DERIVATIONS ON QS-ALGEBRAS

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Abstract. In this paper, we introduce the notions of \((\ell, r)\) \((r, \ell)\)-derivations of a QS-algebras, \((r, \ell)\) \((\ell, r)\)-\(t\)-derivations of a QS-algebras, \(t\)-bi-derivations of a QS-algebras and we investigate several interesting basic properties.

Keywords: QS-algebras; \((\ell, r)\) \((r, \ell)\)-derivations of a QS-algebras; \((r, \ell)\) \((\ell, r)\)-\(t\)-derivations of a QS-algebras; \(t\)-bi-derivations of a QS-algebras.

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1. Introduction

In 1966, Y. Imai and K. Isèki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [10,11,16]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers et al [8] introduced a notions, called Q-algebras, which is a generalization of BCH / BCI / BCK-algebras and generalized some theorems discussed in BCI-algebras. Moreover, Ahn and Kim [15] introduced the notions of QS-algebras which is a proper subclass of Q-algebras. Kondo [13] proved that, each theorem of QS-algebras is provable in the theory of Abelian groups and conversely each theorem of Abelian groups is provable in the theory of QS-algebras. Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. Several authors [2,6,7,13,14] have studied derivations in rings and near rings. Jun and Xin [17] applied the notions of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular

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derivations in $BCI$-algebras. They investigated some of its properties, defined a $d$-derivations ideal and gave conditions for an ideal to be $d$-derivations. Later, Abujabal and Al-Shehri [5], defined a left derivations in $BCI$-algebras and investigated a regular left derivations. Zhan and Liu [9] studied f-derivations in BCI-algebras and proved some results. Muhiuddin and Al-roqi [3,4] introduced the notions of $(\alpha, \beta)$-derivations in a BCI-algebras and investigated related properties. They provided a condition for a $(\alpha, \beta)$-derivations to be regular. They also introduced the concepts of a $d_{(\alpha, \beta)}$- invariant $(\alpha, \beta)$-derivations and $\alpha$-ideal, and then they investigated their relations. Furthermore, they obtained some results on regular $(\alpha, \beta)$-derivations. Moreover, they studied the notions of $t$-derivations on BCI-algebras [4] and obtained some of its related properties. In this paper we introduce the notions of $\mathcal{D}_{(\alpha, \beta)}$-derivations of a $QS$-algebra and investigate some related properties.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

**Definition 2.1** [15] A $QS$-algebra $(X, *, 0)$ is a non-empty set $X$ with a constant $0$ and a binary operation $*$ such that for all $x, y, z \in X$ satisfying the following axioms:

- **(QS-1)** $(x \ast y) \ast z = (x \ast z) \ast y$.
- **(QS-2)** $x \ast 0 = x$.
- **(QS-3)** $x \ast x = 0$.
- **(QS-4)** $(x \ast y) \ast (x \ast z) = z \ast y$.

**Definition 2.2** [15] Let $(X, *, 0)$ be a $QS$-algebra, we can define a binary relation $\leq$ on $X$ as, $x \leq y$ if and only if $x \ast y = 0$, this makes $X$ as a partially ordered set.

**Proposition 2.3** [15] Let $(X, *, 0)$ be a $QS$-algebra. Then the following hold: $\forall x, y, z \in X$. 


1. \( x \leq y \) implies \( z \cdot y \leq z \cdot x \).
2. \( x \leq y \) and \( y \leq z \) imply \( x \leq z \).
3. \( x \cdot y \leq z \) implies \( x \cdot z \leq y \).
4. \((x \cdot z) \cdot (y \cdot z) \leq x \cdot y \).
5. \( x \leq y \) implies \( x \cdot z \leq y \cdot z \).
6. \( 0 \cdot (0 \cdot (0 \cdot x)) = 0 \cdot x \).

**Lemma 2.4** Let \((X, \cdot, 0)\) be a QS-algebra. If \( x \cdot y = z \), then \( x \cdot z = y \) \( \forall x, y, z \in X \).

**Lemma 2.5** Let \((X, \cdot, 0)\) be a QS-algebra. \( 0 \cdot (x \cdot y) = y \cdot x \) \( \forall x, y \in X \).

**Corollary 2.6** Let \((X, \cdot, 0)\) be a QS-algebra. \( 0 \cdot (0 \cdot x) = x \) \( \forall x \in X \).

**Lemma 2.7** Let \((X, \cdot, 0)\) be a QS-algebra. \( x \cdot (0 \cdot y) = y \cdot (0 \cdot x) \) \( \forall x, y \in X \).

**Proposition 2.8** Let \((X, \cdot, 0)\) be a QS-algebra. Then the following hold: \( \forall x, y, z \in X \).

1. \( x \cdot (x \cdot y) = y \).
2. \( x \cdot (x \cdot (x \cdot y)) = x \cdot y \).
3. \((x \cdot (x \cdot y)) \cdot y = 0 \).
4. \((x \cdot z) \cdot (y \cdot z) = x \cdot y \).
5. \((x \cdot y) \cdot x = 0 \cdot y \).
6. \( x \cdot 0 = 0 \Rightarrow x = 0 \).
7. \( 0 \cdot (x \cdot y) = (0 \cdot x) \cdot (0 \cdot y) \).
8. \( x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y \).

**Proof.**
1. \( x \cdot (x \cdot y) = (x \cdot 0) \cdot (x \cdot y) = y \cdot 0 = y \).

**Proposition 2.8.1**
2. \( x \cdot (x \cdot (x \cdot y)) = x \cdot y \).
3. \((x \cdot (x \cdot y)) \cdot y = y \cdot y = 0 \).
4. \((x*z)*(y*z) \leq x*y\) \textit{clear from Proposition 2.3.4}

\[
(x*y)((x*z)*(y*z)) = (x*y)((0*(z*x))*(0*(z*y))) = (x*y)((z*y)*(z*x)) = (x*y)*(x*y) = 0, \text{ then } x*y \leq (x*z)*(y*z).
\]

\textit{Hence} \((x*z)*(y*z) = x*y\).

5. \((x*y)*x = (x*x)*y = 0*y\).

6. If \(x*0 = 0\), then \(x = 0\).

7. \(0*(x*y) = (x*x)*(x*y) = y*x = (0*x)*(0*y)\).

8. \(x*y = 0 \Rightarrow x \leq y\) and \(y*x = 0 \Rightarrow y \leq x\), \textit{then} \(x = y\).

\textbf{Example 2.9} [12] Let \(X = \{0,1,2\}\) be a set in which the operation \(*\) is defined as follows:

\[
\begin{array}{ccc}
* & 0 & 1 & 2 \\
0 & 0 & 2 & 1 \\
1 & 1 & 0 & 2 \\
2 & 2 & 1 & 0 \\
\end{array}
\]

Then \((X,* ,0)\) is a QS-algebra.

\textbf{Definition 2.10} Let \((X,* ,0)\) be a QS-algebra and \(S\) be a non-empty subset of \(X\), then \(S\) is called \textit{subalgebra of} \(X\) if \(x*y \in S\) \(\forall x,y \in S\).

\textbf{Definition 2.11} \((X,* ,0)\) is a QS-algebra, \(x, y \in X\) we denote \(x \wedge y = y*(y*x)\).

\section{3. Derivations of QS-algebras}
Definition 3.1 Let \((X, *, 0)\) be a QS-algebra. A map \(d : X \to X\) is called a left-right derivation (briefly \((l, r)\)-derivation) of \(X\) if \(d(x*y) = (d(x)*y) \land (x*d(y)) \ \forall x, y \in X\).

Similarly, a map \(d : X \to X\) is called a right-left derivation (briefly \((r, l)\)-derivation) of \(X\) if \(d(x*y) = (x*d(y)) \land (d(x)*y) \ \forall x, y \in X\). A map \(d : X \to X\) is called a derivation of \(X\) if \(d\) is both a \((l, r)\)-derivation and a \((r, l)\)-derivation of \(X\).

Example 3.2 Let \(X = \{0, 1, 2\}\) be a QS-algebra, in which the operation * is defined as follows:

\[
\begin{array}{c|ccc}
* & 0 & 1 & 2 \\
\hline
0 & 0 & 2 & 1 \\
1 & 1 & 0 & 2 \\
2 & 2 & 0 & 0 \\
\end{array}
\]

Define a map \(d : X \to X\) by

\[
d(x) = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x = 1 \\
2 & \text{if } x = 2 
\end{cases}
\]

Then it is clear that \(d\) is a derivation of \(X\).

Definition 3.3 Let \((X, *, 0)\) be a QS-algebra and \(d : X \to X\) be a map of a QS-algebra \(X\), then \(d\) is called regular if \(d(0) = 0\).

Proposition 3.4 Let \((X, *, 0)\) be a QS-algebra

1. If \(d\) is a \((l, r)\)-derivation of \(X\), then \(d(x) = d(x) \land x \ \forall x \in X\).

2. If \(d\) is a \((r, l)\)-derivation of \(X\), then

\(d\) is regular \(\iff d(x) = x \land d(x) \ \forall x \in X\).

Proof. 1. Let \(d\) be a \((l, r)\)-derivation of \(X\). Then

\[
d(x) = d(x*0) = d(x*0) \land (x*0) = d(x) \land (x*0) = (x*d(0))*((x*d(0))*d(x))
\]

(from Def 2.1 (QS-1))

\[
= (x*d(0))*((x*d(x))*d(0)) = x*(x*d(x)) = d(x) \land x.
\]

(from Prop 2.8. 4)
2. Let \( d \) be regular \((r,l)\)-derivation of \( X \). Then
\[
d(x) = d(x*0) = (x*d(0)) \wedge (d(x)*0) = (x*0) \wedge d(x) = x \wedge d(x).
\]
Conversely, let \( d \) be a \((r,l)\)-derivation of \( X \) and \( d(x) = x \wedge d(x) \quad \forall \, x \in X \), then we get
\[
d(0) = 0 \wedge d(0) = d(0)*(d(0)*0) = d(0)*d(0) = 0.
\]
Hence \( d \) is regular.

**Lemma 3.5** Let \((X,*,0)\) be a QS-algebra and \( d \) be a \((l,r)\)-derivation of \( X \). Then the following hold \( \forall \, x,y \in X \).
1. \( d(x*y) = d(x)*y \).
2. \( d(0) = d(x)*x \) and if \( d \) is regular then \( d(x) \leq x \).

**Proof.** Clear.

**Lemma 3.6** Let \((X,*,0)\) be a QS-algebra and \( d \) be a \((r,l)\)-derivation of \( X \). Then
1. \( d(x*y) = x*d(y) \quad \forall \, x,y \in X \).
2. \( d(0) = x*d(x) \) and if \( d \) is regular then \( x \leq d(x) \).

**Proof.** Clear.

**Theorem 3.7** Let \((X,*,0)\) be a QS-algebra and \( d \) be a regular \((r,l)\)-derivation of \( X \). Then the following hold: \( \forall \, x,y \in X \).
1. \( d(x) = x \).
2. \( d(x)*y = x*d(y) \).
3. \( d(x*y) = d(x)*y = x*d(y) = d(x)*d(y) \).
4. \( \text{Ker}(d) = \{ x \in X : d(x) = 0 \} \) is a subalgebra of \( X \).

**Proof.** 1. Since \( d \) is a regular \((r,l)\)-derivation of \( X \), we have
\[
d(x) = d(x*0) = (x*d(0)) = x*d(0) = x.
\]
2. Since \( d \) is a regular \((r,l)\)-derivation of \( X \), then by Theorem 3.7. 1, we have
\[
d(x) = x \quad \forall \, x \in X.
\]
Then \( d(x)*y = x*y = x*d(y) \).
3. Since \( d \) is a regular \((r,l)\)-derivation of \( X \), then by Theorem 3.7, we have
\[
d(x) = x \ \forall x \in X.
\]
Then
\[
d(x \ast y) = d(x) \ast d(y) = d(x) \ast y = x \ast y.
\]

4. Since \( d \) is a regular, \( d(0) = 0 \), then \( 0 \in \text{Ker}(d) \), which implies that
\[
\text{Ker}(d) \text{ is non-empty set. Let } x, y \in \text{Ker}(d), \text{ then } d(x) = 0, \ d(y) = 0, \text{ hence we have}
\]
\[
d(x \ast y) = x \ast y = d(x) \ast d(y) = 0 \ast 0 = 0,
\]
therefore \((x \ast y) \in \text{Ker}(d)\) and \( \text{Ker}(d) \) is a subalgebra of \( X \).

**Lemma 3.8** Let \((X, \ast, 0)\) be a QS-algebra and \( d \) be a derivation on \( X \). If
\[
x \leq y \ \forall x, y \in X. \text{Then } d(x) = d(y).
\]
**Proof.** We have
\[
\text{from Def 2.1. (QS–2)} \quad x \leq y \iff x \ast y = 0, \text{ then } d(x) = \overline{d(x \ast 0)} = d(x \ast (x \ast y)) = \overline{d(y)}.
\]

4. **\( t \)-Derivations on QS-Algebras**

**Definition 4.1** Let \((X, \ast, 0)\) be a QS-algebra. Then for any \( t \in X \), we define a self map
\[
d_t : X \to X \text{ by } d_t(x) = x \ast t \ \forall x \in X.
\]

**Definition 4.2** Let \((X, \ast, 0)\) be a QS-algebra. Then for any \( t \in X \), a self map \( d_t : X \to X \) is called a \((t, l)\)-derivation of \( X \) if it satisfies the condition
\[
d_t(x \ast y) = (d_t(x) \ast y) \land (x \ast d_t(y)) \ \forall x, y \in X.
\]
Similarly for any \( t \in X \), a self map \( d_t : X \to X \) is called a \( t - (t, l) \)-derivation of \( X \) if it satisfies the condition
\[
d_t(x \ast y) = (x \ast d_t(y)) \land (d_t(x) \ast y) \ \forall x, y \in X.
\]
And for any \( t \in X \), a self map \( d_t : X \to X \) is called a \( t \)-derivation of \( X \) if \( d_t \) is both a \( t-(l, r) \)-derivation and a \( t-(r, l) \)-derivation of \( X \).

**Example 4.3** Let \( X = \{0, 1, 2\} \) be a QS-algebra in which the operation \( \ast \) is defined as follows:
Define a map \( d_t : X \to X \) by

\[
d_t(x) = \begin{cases} 
  x & \forall x \in X \\
  0 & \forall x \in X 
\end{cases}
\]

if \( t = 0 \), \( x \in X \), then \( x \in X \).

Then it is clear that \( d_t \) is a derivation of \( X \).

**Definition 4.4** Let \((X, *, 0)\) be a QS-algebra and \( d_t : X \to X \) be a map of a QS-algebra \( X \), then \( d_t \) is called \( t \)-regular if \( d_t(0) = 0 \).

**Proposition 4.5** Let \((X, *, 0)\) be a QS-algebra.

1. If \( d_t \) is a \( t-(l, r) \)-derivation of \( X \), then \( d_t(x) = d_t(x) \land x \forall x \in X \).
2. If \( d_t \) is a \( t-(r, l) \)-derivation of \( X \), then
   \( d_t \) regular \( \iff d_t(x) = x \land d_t(x) \forall x \in X \).

**Proof.**

1. Let \( d_t \) be a \( t-(l, r) \)-derivation of \( X \). Then
   \[
d_t(x) = d_t(x \ast 0) = (d_t(x) \ast 0) \land (x \ast d_t(0)) = d_t(x) \land (x \ast d_t(0)) = (x \ast d_t(0)) \ast ((x \ast d_t(0)) \ast d_t(x))
   \]
   from Def 2.1,(QS)
   \[
   = (x \ast d_t(0)) \ast ((x \ast d_t(0)) \ast d_t(x)) = x \ast (x \ast d_t(x)) = d_t(x) \land x.
   \]
2. Let \( d_t \) be regular \( t-(r, l) \)-derivation of \( X \). Then
   \[
d_t(x) = d_t(x \ast 0) = (x \ast d_t(0)) \land (d_t(x) \ast 0) = (x \ast 0) \land d_t(x) = x \land d_t(x).
   \]
   Conversely, let \( d_t \) be a \( t-(r, l) \)-derivation of \( X \) and satisfied \( d_t(x) = x \land d_t(x) \forall x \in X \), then we get \( d_t(0) = 0 \land d_t(0) = d_t(0) \ast d_t(0) = d_t(0) \ast d_t(0) = 0 \).

**Theorem 4.6** Let \((X, *, 0)\) be a QS-algebra and \( d_t \) be a \( t-(l, r) \)-derivation of \( X \). Then the following hold: \( \forall x, y \in X \).

1. \( d_t(x \ast y) = d_t(x) \ast y \).
2.  \( d_i(0) = d_i(x) \cdot x \).

3. If \( x \leq y \), then \( d_i(x) \leq d_i(y) \).

Proof. 1. \( d_i(x \cdot y) = (d_i(x) \cdot y) \land (x \cdot d_i(y)) = \) from Proposition 2.8. 1 
\[ (x \cdot d_i(y)) \cdot ((x \cdot d_i(y)) \cdot (d_i(x) \cdot y)) = d_i(x \cdot y) \).

2. \( d_i(0) = d_i(x \cdot x) = d_i(x) \cdot x \) from Prop 2.8. 4.

3. Let \( x \leq y \), then \( d_i(x) \cdot d_i(y) = (x \cdot t) \cdot (y \cdot t) = (x \cdot y) = 0 \). Thus \( d_i(x) \leq d_i(y) \).

Lemma 4.7 Let \((X, \cdot, 0)\) be a QS -algebra and \( d_i \) be a \( t\-(r,l) \)-derivation of \( X \). Then \( d_i(x \cdot y) = x \cdot d_i(y) \) \( \forall x, y \in X \).

Proof. Clear.

Theorem 4.8 Let \((X, \cdot, 0)\) be a QS -algebra and \( d_i \) be a regular \( t\-(r,l) \)-derivation of \( X \). Then the following hold \( \forall x, y \in X \).

1. \( d_i(x) = x \).

2. \( d_i(x) \cdot y = x \cdot d_i(y) \).

3. \( d_i(x \cdot y) = d_i(x) \cdot y = x \cdot d_i(y) = d_i(x) \cdot d_i(y) \).

4. \( \text{Ker}(d_i) = \{ x \in X : d_i(x) = 0 \} \) is a subalgebra of \( X \).

Proof. 1. Since \( d_i \) is a regular \( t\-(r,l) \)-derivation of \( X \), \( \forall x, y \in X \), we have \( d_i(x) = d_i(x \cdot 0) = x \cdot d_i(0) = x \cdot 0 = x \).

2. Since \( d_i \) is a regular \( t\-(r,l) \)-derivation of \( X \), then by Theorem 4.8. 1, we have \( d_i(x) = x \forall x \in X \). Then \( d_i(x) \cdot y = x \cdot y = x \cdot d_i(y) \).

3. Since \( d_i \) is a regular \( t\-(r,l) \)-derivation of \( X \), then by Theorem 4.8. 1 \( d_i(x) = x \forall x \in X \), hence we have \( d_i(x \cdot y) = d_i(x) \cdot y = x \cdot d_i(y) = d_i(x) \cdot d_i(y) = x \cdot y \).

4. Since \( d_i \) is a regular, \( d_i(0) = 0 \), then \( 0 \in \text{Ker}(d_i) \), hence we have
\( \text{Ker}(d_i) \) is a non-empty set.

Let \( x, y \in \text{Ker}(d_i) \), then \( d_i(x) = 0, \ d_i(y) = 0 \), hence we have
\[
d_i(x \ast y) = x \ast y = d_i(x) \ast d_i(y) = 0 \ast 0 = 0, \text{ therefore } (x \ast y) \in \text{Ker}(d_i).
\]

Then \( \text{Ker}(d_i) \) is a subalgebra of \( X \).

**Lemma 4.9** Let \((X, \ast, 0)\) be a QS-algebra and \( d_i \) be a derivation on \( X \). If \( x \leq y \ \forall x, y \in X \). Then \( d_i(x) = d_i(y) \).

**Proof.** We know
\[
x \leq y \iff x \ast y = 0, \text{ then } d_i(x) = d_i(x \ast 0) = d_i(x \ast (x \ast y)) = d_i(y).
\]

5. **Generalized t-Derivations of QS-Algebras**

**Definition 5.1** Let \( X \) be a QS-algebra. A mapping \( D_i : X \times X \to X \) is called a generalized \( t \)-\((l, r)\)–derivation if there exists an \( t \)-\((l, r)\)–derivation \( d_i : X \to X \) such that \( D_i(x \ast y) = (D_i(x) \ast y) \land (x \ast d_i(y)) \ \forall x, y \in X \). Similarly a mapping \( D_i : X \to X \) is called a generalized \( t \)-\((r, l)\)–derivation if there exists an \( t \)-\((r, l)\)–derivation \( d_i : X \to X \) such that \( D_i(x \ast y) = (x \ast D_i(y)) \land (d_i(x) \ast y) \ \forall x, y \in X \).

Moreover if \( D_i \) is both a generalized \( t \)-\((l, r)\)– and \( (r, l)\)–derivation, we say that \( D_i \) is a generalized \( t \)-derivation.

**Example 5.2** Let \( X = \{0, 1, 2, 3\} \) be a QS-algebra in which the operation \( \ast \) is defined as follows:

\[
\begin{array}{cccc}
\ast & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}
\]
Define a map \( d_t : X \to X \) and a map \( D_t : X \times X \to X \) by
\[
d_t(x) = x * t \quad \text{and} \quad D_t(x) = t * x \quad \forall \ x \in X
\]
Then it is clear that \( D_t \) is a generalized \( t \)-derivation of \( X \).

**Definition 5.3** Let \( X \) be a QS-algebra and \( D_t : X \to X \) be a map of a QS-algebra \( X \), then \( D_t \) is called \( t \)-regular if \( D_t(0)=0 \).

**Proposition 5.4** Let \( D_t \) be a self-map of a QS-algebra \( X \). Then
1. if \( D_t \) is a generalized \( t \)-\( (l,r) \)-derivation of \( X \), then \( D_t(x) = D_t(x) \land x \quad \forall \ x \in X \)
2. if \( D_t \) is a generalized \( t \)-\( (r,l) \)-derivation of \( X \), then
   \[
   D_t \text{ is } t\text{-regular } \iff D_t(x) = x \land d_t(x) \quad \forall \ x \in X.
   \]

**Proof.**
1. if \( D_t \) is a generalized \( t \)-\( (l,r) \)-derivation of \( X \), then there exists an \( t \)-derivation \( d_t \) such that \( D_t(x \cdot y) = (D_t(x) \cdot y) \land (x \cdot d_t(y)) \quad \forall \ x, y \in X \). Hence, we get
   \[
   D_t(x) = D_t(x \cdot 0) = (D_t(x) \cdot 0) \land (x \cdot d_t(0)) = D_t(x) \land (x \cdot d(0)) = \frac{\text{from Def 2.1 (QS-2)}}{(x \cdot d(0)) \land ((x \cdot d(0)) \land D_t(x)) = (x \cdot d(0)) \land (x \cdot D_t(x)) = x \land D_t(x) \land x.}
   \]
2. if \( D_t \) is a generalized \( t \)-\( (r,l) \)-derivation of \( X \), then there exists an \( t \)-\( (r,l) \)-derivation \( d_t \) such that \( D_t(x \cdot y) = (x \cdot D_t(y)) \land (d_t(x) \cdot y) \quad \forall \ x, y \in X \). Hence, we get
   \[
   D_t(x) = D_t(x \cdot 0) = (x \cdot D_t(0)) \land (d_t(x) \cdot 0) = (x \cdot 0) \land d_t(x) = x \land d_t(x).
   \]

**Proposition 5.5** Let \( X \) be a QS-algebra and \( D_t \) is a generalized \( t \)-\( (l,r) \)-derivation of \( X \), then the following hold \( \forall x, y \in X \):
1. \( D_t(x \cdot y) = d_t(x) \cdot y \).
2. \( D_t(0) = D_t(x) \cdot x \).
3. \( D_t(x \cdot D_t(x)) = 0 \).

**Proof.** Clear.
**Proposition 5.6** Let $X$ be a QS-algebra and $D_t$ is a generalized $t$-$(r,l)$–derivation of $X$, then the following hold $\forall x, y \in X$:
1. $D_t(x) = d_t(x)$.
2. $D_t(x \ast y) = x \ast d_t(y)$.
3. $D_t(D_t(x \ast y) = 0$.

**Proof.** Clear.

6. **On t-Bi-Derivations of QS –Algebras**

**Definition 6.1** Let $X, Y$ be QS - algebras. We define an operation $\ast$ on the Cartesian product $X \times Y$ of $X$ and $Y$ as follows $(x_i, y_i) \ast (x_j, y_j) = (x_i \ast x_j, y_i \ast y_j)$ $\forall (x_i, y_i) \in X \times Y, i = 1, 2$.

Then it is clear $(X \times Y, \ast, (0,0))$ a QS -algebra, and it is called the product of $X, Y$.

**Lemma 6.2** If $(X, \ast, 0)$ is a QS -algebra, then $(X \times Y, \ast, 0)$ is a QS –algebra.

**Proof.** Clear.

**Definition 6.3** Let $X$ be a QS -algebra and $d_t : X \rightarrow X$ be a mapping. A mapping $D_t : X \times X \rightarrow X$ is defined by $D_t(x, y) = (x \ast y) \ast t$.

**Definition 6.4** Let $(X, \ast, 0)$ is a QS-algebra and $D_t : X \times X \rightarrow X$ be a mapping. If $D_t$ satisfies the identity $D_t(x \ast y, z) = (D_t(x, z) \ast y) \wedge (x \ast D_t(y, z))$ for all $x, y, z \in X$, then $D_t$ is called $t$-left–rightbi- derivation (briefly $t$- $(l, r)$–bi- derivation). Similarly if $D_t$ satisfies the identity $D_t(x \ast y, z) = (x \ast D_t(y, z)) \wedge (D_t(x, z) \ast y)$ for all $x, y, z \in X$, then $D_t$ is called $t$-right–leftbi- derivation (briefly $t$- $(r, l)$–bi- derivation). Moreover if $D_t$ is both an $(r, \ell)$ and $(\ell, r)$ t- bi- derivation, it is called that $D_t$ is $t$- bi- derivation.
Example 6.5 Let $X = \{0, 1, 2, 3\}$ be a QS-algebra in which the operation $*$ is defined as follows:

<table>
<thead>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
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<td>1</td>
<td>2</td>
<td>3</td>
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<td>3</td>
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</tbody>
</table>

Define a map $D_t : X \times X \to X$ by

$$D_t(x, y) = t \ast (x \ast y) \quad \forall \ x, y, z, t \in X$$

Then it is clear that $D_t$ is $t$-bi-derivation of $X$.

Definition 6.6 Let $X$ be a QS-algebra and $D_t : X \times X \to X$ be a mapping. If $D_t(0, z) = 0$, $\forall z \in X$, $D_t$ is called component wise regular. In particular if $D_t(0, 0) = d_t(0) = 0$, $D_t$ is called $d_t$-regular.

Proposition 6.7 Let $X$ be a QS-algebra and $D_t : X \times X \to X$ be a mapping. Then

1. If $D_t$ is a $t$-($l, r$)–bi-derivation, then $D_t(x, z) = D_t(x, z) \land x \quad \forall x, z \in X$
2. If $D_t$ is a $t$-($r, l$)–bi-derivation, then $D_t$ is component wise regular $\iff D_t(x, z) = x \land D_t(x, z) \forall x, z \in X$.

Proof: 1. Let $D_t$ be a $t$-($l, r$)–bi-derivation. Then $\forall x, z \in X$

$$D_t(x, z) = D_t(x \ast 0, z) = (D_t(x, z) \ast 0) \land (x \ast D_t(0, z))$$

from Def 2.1 (QS–2)

$$= D_t(x, z) \land (x \ast D_t(0, z))$$

$$= (x \ast D_t(0, z)) \ast ((x \ast D_t(0, z)) \ast (D_t(x, z)))$$

from Def 2.1 (QS–1)

$$= (x \ast D_t(0, z)) \ast ((x \ast D_t(x, z)) \ast D_t(0, z))$$

from Proposition 2.3

$$= x \ast (x \ast D_t(x, z)) = D_t(x, z) \land x.$$
2. Let $D_i$ be component wise regular $t$-$(r,l)$--bi- derivation.

Then $D_i(x, z) = D_i(x*0, z) = (x*D_i(0, z)) \wedge (D_i(x, z)*0) =$ 

\[
\text{from Def 2.1. (QS-2)} \quad (x*0) \wedge (D_i(x, z)*0) = \left( x \wedge \right. D_i(x, z) .
\]

Conversely, let $D_i$ be a $t$-$(r,l)$--bi- derivation and $D_i(x, z) = x \wedge D_i(x, z) \quad \forall x, z \in X$. Then we get

$D_i(0, z) = 0 \wedge D_i(0, z) = D_i(0, z)* (D_i(0, z)*0) = D_i(0, z)* D_i(0, z) = 0.$

**Theorem 6.8** Let $X$ be a QS- algebra and $D_i : X \times X \to X$ be a $t$-$(l,r)$--bi- derivation. Then

1. $D_i(x*y, z) = x*D_i(y, z) \quad \forall x, y, z \in X$.
2. $x*D_i(x, z) = y*D_i(y, z) \quad \forall x, y, z \in X$.

**Proof.** 1. Let $D_i$ be a $t$-$(l,r)$--bi- derivation. Then $\forall x, y, z \in X$

\[
D_i(x*y, z) = (x*D_i(y, z)) \wedge (y*D_i(x, z)) = (y*D_i(x, z))*(y*D_i(x, z))*(x*D_i(y, z))
\]

\[
= x*D_i(y, z) .
\]

2. Let $D_i$ be a $t$-$(l,r)$--bi- derivation. Then $\forall x, z \in X$

\[
D_i(0, z) = D_i(x*x, z) = (x*D_i(x, z)) \wedge (x*D_i(x, z))
\]

\[
\text{from Def 2.1. (QS-3)} \quad D_i(0, z) = (x*D_i(x, z)) \wedge (x*D_i(x, z))
\]

\[
= (x*D_i(x, z))*(x*D_i(x, z))*(x*D_i(x, z)) = (x*D_i(x, z)) * 0 = x*D_i(x, z).
\]

Thus, we can write $D_i(0, z) = x*D_i(x, z) = y*D_i(y, z) \quad \forall y \in X$.

**Lemma 6.9** Let $X$ be a QS-algebra and $D_i : X \times X \to X$ be a component wise regular $t$-$(l,r)$--bi- derivation. Then $D_i(x, z) = x \quad \forall x, z \in X$.

**Proof.** Since $D_i$ is a component wise regular, then $D_i(0, z) = 0, \ \forall z \in X$. Then

\[
D_i(x, z) = D_i(x*0, z) = (x*D_i(0, z)) \wedge (0*D_i(x, z)) = (x*0) \wedge (0*D_i(x, z))
\]

\[
= x \wedge (0*D_i(x, z)) = (0*D_i(x, z)) * ((0*D_i(x, z)) * x) = _x .
\]
Proposition 6.10 Let $X$ be a QS-algebra and $D_r : X \times X \rightarrow X$ be a $t-(l,r)-bi$-derivation. If there exist $a \in X$ such that $D_r(x, z) \ast a = 0 \quad \forall x, z \in X$, then $D_r(x \ast a, z) = 0$.

**Proof.** Since $D_r$ is a $t-(l,r)-bi$-derivation, we get

$$D_r(x \ast a, z) = (D_r(x, z) \ast a) \land (x \ast D_r(a, z)) = 0 \land (x \ast D_r(a, z))$$

$$= (x \ast D_r(a, z)) \ast ((x \ast D_r(a, z)) \ast 0) = (x \ast D_r(a, z)) \ast (x \ast D_r(a, z)) = 0.$$  

Proposition 6.11 Let $X$ be a QS-algebra and $D_r : X \times X \rightarrow X$ be a $t-(r,l)-bi$-derivation. If there exist $a \in X$ such that $a \ast D_r(x, z) = 0 \quad \forall x, z \in X$, then $D_r(a \ast x, z) = 0$.

**Proof.** Since $D_r$ is a $t-(r,l)-bi$-derivation, we get

$$D_r(a \ast x, z) = (a \ast D_r(x, z)) \land (D_r(a, z) \ast x) = 0 \land (D_r(a, z) \ast x)$$

$$= (D_r(a, z) \ast x) \ast ((D_r(a, z) \ast x) \ast 0) = (D_r(a, z) \ast x) \ast (D_r(a, z) \ast x) = 0.$$  

7. Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, The notion of $(\ell, r)$ $(r, \ell)$-derivations of a QS-algebras, $(\ell, r)$ $(r, \ell)$ $t$-derivations of a QS-algebras, $t$-bi- derivations of a QS-algebras are introduced and investigated, also some useful properties of these types derivations in QS-algebras. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCH-algebras, Hilbert algebras, BF-algebras, J-algebras, WS-algebras, CI-algebras, SU-algebras, BCL-algebras, BP-algebras and BO-algebras , PU- algebras and so forth. The main purpose of our future work is to investigate the fuzzy derivations ideals in QS-algebras, which may have a lot of applications in different branches of theoretical physics and computer science.

Conflict of Interests

The author declares that there is no conflict of interests.
REFERENCES

Appendix

Algorithm for QS-algebras.

Input (\( X \): set, \(*\): binary operation)
Output (“\( X \) is a QS-algebra or not”)

Begin
If \( X = \emptyset \) then go to (1.);
End If
If \( 0 \not\in X \) then go to (1.);
End If
Stop: = false;
\( i := 1; \)

While \( i \leq |X| \) and not (Stop) do
If \( x_i * x_i \neq 0 \), \( x_i * 0 \neq x_i \) then
Stop: = true;
End If
\( j := 1, k := 1 \)

While \( j, k \leq |X| \) and not (Stop) do
If \((x_i * y_j) * z_k \neq (x_i * z_k) * y_j \), \((x_i * y_j) * (x_i * z_k) \neq z_k * y_j \) then
Stop: = true;
End If
End While
End While
End If

(1.) Output (“\( X \) is not a QS-algebra”)
Else
Output (“\( X \) is a QS-algebra”)
End If
End