Available online at http://scik.org
J. Semigroup Theory Appl. 2018, 2018:1
https://doi.org/10.28919/jsta/3450
ISSN: 2051-2937

SOME SOLUTIONS OF FRACTIONAL INVERSE PROBLEMS

R. KHALIL* AND R. ABDELGANI

University of Jordan, Amman, Jordan

Abstract. In this paper we find some types of solutions for certain degenerate and non degenerate fractional inverse problems. The main idea of the proofs is based on theory of tensor product of Banach spaces.

Keywords: tensor product; Banach spaces; non-degenerate Cauchy problem; fractional derivative.

2010 AMS Subject Classification: 47D06.

1. Introduction

Fractional derivatives proved to be very fruitful in applied sciences. In the literature, there are many definitions of fractional derivatives. The most classical and well known are the Riemann-Liouvill and the Caputo definitions [7]. Conformable fractional definition was introduced in [2], and it coincides with classical ones on polynomials.

For $f:[0,\infty)\rightarrow R$, and $0<\alpha\leq 1$, we let

$$f^{(\alpha)}(t)=\lim_{\varepsilon\longrightarrow 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}..........(1)$$

*Corresponding author

E-mail address: roshdi@ju.edu.jo

Received August 2, 2017

denote the $\alpha-$derivative of f at $t>0$. Further, we let

$$f^{(\alpha)}(0)=\lim_{t\to 0} f^{(\alpha)}(t)$$

We refer to [1] and [2] for more properties and results on such derivative.

It should be noticed and clear to see that the derivative in (1) can be carried on when f takes values in a normed space.

Let X be a Banach space and $I=[0,1]$. Let $C(I)$ be the Banach space of all continuous real valued functions defined on I , and $C(I,X)$ be the set of all continuous functions from I to X. It is well

known, [5], that $C(I,X)$ is isometrically isomorphic to $C(I)\overset{\vee}{\otimes}X$, the complete injective tensor product of $C(I)$ with X.

A classical and important problem in differential equations is the so called Abstract Cauchy Problem that takes the form:

$Bu'(t)=Au(t)+f(t)z$, with $u(0)=x_0$, where $u:I\to X$ is a differentiable function and A is a closed densely defined linear operator on X, [6].

In this paper we discuss the fractional Abstract Cauchy Problem. Now, the general form of the $\alpha-$Abstract Cauchy Problem is

$$\begin{aligned} Bu^{(\alpha)}(t) &= Au(t)+f(t)z.............(2) \\ u(0) &= x_0 \end{aligned}$$

Now, If $f(t)=0$ or $z=0$ then equation (2) is called homogeneous, otherwise it is nonhomogeneous. In the homogeneous case u is the only unknown in the

equation.

If B is an invertible linear operator, then Problem (2) is called degenerate otherwise it is nondegenerate.

Now, in the nonhomogeneous case we have two types of problems. The first type only u is unknown ,and f is given. In the second type we have two unknowns f and u.

But in this type we have some initial conditions in order to be able to determine u and f. Usually, $\alpha-$Abstract Cauchy Problem is called $\alpha-$inverse problem.

2. Main results

Let X and Y be Banach spaces. For $x \in X$ and $y \in Y$, the operator $x \otimes y : X^* \to Y$ is called an atom, where X^* is the dual of X. So atoms in $C(I) \overset{\vee}{\otimes} X$ are elements of the form

$F = h \otimes y$ where $h \in C(I)$, and $y \in X$. We are interested in solutions for problem (2) which are atoms.

Theorem 2.1. Consider the problem

$$u^{(\alpha)} \otimes x + u \otimes Ax = f \otimes z ....(3)$$

where $u : I \to R$ is $\alpha-$ differentiable and f is continous on I, A is a densely defined closed linear operator on X (and $x \in X$).

Assume that :

(1) There exists some $x^* \in X^*$, and $g \in C(I, \mathbb{R})$, such that $g^{(\alpha)}(0)$ exists, and $u(t)\langle x, x^* \rangle = g(t)$.

(2) $\ln(\frac{g(1)}{g(0)}) \in \rho(A)$.

Then (3) has a unique solution.

Proof

Since,[2], every $\alpha-$differentiable function is continuous, it follows that g is continous.

Now, since every atom has infinite number of representations (e.g.: if $x \otimes y$ is an atom then $\lambda x \otimes \frac{1}{\lambda} y$ another representation of $x \otimes y$ for any $\lambda \neq 0$), then

without loss of generality we can assume that $u(0) = f(0) = 1$.

From (3), $u^{(\alpha)} \otimes x$ and $u \otimes Ax$ are two atoms whose sum is also an atom $f \otimes z$. Thus, [8], we have two cases: (i) $u^{(\alpha)} = \lambda u$ and (ii) $Ax = \beta x$.

Case (i)

If $u^{(\alpha)} = \lambda u$, then using a result in[2], we have $u(t) = ce^{\frac{\lambda}{\alpha} t^{\alpha}}, c \in \mathbb{R}$.

But since $u(0) = 1$,then we have $c = 1$.

So

$$u(t) = e^{\frac{\lambda}{\alpha} t^{.\alpha}} ....(4)$$

.

Now using condition (1), we have $\langle x, x^* \rangle = g(0)$.

Thus,

$$\begin{aligned} g(t) &= u(t)\langle x,x^{*}\rangle \\ &= u(t)g(0) \\ &= e^{\frac{\lambda}{\alpha}t\cdot\alpha}g(0) \\ &= e^{\frac{\lambda}{\alpha}t\cdot\alpha}\langle x,x^{*}\rangle \end{aligned}$$

Putting $t=1$, we get $g(1)=e^{\frac{\lambda}{\alpha}}g(0)$.

Consequently, $e^{\frac{\lambda}{\alpha}}=\frac{g(1)}{g(0)}$.

By taking logarithm of both sides, we get

$$\lambda=\alpha\ln\left(\frac{g(1)}{g(0)}\right).........(5).$$

Since $u(t)=e^{\frac{\lambda}{\alpha}t\cdot\alpha}$ by (4), hence u is determined uniquely.

Now, we want to find f and x uniquely.

substitute (4),in (3) ,and apply x^{*} to both sides, we get

$$\begin{aligned} f(t)\langle z,x^{*}\rangle &= u^{(\alpha)}\langle x,x^{*}\rangle+u\langle Ax,x^{*}\rangle \\ &= \frac{\lambda}{\alpha}e^{\frac{\lambda}{\alpha}t\cdot\alpha}\langle x,x^{*}\rangle+e^{\frac{\lambda}{\alpha}t\cdot\alpha}\langle Ax,x^{*}\rangle \\ &= g^{(\alpha)}(t)+e^{\frac{\lambda}{\alpha}t\cdot\alpha}\langle Ax,x^{*}\rangle \end{aligned}$$

Now, for $t=0$, we have

$$\langle Ax,x^{*}\rangle=\langle z,x^{*}\rangle-g^{(\alpha)}(0).........(6)$$

Thus

$$f(t)\langle z,x^{*}\rangle=g^{(\alpha)}(t)+e^{\frac{\lambda}{\alpha}t\cdot\alpha}\left(\langle z,x^{*}\rangle-g^{(\alpha)}(0)\right)$$

Hence f is determined uniquely.

Now to show that x is determined uniquely:

Let $t=0$. Then we have

$$u^{(\alpha)}(0)x=u(0)Ax+f(0)z........(7)$$

But $u^{(\alpha)}(0) = \frac{\lambda}{\alpha}$, and $u(0) = f(0) = 1$. Hence, $\frac{\lambda}{\alpha}x = Ax + z$.

So by (4)

$$\begin{aligned} z &= \left(\frac{\lambda}{\alpha}I - A\right)x \\ &= \left(\ln\left(\frac{g(1)}{g(0)}\right)I - A\right)x \end{aligned}$$

Thus, by condition (2) we have

$$x = z\left(\ln\left(\frac{g(1)}{g(0)}\right)I - A\right)^{-1}$$

So x is unique.

Case (ii)

If $Ax = \beta x$, then (3) becomes

$$u^{(\alpha)} \otimes x + u \otimes \beta x = f \otimes z........(8)$$

Apply x^* to both side to get

$$u^{(\alpha)}(t)\langle x, x^*\rangle + \beta u(t)\langle x, x^*\rangle = f(t)\langle z, x^*\rangle ....(9)$$

So

$$g^{(\alpha)}(t) + \beta g(t) = f(t)\langle z, x^*\rangle ....(10)$$

In (9) let $t = 0$. Then we get $g^{(\alpha)}(0) + \beta g(0) = \langle z, x^*\rangle$, and $\beta = \frac{\langle z, x^*\rangle - g^{(\alpha)}(0)}{g(0)}$.

And so β is determined uniquely.

Since g is given, using (9), we get $f(t)$ is determined uniquely.

Back to equation (8), we have

$$\left(u^{(\alpha)} + \beta u\right) \otimes x = f \otimes z.......(11)$$

Consequently,

$$\left(u^{(\alpha)} + \beta u\right) = \gamma f.......(12)$$

And

$$x = \frac{1}{\gamma}z........(13)$$

Equation (12) is a first order linear differential equation, whose solution, [ ], is

$$u(t)=e^{\frac{-\beta}{\alpha}t^{\alpha}}\left[\int_{0}^{t}\gamma\tau^{\alpha-1}e^{\frac{\beta}{\alpha}\tau^{\alpha}}f(\tau)d\tau\right]+ce^{\frac{-\beta}{\alpha}t^{\alpha}}........(15)$$

Further, from (12) and (13), we get

$$\left(u^{(\alpha)}(t)+\beta u(t)\right)x=\gamma f(t)x.....(16)$$

Now, applying x^* to both sides of (16), we get

$$u^{(\alpha)}(t)\langle x,x^*\rangle+\beta u(t)\langle x,x^*\rangle=\gamma f(t)\langle x,x^*\rangle$$

Thus,

$$g^{(\alpha)}(t)+\beta g(t)=\gamma f(t)g(0)........(17)$$

Put $t=0$ in (17) we get $g^{(\alpha)}(0)+\beta g(0)=\gamma f(t)g(0)$.

Thus, $\gamma=\frac{g^{(\alpha)}(0)}{g(0)}+\beta$,and γ is determined uniquely.

Now, by (13) x is also determined uniquely.

And we can find the value of c by using $u(0)=1$.

So u is also determined uniquely, and this completes the proof.

Theorem 4.2.

Consider the problem

$$Bu^{(\alpha)}(t)x+Au(t)x=f(t)z.........(18)$$

where A,B are two densely defined closed linear operator defined on X, and $x\in X$.

Assume the following two conditions are satisfied :

(1) There exist some $x^*\in X^*$, and $g\in C(I,\mathbb{R})$, such that g is $\alpha-$differentiable function on I, where $g^{(\alpha)}(0)$ exist, and $u(t)\langle x,x^*\rangle=g(t)$.

(2) The element z is a uniquely imaged element in X, for the operators A and $\ln\left(\frac{g(1)}{g(0)}\right)B+A$.

Then (18) has a unique solution.

Proof

Without loss of generality we can assume that $u(0)=f(0)=1$.

Now, since $u(t)\langle x,x^*\rangle=g(t)$,then we have$g(0)=u(0)\ \langle x,x^*\rangle=\langle x,x^*\rangle$.

So $u(t)=\frac{g(t)}{g(0)}$, and u is determined uniquely.

Now, we want to determine f and x uniquely.

First, since we are looking for atomic solution, then we can write (18) as

$$u^{(\alpha)} \otimes Bx + u \otimes Ax = f \otimes z........(19)$$

Again, using [8], there are two cases to consider: (i) $u^{(\alpha)} = \lambda u$, and (ii) $Bx = \beta Ax$.

Case (i)

If $u^{(\alpha)} = \lambda u$, then [2], we have $u(t) = ce^{\frac{\lambda}{\alpha}t^{\alpha}}, c \in \mathbb{R}$.Since $u(0) = 1$,then we have $c = 1$.
So $u(t) = e^{\frac{\lambda}{\alpha}t^{.\alpha}}$. But $u(t) = \frac{g(t)}{g(0)}$.
Hence, $\frac{g(1)}{g(0)} = e^{\frac{\lambda}{\alpha}}$. Take logarithm of both side of $\frac{g(1)}{g(0)} = e^{\frac{\lambda}{\alpha}}$ to get

$$\lambda = \alpha \ln\left(\frac{g(1)}{g(0)}\right).......(20)$$

.

Thus, λ is determined uniquely.

Now, Substitute $u(t) = e^{\frac{\lambda}{\alpha}t^{.\alpha}}$in (18), we get

$$\frac{\lambda}{\alpha}e^{\frac{\lambda}{\alpha}t^{.\alpha}}Bx + e^{\frac{\lambda}{\alpha}t^{.\alpha}}Ax = f(t)z........(21)$$

So

$$e^{\frac{\lambda}{\alpha}t^{.\alpha}} \otimes \left(\frac{\lambda}{\alpha}Bx + Ax\right) = f(t) \otimes z..........(22)$$

Now, since we have two atoms which are equal, then we have either

$$e^{\frac{\lambda}{\alpha}t^{.\alpha}} = \mu f(t).........(23)$$

or

$$\left(\frac{\lambda}{\alpha}Bx + Ax\right) = \frac{1}{\mu}z...........(24)$$

To find the value of μ, take $t = 0$ in (23), to get $1 = \mu f(0)$.

Thus, μ is determined, and thus $f(t)$ is determined uniquely.

Substitute (20) in (24), we get

$$\left(\ln\left(\frac{g(1)}{g(0)}\right)Bx + Ax\right) = \frac{1}{\mu}z........(25)$$

By condition (2) and equation (25), we get x is uniquely determined.

Case (ii)

If $Bx = \beta Ax$, then (18) takes the form:

$$u^{(\alpha)}(t)\beta Ax + u(t)Ax = f(t)z........(26)$$

Using tensor product we have

$$\begin{aligned} f \otimes z &= u^{(\alpha)} \otimes \beta Ax + u \otimes Ax \\ &= \left(\beta u^{(\alpha)} + u\right) \otimes Ax.............(27) \end{aligned}$$

Now, since we have two atoms are equal, then we have either
(1) $\left(\beta u^{(\alpha)} + u\right) = \lambda f$ or (2) $Ax = \frac{1}{\lambda} z$
In case (1), substitute $t = 0$ to get

$$\left(\beta u^{(\alpha)}(0) + 1\right) = \lambda.......(28)$$

Using condition (2) and (25), we have x is uniquely determined.

The rest of the proof is similar to Theorem 2.1.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] T. Abdeljawad, Conformable Fractional Calculus , J. Comput. Appl. Math. 279 (2015), 57-66.

[2] R. Khalil, M. Al Horani, A.Yousef and M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264(2014), 65-70.

[3] Abdeljawad, T. Horani,M AL. and Khalil, R. Conformable fractional semigroups of operators, J. Semigroup Theory Appl. 2015 (2015), Article ID 7.

[4] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons, Inc. (1978).

[5] W. Light, and E. Cheny, Approximation Theory in Tensor Product Spaces, Lecture notes in math. 1169. Springer-Verlag, New York (1985).

[6] Pazy, A., Semigroup of linear Operators and Applications to Partial Differential Equations, Springer-Verlag New York, Inc. (1983).

[7] A. Kilbas, H. Srivastava, and H. Trujillo, Theory and Applications of Fractional Differential Equations, in: Math. Studies., North-Holland, New York, (2006).

[8] R. Khalil, Isometries on Lp $\otimes$ Lp, Tamkang J. Math. 16 (2)(1985), 77-85.