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# A STRUCTURE THEOREM FOR LEFT RESTRICTION SEMIGROUPS OF TYPE F

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**Abstract.** A structure theorem for right adequate semigroups that are F-right abundant otherwise known as right adequate semigroups of type F was given by Cui and Guo. Here we obtain an analogous structure theorem for that of left restriction semigroups: a semigroup is a left restriction semigroup of type F if and only if it is isomorphic to some $\mathcal{F}(T,Y)$, where $(T,Y)$ is an $\mathcal{F}$-pair.

**Keywords:** left restriction semigroup; F-left restriction semigroup; right cancellative semigroup; distinguished semilattice.

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## 1. INTRODUCTION

Left restriction semigroups are class of semigroups which generalize inverse semigroups and which emerge very naturally from the study of partial transformation of a set. A more detailed description of left restriction semigroups can be found in [7], [8].

An F-inverse semigroup is an inverse semigroup S in which every $\sigma$ - class (where $\sigma$ is the group congruence on S) has greatest element with respect to the natural partial order $\leq$ on S. McFadden and O'Caroll [5] have pointed out that the concept of F-inverse semigroups is indeed a generalization of resituated inverse semigroups. Edwards [1] defined analogously F-regular semigroups and F-orthodox semigroups and showed that an F-regular semigroup is indeed an F-orthodox semigroup.

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Following Fountain [3] a semigroup S is called left abundant if each $\mathcal{R}^*$ - class of S contains atleast one idempotent. Dually, right abundant semigroup can be defined. The semigroup S is called abundant if S is both left abundant and right abundant. As in [2], a left (right) abundant semigroup is called a left (right) adequate semigroup if the set of idempotents of S (i.e. $(S)$ ) form a semilattice. Regular semigroups are abundant semigroups and inverse semigroups are adequate semigroups.

In order to generalize the F-regular semigroups, Guo [10] defined F-abundant semigroup as an abundant semigroup in which there exists a cancellative congruence $\sigma$ such that each $\sigma$ – class contains a greatest element with respect to the Lawson order $\leq$. Further investigation by Cui and Guo [6] obtained a structure theorem for right adequate semigroups of type F (i.e. right adequate semigroups that are F-right abundant).

As an analogue of [6], we obtain a structure theorem for left restriction semigroups which are F-left restriction, called left restriction semigroups of type F.

## 2. PRELIMINARIES

In this section we recall some definitions as well as some known results which will be useful in the sequel.

**Definition 2.1.** Let $S$ be a semigroup. Then $S$ is said to be left (right) ample if

i) every element $a \in S$ is $\mathcal{R}^*(\mathcal{L}^*) -$ related to an idempotent, denoted by $a^{\dagger}$ $(a^*)$

ii) for all $a \in S$ and all $e \in E(S)$,

$$ae = (ae)^{\dagger}a \qquad ( ea = a(ea)^* ).$$

**Definition 2.2.** Let $S$ be a semigroup and let $E \subseteq E(S)$ (E is the distinguished semilattice of idempotents).

Let $a, b \in S$, we have following relations on $S$

$$a\widetilde{\mathcal{R}}_E b \Leftrightarrow \forall \; e \in E, \quad ea = a \Leftrightarrow eb = b$$

$$a\widetilde{\mathcal{L}}_E b \Leftrightarrow \forall \; e \in E, \quad ae = a \Leftrightarrow be = b \; .$$

**Definition 2.3.** Let $S$ be a semigroup and let $E \subseteq E(S)$. Then $S$ is said to be left (right) restriction semigroup if

i) $E$ is a semilattice

ii) every element $a \in S$ is $\widetilde{\mathcal{R}}_E$ $(\widetilde{\mathcal{L}}_E)$– related to an idempotent of $E$, denoted by $a^{\dagger}$ $(a^*)$

iii) the relation $\widetilde{\mathcal{R}}_E$ $(\widetilde{\mathcal{L}}_E)$ is a left (right) congruence

iv) the left (right) ample condition holds:

$$ae = (ae)^{\dagger} a \qquad ( ea = a(ea)^{*} ).$$

**Definition 2.4.** Let $S$ be a semigroup and $E$ be a set of idempotents contained in S. Then for $a, b \in S$, the relation $\sigma_E$ is defined to be the smallest (semigroup) congruence on S identifying the elements E.

If $E = E(S)$, then we may write $\sigma$ for $\sigma_E$ and if S is either left or right restriction we shall denote $\sigma_{E_S}$ by $\sigma_S$, where $E_S$, is the distinguished semilattice of S.

**Lemma 2.5** [8]. Let S be a left restriction semigroup with distinguished semilattice E. Then for all $a, b \in S$,

$$a \, \sigma_S \, b \iff ea = eb \quad \text{for some } e \in E.$$

The following Lemmas are due to Fountain [3] and Gould [8], [9].

**Lemma 2.6**. Let $S$ be a semigroup and $e$ be an idempotent in $S$. Then the following are equivalent for $a \in S$.

i) $a \, \mathcal{R}^{*} e$

ii) $ea = a$, and for all $x, y \in S^1$, $xa = ya \Rightarrow xe = ye$.

**Lemma 2.7**. Let S be a semigroup and $E \subseteq E(S)$, let $a \in S, e \in E$. Then the following conditions are equivalent:

i) $a \, \tilde{\mathcal{R}}_E \, e$

ii) $ea = a$ and for all $f \in E, fa = a \Rightarrow fe = e$.

In a similar way to the *-relations, the ~ -relations are also related to the Green's relations as follows:

**Lemma 2.8**. In any semigroup S we have $\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \tilde{\mathcal{R}}_E$ . If S is regular, and $E = E(S)$ then $\tilde{\mathcal{R}}_E \subseteq \mathcal{R}$ and so $\tilde{\mathcal{R}}_E \subseteq \mathcal{R}^{*}$.

Dually we have $\mathcal{L} \subseteq \mathcal{L}^{*} \subseteq \tilde{\mathcal{L}}_E$, and if S is regular, and $E = E(S)$ then $\tilde{\mathcal{L}}_E \subseteq \mathcal{L}$ and so $\tilde{\mathcal{L}}_E \subseteq \mathcal{L}^{*}$.

Let S be a left restriction semigroup with distinguished semilattice of idempotents E, we define the relation $\leq$ on S by $a \leq b$ if and only if $a = eb$ for some $e \in E$.

If S is a right restriction semigroup with distinguished semilattice of idempotents E, we define the relation $\leq$ on S by $a \leq b$ if and only if $a = bf$ for some $f \in E$.

It can be easily checked that these relations are partial orders.

However, we note that if S is a restriction semigroup, then these two definitions are infact equivalent by the ample conditions.

When considering a left restriction semigroup S with distinguished semilattice of idempotents E we have that for $a, b \epsilon S$,

$$a^{\dagger} \leq b^{\dagger} \Leftrightarrow a^{\dagger} = a^{\dagger} b^{\dagger},$$

which is the usual order on E, and

$$a \leq b \Longrightarrow a^{\dagger} b \Longrightarrow a^{\dagger} = a^{\dagger} b^{\dagger} \Longrightarrow a^{\dagger} \leq b^{\dagger}.$$

We note the following useful Lemma, the proof for which in [2] for left adequate semigroups can be easily adapted for left restriction semigroups.

**Lemma 2.9.** Let S be a left restriction semigroup. Then

i) $(ab)^{\dagger} = (ab^{\dagger})^{\dagger}$ for all $a, b \epsilon S$

ii) $(ea)^{\dagger} = ea^{\dagger}$ for all $a \epsilon S$ and $e \epsilon E$.

A congruence $\rho_S$ on a left restriction semigroup S is called a right cancellative semigroup congruence on S if $S/\rho_S$ is a right cancellative semigroup.

By an F-left restriction semigroup, we mean a left restriction semigroup S in which there exists a right cancellative semigroup congruence $\sigma_S$ on $S$ such that each $\sigma_S$ -class of S contains a greatest element with respect to $\leq$ (for a left restriction semigroup). From [8], we know that $\sigma_S$ is indeed the smallest right cancellative semigroup congruence on S.

Let S be an $F$-left restriction semigroup. We denote by T the set of greatest elements in all $\sigma_S$ -classes of S. We define a multiplication $\circ$ as follows:

$$t \circ u = \text{the greatest element of the } \sigma_S \text{-class of S containing } tu.$$

It can be easily checked that $(T, \circ)$ is a semigroup isomorphic to $S/\sigma_S$ . We shall employ this fact in our next section.

## 3. THE STRUCTURE THEOREM

The aim of this section is to establish the structure theorem for left restriction semigroups of type F. First, we consider the following structure which is taken from [6].

Let T be a right cancellative semigroup with identity 1 and $Y$ be a semilattice with identity j. Let T act on the left of $Y$ and let $t, u \epsilon T,\ A, B \epsilon Y$.

Suppose that the following hold:

$F_1)$ $1A = A$

$F_2)$ $t(A \wedge B) = tA \wedge tB$.

Then $(T, Y)$ is called an F-pair. This pair induces the semigroup

$$\mathcal{F} = \mathcal{F}(T, Y) = \{ (A, t) \in Y \times T : A \leq tj \},$$

with product defined by the rule

$$(A, t)(B, u) = (A \wedge tB, tu),$$

for $(A, t), (B, u) \in \mathcal{F}(T, Y)$.

**Remark 3.1.** In $\mathcal{F}(T, Y), E = E_{\mathcal{F}} = \{ (A, 1) : A \in Y \} \cong Y$. Furthermore, $E_{\mathcal{F}}$ is a semilattice.

**Lemma 3.2.** $\mathcal{F}(T, Y)$ is left adequate.

**Proof.** It follows from Remark 3.1.

**Lemma 3.3.** $\mathcal{F}(T, Y)$ is left ample.

**Proof.** Let $(A, t) \in \mathcal{F}(T, Y)$ and $E_{\mathcal{F}} \in \mathcal{F}(T, Y)$, then

$$(A, t)(B, 1) = (A \wedge tB, t) = (A \wedge tB, 1)(A, t)$$

$$= [(A, t)(B, 1)]^{\dagger}(A, t).$$

**Lemma 3.4.** $\mathcal{F}(T, Y)$ is left restriction.

**Proof.** We know from Lemma 3.2 that $E_{\mathcal{F}}$ is a semilattice. Let $(A, t) \in \mathcal{F}(T, Y)$ and $(A, 1) \in E_{\mathcal{F}}$. We wish to show that $(A, t)\tilde{\mathcal{R}}_{E_{\mathcal{F}}}(A, 1)$ for $(A, t) \in \mathcal{F}(T, Y)$.

We have

$$(A, 1)(A, t) = (A \wedge A, t) = (A, t).$$

For $(B, 1) \in E_{\mathcal{F}}$,

$$(B, 1)(A, t) = (A, t) \Longrightarrow (B \wedge A, t) = (A, t)$$

$$\Longrightarrow B \wedge A = A$$

$$\Longrightarrow (B, 1)(A, 1) = (A, 1)$$

So $(A, t)\tilde{\mathcal{R}}_{E_{\mathcal{F}}}(A, 1)$ and we shall let $(A, t)^{\dagger} = (A, 1)$.

Now we wish to show that $\tilde{\mathcal{R}}_{E_{\mathcal{F}}}$ is a left congruence,

For $(A, t), (B, u) \in \mathcal{F}(T, Y)$,

$$(A, t)\tilde{\mathcal{R}}_{E_{\mathcal{F}}}(B, u) \Longleftrightarrow (A, t)^{\dagger} = (B, u)^{\dagger}$$

$$\Longleftrightarrow (A, 1) = (B, 1)$$

$$\Longleftrightarrow A = B.$$

We now have that $(A,t)\,\tilde{\mathcal{R}}_{E_{\mathcal{F}}}(B,u) \Longrightarrow A = B$

$$\Longrightarrow vA = vB \text{ for any } v \in \mathrm{T}$$
$$\Longrightarrow (vA,t) = (vB,u)$$
$$\Longrightarrow (vA,vt) = (vB,vu)$$
$$\Longrightarrow (C \wedge vA, vt) = (C \wedge vB, vu)$$
$$\Longrightarrow [(C,v)(A,t)]^{\dagger} = [(C,v)(B,u)]^{\dagger} \text{ for any } (C,v) \in \mathcal{F}(T,Y)$$
$$\Longrightarrow (C,v)(A,t)\,\tilde{\mathcal{R}}_{E_{\mathcal{F}}}(C,v)(B,u) \text{ for any } (C,v) \in \mathcal{F}(T,Y)$$

Thus $\tilde{\mathcal{R}}_{E_{\mathcal{F}}}$ is a left congruence. The ample condition follows from Lemma 3.3.

**Lemma 3.5.** For $(A,t),(B,u) \in \mathcal{F}(T,Y)$, $(A,t)\sigma_{\mathcal{F}}(B,u)$ if and only if $t = u$. Thus $\mathcal{F}/\sigma_{\mathcal{F}} \cong T$.

**Proof.** Let $(A,t)\sigma_{\mathcal{F}}(B,u)$, then there exists $(C,1) \in E_{\mathcal{F}}$ such that

$$(C,1)(A,t) = (C,1)(B,u),$$

i.e. $$(C \wedge A, t) = (C \wedge B, u)$$

and so $t = u$.

Conversely, let $t = u$. We wish to show that $(A,t)\sigma_{\mathcal{F}} = (B,u)\sigma_{\mathcal{F}}$. Let $(A \wedge B, 1) \in E_{\mathcal{F}}$. We have

$$(A \wedge B, 1)(A,t) = (A \wedge B \wedge A, t)$$
$$= (A \wedge B, t) = (A \wedge B \wedge B, u)$$
$$= (A \wedge B, 1)(B,u).$$

So $(A,t)\sigma_{\mathcal{F}} = (B,u)\sigma_{\mathcal{F}}$ when $t = u$.

The rest of the proof follows from the fact that the map $\theta : \mathcal{F}/\sigma_{\mathcal{F}} \to T$ defined by $[(A,t)\sigma_{\mathcal{F}}]\theta = t$ is an isomorphism.

**Lemma 3.6.** For $(A,t),(B,u) \in \mathcal{F}(T,Y)$, $(A,t) \leq (B,u)$ if and only if $t = u$ and $A \leq B$.

**Proof.** Let $(A,t) \leq (B,u)$, then there exists $(A,1) \in E_{\mathcal{F}}$ such that

$$(A,t) = (A,1)(B,u) = (A \wedge B, u).$$

By comparing components, $t = u$ and $A = A \wedge B$, i.e $\mathrm{A} \leq \mathrm{B}$.

Conversely, let $t = u$ and $\mathrm{A} \leq \mathrm{B}$, then

$$(A,t) = (A,1)(B,u)$$
$$\Longrightarrow (A,1) = (A,1)(B,1)$$
$$\Longrightarrow (A,1) \leq (B,1)$$
$$\Longrightarrow (A,t) \leq (B,u).$$

**Lemma 3.7.** $(A,t)\left(\tilde{\mathcal{R}}_{E_{\mathcal{F}}} \cap \sigma_{\mathcal{F}}\right)(B,u) \Leftrightarrow (A,t) = (B,u)$.

**Proof.** We have that

$$\begin{aligned}(A,t)\left(\tilde{\mathcal{R}}_{E_{\mathcal{F}}} \cap \sigma_{\mathcal{F}}\right)(B,u) &\Leftrightarrow (A,t)\ \tilde{\mathcal{R}}_{E_{\mathcal{F}}}(B,u) \text{ and } (A,t)\ \sigma_{\mathcal{F}}\ (B,u)\\ &\Leftrightarrow A = B \text{ and } t = u\\ &\Leftrightarrow (A,t) = (B,u).\end{aligned}$$

**Lemma 3.8.** $\mathcal{F}(T,Y)$ is a left restriction semigroup of type F.

**Proof.** It follows from Lemma 3.5 and Lemma 3.6.

Now, we can establish the structure theorem for left restriction semigroups of type F.

**Theorem 3.9.** The semigroup $\mathcal{F}(T,Y)$ is a left restriction semigroup of type F. Conversely, a left restriction semigroup of type F is isomorphic to $\mathcal{F}(T,Y)$ for some F-pair $(T,Y)$.

**Proof.** The direct part of the proof follows from Lemma 3.8, so we only need to prove the converse part. Let S be a left restriction semigroup of type F and $E_S$ be the distinguished semilattice of S. Then $E_S$ is a distinguished semilattice with identity j. We let T denote the set of greatest elements in all the $\sigma_S$ -classes of S. Define multiplication $\circ$ as follows:

$$t \circ u = \text{the greatest element of the } \sigma_S \text{-class of S containing } tu.$$

It can be easily checked that $(T,\circ)$ is a semigroup isomorphic to $S/\sigma_S$ . Hence T is a right cancellative semigroup. Let 1 be the identity of T. It is not difficult to see that 1 = j. Also, we define an action of T on Y by $tA = (tA)^{\dagger}$ for $A \in E_S$ and $t \in T$.

We now show that $(T,E_S)$ is an F-pair. Let $t,u \in T$ and $A,B \in E_S$.

$$1A = (jA)^{\dagger} = A, \text{ Thus } F_1) \text{ holds.}$$

For the condition $F_2$), we have

$$\begin{aligned}t(A \wedge B) &= (t(A \wedge B))^{\dagger} = (tAB)^{\dagger}\\ &= (tA(tB)^{\dagger})^{\dagger} = (tA)^{\dagger}(tB)^{\dagger}\\ &= tA \wedge tB\ .\end{aligned}$$

Therefore $(T,E_S)$ is an F-pair.

It remains to show that $S \cong \mathcal{F}(T,E_S)$ .

We now define a map $\varphi : S \to \mathcal{F}(T,E_S)$ by

$$s\varphi = (s^{\dagger}, u_s)\ ,$$

where $u_s$ is the greatest element of the $\sigma_S$ –class containing s with respect to $\leq$ . It is clear that $\varphi$ is well defined. The conclusion of the last paragraph in effect is that $\varphi$ is onto.

It is clear that $\varphi$ is also one-one, since (for all $s, t \epsilon S$)

$$s\varphi = t\varphi \Longrightarrow (s^{\dagger}, u_s) = (t^{\dagger}, u_t)$$
$$\Longrightarrow s^{\dagger} = t^{\dagger},\ u_s = u_t\ .$$

Hence $(s,t) \epsilon \tilde{\mathcal{R}}_{E_{\mathcal{F}}} \cap\ \sigma_S$ and by Lemma 3.7, $s = t$ .

Taking $s, t \epsilon S$, we have

$$\begin{aligned} s\varphi t\varphi &= (s^{\dagger}, u_s)(t^{\dagger}, u_t) = (s^{\dagger} \wedge (u_s t^{\dagger}), u_s \circ u_t) \\ &= (s^{\dagger} \wedge (u_s t^{\dagger})^{\dagger}, u_s \circ u_t) = ((s^{\dagger} u_s t^{\dagger})^{\dagger}, u_{st}) \\ &= ((st^{\dagger})^{\dagger}, u_{st}) = ((st)^{\dagger}, u_{st}) \\ &= (st)\varphi. \end{aligned}$$

This completes the proof.☐

## Conflict of Interests

The authors declare that there is no conflict of interests.

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