \(R\) contains 1. Then \(R\) contains a copy of \(\mathbb{Z}\). Now we localize \(R\) at the multiplicatively closed set \(\mathbb{Z}\setminus\{0\}\). (See [2] for definition of localization). The resulting *-ring denoted by \((R_1, *)\) contains a copy of \(Q\) and it is a proper *-ring. For if \([[(r, m)]][(r, m)]^* = [(0, 1)]\) then \(rr^* = 0\) and so \(r = 0, [(r, m)] = [(0, 0)]\).

Now we prove the following.

**Proposition 19.** Let \((R, *)\) be a *-ring. Let \(I_1\) be the ideal generated by all \(A\) in \((R, *)\) such that \(AA^*\) or \(A^*A\) is 0 and, for \(k > 1\), let \(I_k\) be the ideal generated by all \(A \in (R, *)\) such that \(AA^*\) or \(A^*A\) is in \(I_{k-1}\). Then \(I_k\) is a *-ideal, \(I_k \subseteq I_{k+1}\), and if \(I\) is the union of all \(I_k\), \(k > 0\), then \(I\) is a *-ideal and \((R/I, *)\) is a p*-ring.

**Proof.** That \(I_k\) is a *-ideal and that \(I_k \subseteq I_{k+1}\) are trivial to verify. Also \(I\) is a *-ideal. If \(AA^*\) is in \(I\) then it is in some \(I_k\) and so \(A\) is in \(I_{k+1}\) and hence \(A\) is in \(I\). Thus \((R/I)\) is a p*-ring. \(\blacksquare\)

**Corollary 20.** Let \((S, *)\) be a *-semigroup, not necessarily a p*-semigroup, and let \((Z[S], *)\) be the corresponding *-semigroup ring of \((S, *)\) over \(Z\). Let \(I_{k}, k > 0\), and \(I\) be the ideals as in the preceding proposition. Then \((Z[S]/I, *)\) is a p*-ring. If \((S, *)\) is a finite p*-semigroup then it is *-embeddable in a p*-ring if and only if there are no distinct elements \(s, t\) in \(S\) such that \(s - t\) in any \(I_k\). In this case if \(S\) is commutative then \((S, *)\) is *-embeddable in a subdirect product of fields. Also in this case if \(Z[S]/I\) is finite then \((S, *)\) is *-embeddable in a finite direct product of matrix rings each over a finite field.

**Proof.** The proof is a direct consequence of the proposition (19), remarks 3 and 2 and Wedderburn’s Theorem since \((S, *)\) in case of \(S\) is finite and hence the corresponding algebra is Artinean. For then \((S, *)\) is a finite p*-semigroup such that \((R, *) = (Z[S]/I, *)\) is infinite and
there are no distinct elements $s,t$ in $S$ such that $s - t$ is in any $I_k$. Then $(Q[S]/I_*,\ast)$ is isomorphic to a finite direct product of matrices over division ring and hence $(S,\ast)$ is represented as a $p\ast$-semigroup of matrices over a division ring.

**Proposition 21.** Let $(S,\ast)$ be an mp semigroup and let $(D,\ast)$ be a 0-characteristic integral domain with proper involution $\ast$. If $S$ is finite, or if $i \in D$ then $D[S]$ is nil-semisimple while $(D[S],\ast)$ need not be a proper $\ast$-ring and the extended involution need not be a proper ring involution.

**Proof.** We can assume that $D$ is contained in the complex number field $C$. Assume first that $i \in D$. Then $D$ is closed under complex conjugation which is a proper involution. Since $(S,\ast)$ is an mp-semigroup it follows from Theorem A that $(D[S],\ast)$ is proper $\ast$ and nil-semisimple. Now assume that $i \notin D$ and assume that $S$ is finite. Let $J$ be a nil ideal in $D[S]$. Since $S$ is finite and the $D$-module $D[S]$ is isomorphic to the direct sum of $|S|$ copies of the Noetherian left $D$-modules (each is isomorphic to $D$), then $D[S]$ is a Noetherian left $D$-module. Hence it is also a Noetherian left $D[S]$-module and thus it is a left-Noetherian ring. By theorem B, $J$ is nilpotent and there is a positive integer $n$ such that $J^n = 0$. Then $I = J + iJ$ is a nilpotent ideal in $D[i][S]$ which is nil-semisimple. Thus $I$ is 0 and hence $J$ is a 0 ideal.

**Proposition 22.** Let $(S,\ast)$ be a finite mp-semigroup and let $F$ be a 0-characteristic field. Then $F[S]$ is a finite direct product of matrices over a skew field and $(S,\ast)$ is $\ast$-embeddable in the $\ast$-ring $(F[S],\ast)$ where $\ast$ is the natural involution inherited from the involution $\ast$ in $(S,\ast)$. If the field $F$ has a non zero characteristic then $F[S]$ is a finite direct product of matrices over a field.

**Proof.** We can assume without loss of generality that $S$ has an identity element 1 (This easy to prove). Since $F[S]$ is a nil-semisimple ring by proposition 21 and since it is a finite dimensional $F$-vector space, it follows that it is a finite direct product of matrix rings over a skew field. Let $(S,\ast)$ be a finite mp-semigroup and let $F$ be a field of 0-characteristic. Then the involution on $S$ gets extended to an involution on $F[S]$ in a natural way: $(\sum a_i s_i)^\ast = \sum a_i s_i^\ast$. (But there is no guarantee that this involution is proper on $R[S]$, unless $R$ is formally complex). If $ch(F) \neq 0$ the prime field is $Z_p$ and the subring generated by $Z_p$ and $S$ is finite and has a proper involution and so it is a finite direct sum of matrix rings over a finite skew field (a field then).
**Proposition 23.** Let \((R,*)\) be a finite proper \(*\)-ring. Then \((R,*)\) is \(*\)-isomorphic with a finite direct product of matrix rings over a field.

**Proof.** We show that \(R\) has a 0-radical \(I\). For let \(A\) be in \(I\). Then \(AA^*\) is in \(I\). But then there is a natural number \(n\) such that \((AA^*)^n = 0\). By properness of \(*\) it follows that \(AA^* = 0\) and hence \(A = 0\). Thus \(I\) is the zero ideal. From Wedderburn Theorem it follows that \(R\) is isomorphic with a finite direct product of matrix rings over a skew field. Since \(R\) is finite the skew fields are fields. \(\blacksquare\)

**Proposition 24.** Let \((S,*)\) be a proper \(*\)-semigroup \(*\)-embeddable in a proper \(*\)-ring \((R,*)\). Then

1. There is a \(*\)-ideal \(I\) in \((Z[S],*)\) such that \((Z[S]/I,*)\) is a p*-ring which \(*\)-embeds \((S,*)\).
2. If \(ch(R) = 0\) and \(S\) is finite then \((S,*)\) is \(*\)-embeddable in a finite direct sum of matrix rings over a division ring with proper involution.
3. If \(ch(R) = m \neq 0\) and \(S\) is finite then \((S,*)\) is \(*\)-embeddable in a finite direct sum of matrix rings over a finite prime-characteristic field with proper involution.

**Proof.** (1) There is a natural \(*\)-mapping \(f : (Z[S],*) \to (R,*)\) given by \(f(\sum m_i s_i) = \sum m_i g(s_i)\), where \(g\) is the \(*\)-embedding of \((S,*)\) into \((R,*)\). If \((Z[S],*)\) is p* then we can take \(I = 0\). If there is \(A\) not 0 in \(Z[S]\) such that \(AA^*\) or \(A^*A = 0\) then we take the ideal \(I_1\) generated by all such \(A\) and we consider the \(*\)-ring \(Z[S]/I_1\). We notice that there can be no two different elements \(s, t\) in \(S\) such that \(s - t\) is in \(I_1\) lest \(s - t = 0\) in \(R\) which would imply non \(*\)-embeddability of \((S,*)\) in \((R,*)\). If this \(*\)-ring is p* then we are done with getting the required p*-ring \(Z[S]/I\). Otherwise there is \(A\) not in \(I_1\) such that \(AA^*\) is in \(I_1\). We take all such \(A\) and all \(B\) such that \(B^*B\) is in \(I_1\) and form the ideal \(I_2\). These are 0 in \(R\) of course. Now we form the \(*\)-ring \(R/I_2\). There can be no two different elements \(s, t\) in \(S\) such that \(s - t\) is in \(I_2\) lest that would contradict \(*\)-embeddability of \((S,*)\) into \((R,*)\). If this \(*\)-ring is p* then we are finished by getting a \(p^*\)-ring \(R/I_2\) which \(*\)-embeds \((S,*)\). We continue this way. The union of these \(*\)-ideals is clearly a \(*\)-ideal \(I\) and \((R/I,*)\) is a \(p^*\)-ring which \(*\)-embeds \((S,*)\).

(2) If \(ch(R) = 0\) and \(S\) is finite we can assume that \(R\) contains a copy of \(Q\). Let \(R' = \langle Q, S \rangle\) be the set of all rational linear combinations of elements of \(S\) in \(R\). Then \(R'\) is a proper \(*\)-ring which
*-embds \((S, \ast)\). Being a homomorphic image of the Artinian ring \(Q[S]\), \(R'\) is Artinian. Since a proper *-ring has 0 nil-radical, by Wedderburn’s Theorem \(R'\) is isomorphic to a finite direct sum \(R_2\) of matrix rings over a skew field. We define an involution * on \(R'\) as follows. Let \(f\) be the isomorphism of \(R'\) onto \(R_2\). Take \(b\) in \(R'\). Then \(b = f(a)\) for a unique element \(a \in R'\). Define \(b^* = f(a^*)\). We show that * is a proper involution. Let \(b, c \in R_2\) and let \(b = f(a_1), c = f(a_2)\). Then \((b + c)^* = (f(a_1) + f(a_2))^* = (f(a_1 + a_2))^* = f(a_1^* + a_2^*) = f(a_1^*) + f(a_2^*) = (f(a_1))^* + (f(a_2))^* = b^* + c^*, (bc)^* = (f(a_1a_2))^* = f(a_2^*)f(a_1^*)
\(= (f(a_2))^*(f(a_1))^* = c^*b^*, b^* = (f(a_1^*))^* = f(a_1^{**}) = f(a_1) = b.\) And if \(bb^* = 0\) then \(f(a_1)(f(a_1^*) = f(a_1a_1^*) = 0\) and so \(a_1a_1^* = 0\) which implies that \(a_1\) and hence \(b = 0\).

(3) If \(ch(R) = m \neq 0\) and \(S\) is finite we can argue similarly that there is a copy of \(Z_m\) in \(R\) and \(R'' = \langle Z_m, S\rangle\) is proper *. Since \(R''\) is finite it is isomorphic to a finite direct sum of matrix rings over a prime characteristic finite field. This is because a finite skew field is a field. The same argument as above applies to show that the involution inherited from \(S\) on the finite sum of matrix rings is proper. This completes the proof. ☑

**Proposition 25.** Let \((S, \ast)\) be a simple *-semigroup. Then it is a \(p^*\)-semigroup and it is *-embeddable in a \(p^*\)-ring.

**Proof.** There is a natural *-homomorphism \(f : (S, \ast)\) → \((Z[S]/I, \ast)\) of \((S, \ast)\) into the proper *-ring \((Z[S]/I, \ast)\). Now the kernel of \(f\) gives rise to a *-ideal in \((S, \ast)\) which is *-simple. This ideal must be zero and so \(f\) is a *-embedding and \((S, \ast)\) is a \(p^*\)-semigroup which is *-embedded in a \(p^*\)-ring. ☑

**Strategy 1.** Assume we have a finite proper *-semigroup \((S, \ast)\) with \(I\) and assume that we would like to know if \((S, \ast)\) is *-embeddable in a proper *-ring \((R, \ast)\) of matrices of characteristic 0. Then we form the algebra \((R, \ast) = (Q[S], \ast)\) where \(\ast\) is the natural involution. If \((R, \ast)\) is \(p^*\) then we are done. If not then we form the ideal \(I_1\) generated by all \(A \in R\) such that \(AA^*\) or \(A^*A = 0\). Then \(I_1\) is closed under the involution \(\ast\) and so \((R_1, \ast) = (R/I, \ast)\) is an algebra with involution and with dimension \(n_1 < n = \mid S \mid \). If there are elements \(s \neq t\) in \(S\) such that \(s - t \in I\) then \((S, \ast)\) is not *-embeddable in a \(p^*\)-ring of characteristic 0. If there is no such pair we check if \((R_1, \ast)\) is \(p^*\). If it is \(p^*\) then we are done and If not then we look for all \(A \in R\) such
that \( A \notin I_1 \) such that \( AA^* \) or \( A^*A \notin I_1 \) and we form the ideal \( I_2 \) generated by all such \( A \) and its involution \( A^* \). This ideal \( I_2 \) is closed under involution. Then we form \((R_2, *) = (R/I_2, *)\) and with dimension \( n_2 < n_1 \). If there are distinct \( s, t \in S \) such that \( s - t \in I_2 \) then \( (S, *) \) is not \(*\)-embeddable in a \( p^*\)-ring of characteristic 0. If there is no such pair we check is \((R_2, *) \) is \( p^* \). If so then we are done and If not we look for all \( A \neq 0 \) in \( R \) such that \( AA^* \) or \( A^*A \) is in \( I_2 \) and form the ideal \( I_3 \) generated by these \( A \). This is closed under taking \( * \) and we form \((R_3, *) = (R/I_3, *)\). This has dimension \( n_3 < n_2 < n_1 < n \). etc. In a finite number of steps either we come up with a \( p^*\)-algebra of 0-characteristic which \(*\)-embeds \((S, *) \) or we conclude that there is no such \( p^*\)-ring. The same procedure we can use to check if there is a \( p^*\)-ring of any prescribed nonzero characteristic or not.

**Strategy 2.** Assume we have a finite proper \(*\)-semigroup \((S, *) \) with 1 which is not \(*\)-embeddable in a \( p^*\)-ring with characteristic 0. It is desired to reform \((S, *) \) to a \( p^*\)-semigroup that is \(*\)-embeddable in a \( p^*\)-ring of characteristic 0. We form as before the \( p^*\)-ring \((Q[S]/I, *)\). Then there is a \( p^*\)-image \((T, *) \) of \((S, *) \) in \((Q[S]/I, *)\). Then there is a \(*\)-congruence \( \sim \) in \( S \) such that the \( p^*\)-semigroup \((S/\sim, *) \) is isomorphic with the \((T, *) \) inside the \( p^*\)-ring \((Q[S]/I, *)\).

**Conflict of Interests**

The author declares that there is no conflict of interests.

**References**


