L-SEQUENCES OF SATURATED NUMERICAL SEMIGROUPS WITH
MULTIPLICITY \( \leq 7 \)

SEDAT İLHAN* AND AHMET ÇELİK

University of Dicle, Department of Mathematics, 21280 Diyarbakır, TURKEY

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Abstract: In this paper, we will investigate Lipman sequences (L-sequences) of saturated numerical semigroups with multiplicity \( \leq 7 \) and conductor \( C \). Also, we will give some results about Frobenius number, determine number and genus in these Lipman sequences.

Keywords: saturated numerical semigroups; Lipman sequences; genus.

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1. Introduction

A numerical semigroup is a subset \( S \) of \( \mathbb{N} \) ( Here \( \mathbb{N} \) denotes the set of nonnegative integers) if \( x + y \in S \), for all \( x, y \in S \), \( 0 \in S \) and \( S \) has finite complement in \( \mathbb{N} \). If \( S \) is a numerical semigroup, then the greatest integer that does not belong to \( S \) is called the Frobenius number of \( S \), denoted by \( F(S) \). If \( S = \langle s_1, s_2 \rangle \), then \( F(S) = s_1 s_2 - s_1 - s_2 \) ( see, for instance [1], [6]). If \( S \) is a numerical semigroup then \( C \) is conductor of \( S \)

*Corresponding author
E-mail address: sedati@dicle.edu.tr
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if \( C = F(S) + 1 \). Also, \( n(S) = \text{Card} \ 0, 1, 2, \ldots, F(S) \cap S \) is called the number determine of \( S \).

Given a numerical semigroup \( S = \langle v_1, v_2, \ldots, v_r \rangle \), we have

\[
S = \langle v_1, v_2, \ldots, v_r \rangle = \left\{ \sum_{i=1}^{r} k_iv_i : k_i \in \mathbb{N} \right\}.
\]

In this case, \( r \) and \( \min x \in S : x > 0 \) is called embedding dimension and multiplicity of \( S \), denoted by \( e(S) \) and \( m(S) \), respectively. In general, \( e(S) \leq m(S) \). If \( e(S) = m(S) \) then \( S \) is called maximal embedding dimension (see, [6]).

If \( S \) is a numerical semigroup such that \( S = \langle v_1, v_2, \ldots, v_n \rangle \), then we write that

\[
S = \langle v_1, v_2, \ldots, v_n \rangle = s_0 = 0, s_1, s_2, \ldots, s_{n-1}, s_n = F(S) + 1, \rightarrow \ldots,
\]

where \( s_i < s_{i+1}, n = n(S) \) and the arrow means that every integer greater than \( F(S) + 1 \) belongs to \( S \) for \( i = 1, 2, \ldots, n = n(S) \). If \( u \in \mathbb{N}\setminus S \) then \( u \) is called gap of \( S \) and we denote the set of gaps of \( S \), by \( H(S) \), i.e., \( H(S) = \mathbb{N}\setminus S \). The cardinality of the set \( H(S) \) is called the genus of \( S \), denoted by \( G(S) \). It is known that \( G(S) = F(S) + 1 - n(S) \) (see, for detail [1], [2], [6]).

A numerical semigroup \( S \) is called Arf if \( a + b - c \in S \), for all \( a, b, c \in S \) such that \( a \geq b \geq c \). It is well known that any Arf numerical semigroup is maximal embedding dimension, but its inverse is not true. A numerical semigroup \( S \) is called saturated if \( s + n_1s_1 + n_2s_2 + \ldots + n_is_i \in S \), where \( s, s_i \in S \) and \( n_i \in \mathbb{Z} \) such that \( n_i \geq 0 \) and \( s_i \leq s \) for \( i = 1, 2, \ldots, t \). A saturated numerical semigroup is Arf, but an Arf numerical semigroup need not be saturated (see, for instance [2], [3], [4], [5], [6]).

Let \( S \) be a numerical semigroup with the maximal ideal \( T = S \setminus 0 \). For each \( k \geq 1 \), we
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define $B(S) = T - T = x \in \mathbb{N} : x + T \subseteq T$ and $kT - kT = a \in \mathbb{N} : a + kT \subseteq kT$. We note that $B(S)$ and $kT - kT$ are numerical semigroups. In this case, $L(S) = \bigcup_{k \geq 1} (kT - kT)$ is a numerical semigroup containing $S$. Evidently, $B(S) \subseteq L(S)$ and $S$ is maximal embedding dimension if and only if $B(S) = L(S)$. If $S$ is a numerical semigroup then we have the following chains,

$$B_0(S) = S \subseteq B_1(S) = B(S) \subseteq B_2(S) = B(B_1(S)) \subseteq \cdots \subseteq B_{r+1}(S) = B(B_r(S)) \subseteq \cdots$$

and

$$L_0 = S \subseteq L_1 = L(S) \subseteq L_2 = L(L_1(S)) \subseteq \cdots \subseteq L_k = L(L_{k-1}(S)) \subseteq \cdots.$$ 

The sequence $L_i(S)$ is called the Lipman sequence of semigroups of $S$. If there exists $\lambda$ such that $L_\lambda(S) = \mathbb{N}$ then $\lambda$ is called as Lipman index of $S$. If $S$ is an Arf numerical semigroup then $B_i(S) = L_i(S)$ for each $i \geq 0$ (see, for detail [2], [11]).

In this paper, we find Lipman sequences of saturated numerical semigroups with multiplicity $\leq 7$ and conductor $C = mk$, for $k \geq 1$, $k \in \mathbb{Z}$ and $m = m(S) = 2, 3, 4, 5, 6, 7$. Also, we write formulas about Frobenius number, determine number and genus in these Lipman sequences and we obtain some results for these numerical semigroups.

2. Main results

Proposition 2.1. ([11]) Let $S = \langle a_1, a_2, \ldots, a_p \rangle$ be a numerical numerical semigroup and $F(S)$ be its Frobenius number. Then we have

1. $F(B_1(S)) = F(S) - a_1$,

2. $L(S) = \langle a_1, a_2 - a_1, \ldots, a_p - a_1 \rangle.$
3. If $S$ is symmetric, then $B_i(S) = \langle a_1, a_2, \ldots, a_p, F(S) \rangle$.

4. $S$ is maximal embedding dimension if and only if $B_i(S) = L_i(S)$.

**Theorem 2.2.** For $k \geq 1$, $k \in \mathbb{Z}$ and $i = 0, 1, 2, \ldots$, we have the following statement:

1. The Lipman semigroups sequence of $S_k = <2, 2k + 1>$ saturated numerical semigroup is $L_i(S_k) = <2, 2k - 2i + 1>$.

2. The Lipman semigroups sequence of $S_k = <3, 3k + 1, 3k + 2>$ saturated numerical semigroup is $L_i(S_k) = <3, 3k - 3i + 1, 3k - 3i + 2>$.

3. The Lipman semigroups sequence of $S_k = <4, 4k + 1, 4k + 2, 4k + 3>$ saturated numerical semigroup is
   
   $L_i(S_k) = <4, 4k - 4i + 1, 4k - 4i + 2, 4k - 4i + 3>$.

4. The Lipman semigroups sequence of $S_k = <5, 5k + 1, 5k + 2, 5k + 3, 5k + 4>$ saturated numerical semigroup is

   $L_i(S_k) = <5, 5k - 5i + 1, 5k - 5i + 2, 5k - 5i + 3, 5k - 5i + 4>$.

5. The Lipman semigroups sequence of $S_k = <6, 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5>$ saturated numerical semigroup is

   $L_i(S_k) = <6, 6k - 6i + 1, 6k - 6i + 2, 6k - 6i + 3, 6k - 6i + 4, 6k - 6i + 5>$.

6. The Lipman semigroups sequence of $S_k = <7, 7k + 1, 7k + 2, 7k + 3, 7k + 4, 7k + 5, 7k + 6>$ saturated numerical semigroup is

   $L_i(S_k) = <7, 7k - 7i + 1, 7k - 7i + 2, 7k - 7i + 3, 7k - 7i + 4, 7k - 7i + 5, 7k - 7i + 6>$.

**Proof.** For $k \geq 1$, $k \in \mathbb{Z}$ and $i = 0$, it is clear that

1. Let $S_k = <2, 2k + 1>$ be saturated numerical semigroup. So, we proof by induction on $i$:
If $i = 1$ then we have $L_i(S_k) = L(S) = L(<2,2k+1>) = <2,2k-1>$. We assume that this expression is true for $i = r$, i.e. $L_r(S_k) = <2,2k-2r+1>$. Now, let be $i = r + 1$. Then we have

$$L_{r+1}(S_k) = L(L_r(S_k)) = L(<2,2k-2r+1>) = <2,2k-2r-1> = <2,2k-2(r+1)+1>.$$ 

Thus, the proof is completed.

2. Let $S_k = <3,3k+1,3k+2>$ be saturated numerical semigroup, then we proof by induction on $i$:

If $i = 1$ then we have

$$L_i(S_k) = L(S) = L(<3,3k+1,3k+2>) = <3,3k-2,3k-1>.$$ 

We assume that this expression is true for $i = r$, so, $L_r(S_k) = <3,3k-3r+1,3k-3r+2>$.

If $i = r + 1$ then we have

$$L_{r+1}(S_k) = L(L_r(S_k)) = L(<3,3k-3r+1,3k-3r+2>)$$

$$= <3,3k-3r-2,3k-3r-1>$$

$$= <3,3k-3(r+1)+1,3k-3(r+1)+2>.$$ 

Thus, the proof is completed. So, we can proof same way expression of (3), (4), (5) and (6).

**Corollary 2.3** If we take $i = 1$ in Theorem 2.2, then we obtain following Lipman semigroups of $S_k$ saturated numerical semigroups which given by Theorem 2.2, respectively:

1. $L(S_k) = <2,2k-1>$,
2. $L(S_k) = <3,3k-2,3k-1>$
3. $L(S_k) = <4,4k-3,4k-2,4k-1>$
4. $L(S_k) = <5,5k-4,5k-3,5k-2,5k-1>$
5. $L(S_k) = <6,6k-5,6k-4,6k-3,6k-2,6k-1>$
Corollary 2.4 The Arf index of each saturated numerical semigroup which given by Theorem 2.2 is $\lambda = k$.

Theorem 2.5 If $S_k = \langle m, mk + 1, mk + 2, \ldots, mk + m - 1 \rangle$ is saturated numerical semigroups, then $F(L(S_k)) = mk - (m + 1)$, where $k \geq 1$, $k \in \mathbb{Z}$ and $m(S) = m = 2, 3, 4, 5, 6, 7$.

Proof.

If $m = 2$ then $S_k = \langle 2, 2k + 1 \rangle$ and $L(S_k) = \langle 2, 2k - 1 \rangle$. Thus, we have

$$F(L(S_k)) = 2(2k - 1) - 2 - 2k + 1 = 2k - 3.$$  

If $m = 3$ then $S_k = \langle 3, 3k + 1, 3k + 2 \rangle$ and $L(S) = \langle 3, 3k - 2, 3k - 1 \rangle$. So, we obtain

$$F(L(S_k)) = F(S_k) - 3 = 3k - 1 - 3 = 3k - 4 \quad \text{from Proposition 2.1 and Theorem 3 in [9].}$$

If $m = 4$ then $S_k = \langle 4, 4k + 1, 4k + 2, 4k + 3 \rangle$ and

$$L(S) = \langle 4, 4k - 3, 4k - 2, 4k - 1 \rangle.$$ 

So, we obtain

$$F(L(S_k)) = F(S_k) - 4 = 4k - 1 - 4 = 4k - 5 \quad \text{from Proposition 2.1 and [3].}$$

If $m = 5$ then $S_k = \langle 5, 5k + 1, 5k + 2, 5k + 3, 5k + 4 \rangle$ and

$$L(S) = \langle 5, 5k - 3, 5k - 2, 5k - 1 \rangle.$$ 

So, we obtain

$$F(L(S_k)) = F(S_k) - 5 = 5k - 1 - 5 = 5k - 6$$

from Proposition 2.1 and Theorem 5 in [9].

If $m = 6$ then $S_k = \langle 6, 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5 \rangle$ and

$$L(S) = \langle 6, 6k - 5, 6k - 4, 6k - 3, 6k - 2, 6k - 1 \rangle.$$ 

So, we obtain

$$F(L(S_k)) = F(S_k) - 6 = 6k - 1 - 6 = 6k - 7 \quad \text{from Proposition 2.1 and [12].}$$

If $m = 7$ then $S_k = \langle 7, 7k + 1, 7k + 2, 7k + 3, 7k + 4, 7k + 5, 7k + 6 \rangle$ and
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\[
L(S) = \langle 7, 7k - 6, 7k - 5, 7k - 4, 7k - 3, 7k - 2, 7k - 1 \rangle.
\]

So, we obtain \( F(L(S_k)) = F(S_k) - 7 = 7k - 1 - 7 = 7k - 8 \) from Proposition 2.1 and [10].

**Theorem 2.6** If \( S_k = \langle m, mk + 1, mk + 2, \ldots, mk + m - 1 \rangle \) is saturated numerical semigroups, then we have

a. \( n(L(S_k)) = k - 1 \)

b. \( G(L(S_k)) = (m - 1)(k - 1) \)

where \( k \geq 1, k \in \mathbb{Z} \) and \( m(S) = m = 2, 3, 4, 5, 6, 7 \).

**Proof.**

a. We write \( n(L(S_k)) = n(S_k) - 1 \) and \( n(S_k) = k \) for \( k \geq 1, k \in \mathbb{Z} \). Thus we have

\[ n(L(S_k)) = n(S_k) - 1 = k - 1 \]

b. We know \( G(L(S_k)) = G(S_k) - (m - 1) \) and \( G(S_k) = (m - 1)k \) for \( k \geq 1, k \in \mathbb{Z} \).

Thus we have \( G(L(S_k)) = G(S_k) - (m - 1) = (m - 1)k - (m - 1) = (m - 1)(k - 1) \).

**Corollary 2.7** If \( S_k = \langle m, mk + 1, mk + 2, \ldots, mk + m - 1 \rangle \) is saturated numerical semigroups, then we have \( G(L(S_k)) = (m - 1)n(L(S_k)) \), where \( k \geq 1, k \in \mathbb{Z} \) and \( m(S) = m = 2, 3, 4, 5, 6, 7 \).

**Example 2.8** We take \( k = 4 \) and \( m = 3 \). Then, we write saturated numerical semigroup \( S = S_4 = \langle 3, 13, 14 \rangle = \langle 0, 3, 6, 9, 12, \ldots \rangle, \quad F(S) = 11 \) and \( n(S) = 4 \). So, \( H(S) = 1, 2, 4, 5, 7, 8, 10, 11 \) and the genus of \( S \) is \( G(S) = \#(H(S)) = 8 \). The Lipman semigroup of \( S \) is \( L(S) = L(\langle 3, 13, 14 \rangle) = \langle 3, 10, 11 \rangle = \langle 0, 3, 6, 9 \rightarrow \ldots \rangle \). In this case, we obtain following Lipman semigroups chain of \( S \):

\[
\begin{align*}
L_0(S) &= S, \\
L_1(S) &= L(S) = \langle 3, 10, 11 \rangle, \\
L_2(S) &= L(L_1(S)) = \langle 3, 7, 8 \rangle, \\
L_3(S) &= L(L_2(S)) = \langle 3, 4, 5 \rangle, \\
L_4(S) &= L(L_3(S)) = \langle 3, 1, 2 \rangle = \langle 1 \rangle = \mathbb{N}.
\end{align*}
\]
Thus, the Arf index of \( S \) is \( \lambda = k = 4 \) since \( L_4(S) = \mathbb{N} \). On the other hand, we obtain

\[
F(L(S)) = mk - (m+1) = 3.4 - (3+1) = 8,
\]

\[
n(L(S)) = k - 1 = 4 - 1 = 3,
\]

\[
G(L(S)) = (m-1)(k-1) = (3-1)(4-1) = 6,
\]

so

\[
G(L(S)) = (k - 1)n(L(S)) = 2.3 = 6.
\]

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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