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CONGRUENCES ON *-BISIMPLE TYPE A I-SEMIGROUPS

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Abstract. This paper studies congruences on a *-bisimple type A I-semigroup in the light of known results in the areas of inverse semigroups and type A ω-semigroups. It turns out that for a *-bisimple type A I-semigroup, we have the idempotent-separating congruence and the minimum cancellative monoid congruence.

Keywords: type A I-semigroups; idempotent-separating; cancellative monoid congruence; generalized Bruck-Reilly *-extension.

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1. Introduction and Summary

Let S be a semigroup and let $E\,(S)$ denote the set of its idempotents. As well known, $E\,(S)$ is partially ordered in the sense that: if $e, f \in E(S), e \leq f$ if and only if $ef = fe = e$. Let I denote the set of all integers and let $\mathbb{N}^0$ denote the set of nonnegative integers. A semigroup S is called an I-semigroup if and only if E (S) is order isomorphic to I under the reverse of the partial order. The *-bisimple type A I-semigroup have been classified by Shang and Wang in [9]. The case in which $\mathcal{D}^* = \widetilde{\mathcal{D}}$ was shown to be the generalized Bruck-Reilly *-extension of a cancellative monoid.

The main purpose of this paper is to present an explicit description of the congruences on *-bisimple type A I-semigroups.

This work is divided into 5 sections; section 2 contains some preliminaries and results concerning *-bisimple type A I-semigroups. The content of section 3 is the characterization of the idempotent-separating congruences on *-bisimple type A I-semigroups. A description of the minimum cancellative monoid congruence on *-bisimple type A I-semigroup is the subject of section 4 while the maximum idempotent-separating congruence is treated in section 5.

Now we recall some definitions which will be useful in the study. Terms not given here can be found in [4], [6] and [9], for more detailed knowledge.

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A semigroup S is said to be

• regular if all its elements are regular. Let S be a semigroup. An element $x \epsilon S$ is said to be regular if there exists $y \epsilon S$ such that $xyx = x$.
• unit regular if for each $x \epsilon S$ there exists a unit y of S for which $x = xyx$.
• An element $x \epsilon S$ is said to be coregular and y its coinverse if $x = xyx = yxy$. S is coregular if all its elements are coregular.
• orthodox if it is regular and the set $E(S)$ of idempotent elements of the semigroup S forms a subsemigroup.

Let S be a semigroup and $a, b \epsilon S$. The elements a and b in S are said to be $\mathcal{R}^*$-related written $a\, \mathcal{R}^*\, b$ if and only if a and b are related in $\mathcal{R}$ in some oversemigroup of S. Dually, we can define the relation $\mathcal{L}^*$. The following Lemma gives an alternative characterization of $\mathcal{R}^*$, the dual for the relation $\mathcal{L}^*$.

Lemma 1.1 [4]. Let S be a semigroup and $a, b \epsilon S$. Then $a\, \mathcal{R}^*\, b$ if and only if for all $x, y \epsilon S^1$, $xa = ya$ if and only if $xb = yb$.

As an easy but useful consequence of Lemma 1.1, we have

Lemma 1.2 [4]. Let S be a semigroup and $a, e^2 = e \epsilon S$. Then $a\, \mathcal{R}^*\, e$ if and only if for any $x, y \epsilon S^1$, $xa = ya$ implies $xe = ye$.

The join of the equivalence relations $\mathcal{R}^*$ and $\mathcal{L}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. Thus $a\, \mathcal{H}^* b$ if and only if $a\, \mathcal{R}^* b$ and $a\, \mathcal{L}^* b$. In general $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$ (see [4]). Basically, $a\, \mathcal{D}^* b$ if and only if there exist elements $x_1, x_2, \dots, x_{2n-1}$ in S such that $a\, \mathcal{L}^* x_1 \mathcal{R}^* x_2 \dots x_{2n-1} \mathcal{R}^*\, b$.
Following Fountain [5] a semigroup is an abundant semigroup if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class in S contain idempotents. An abundant semigroup S is an adequate [5] if $E(S)$ forms a semilattice. In an adequate semigroup every $\mathcal{L}^*$-class $\mathcal{R}^*$-class contains a unique idempotent.
Let a be an element of an adequate semigroup S, and a^* $(a^\dagger)$ denotes the unique idempotent in the $\mathcal{L}^*$-class L_a^* ($\mathcal{R}^*$-class R_a^*) containing a.
Fountain in [3] introduced the concept of right type A semigroup as special type of right PP monoids which is e-cancellable for an idempotent. He followed it in [4] with introduction of type a as an adequate semigroup satisfying certain internal conditions. An adequate semigroup S is a type A semigroup if $ea = a(ea)^*$ and $ae = (ae)^\dagger a$ for all $a \epsilon S$ and $e \epsilon E(S)$. If a type A semigroup S contain precisely one $\mathcal{D}^*$-class it is said to be a *-bisimple type A semigroup. *-bisimple type A semigroup has been studied in [1].

2. The *-Bisimple Type A I-Semigroup

In [9], Yu Shang and Limin Wang considered a similar construction of the one given earlier by Warne [10]. They used this construction to give the structure theorem for *-bisimple type A I-semigroups. We now introduce the construction.

Let M be a monoid with $\mathcal{H}_1^*$ as the $\mathcal{H}^*$-class which contain the identity element 1 of M. Let $S = M\times I\times I$ (where I denotes the set of all integers) with multiplication defined by the rule

$$(x,m,n)(y,p,q) = \begin{cases} \left(x.f_{n-p,p}^{-1}.y\theta^{n-p}.f_{n-p,q}, m, n+q-p\right) & \text{if } n \geq p \\ \left(f_{p-n,m}^{-1}.x\theta^{p-n}.f_{p-n,n}.y, m+p-n, q\right) & \text{if } n \leq p \end{cases}$$

where θ is an endomorphism of M with images in $\mathcal{H}_1^*$. θ^0 denotes the identity automorphism of M, and for $m \in \mathbb{N}^0, n \in I, f_{0,n} = 1$, the identity of M, and for $m > 0$, $f_{m,n} = u_{n+1}\theta^{m-1}.u_{n+2}\theta^{m-2}$ ... $u_{n+(m-1)}\theta.u_{n+m}$ and $f_{m,n}^{-1} = u_{n+m}^{-1}.u_{n+(m-1)}^{-1}\theta \ldots u_{n+2}^{-1}\theta^{m-2}.u_{n+1}^{-1}\theta^{m-1}$, where $\{u_n \mid n \in I\}$ is a collection of elements of H_1 with $u_n = 1$ if $n > 0$.

Under the above multiplication, $S = M\times I\times I$ is a semigroup (see [9]) and this semigroup is called the generalized Bruck-Reilly *-extensions of M determined by θ and it is usually denoted by $S = GBR^*(M,\theta)$.

The following results are proved in [9]. We give a sketch proof of (i), (iii) and (v)

Lemma 2.1. Let $(x,m,n),(y,p,q) \in GBR^*(M,\theta)$. Then

(i) $(x,m,n)\ \mathcal{L}^*\ (y,p,q)$ if and only if $n = q$ and $x\ \mathcal{L}^*(M)\ y$.

(ii) $(x,m,n)\ \mathcal{R}^*\ (y,p,q)$ if and only if $m = p$ and $x\ \mathcal{R}^*(M)\ y$.

(iii) $(x,m,n) \in E\big(GBR^*(M,\theta)\big)$ if and only if $m = n$ and $x \in E(M)$.

(iv) (x,m,n) has an inverse $(y,p,q) \in S$ if and only if $p = n, q = m$ and x is the inverse of $y \in M$.

(v) $GBR^*(M,\theta)$ is a type A semigroup if and only if M is a type A semigroup.

Proof. (i) Let $(x,m,n)\ \mathcal{L}^*\ (y,p,q)$. For $(e,0,0),(e,n,n) \in GBR^*(M,\theta)$ we have

$$(x,m,n)(e,0,0) = (x,m,n)(e,n,n),$$

$$(y,p,q)(e,0,0) = (y,p,q)(e,n,n).$$

Consequently,

$(y,p,q) = (y,p,q)(e,n,n)$. If $q < n$, this gives

$$(y,p,q) = \left(f_{n-q,p}^{-1}.y\theta^{n-q}.f_{n-q,q}.e, p+n-q, n\right).$$

Comparing the third coordinates gives $q = n$, which is a contradiction. Thus $q \geq n$.

Similarly, using the idempotent (e,q,q) we have

$$(x,m,n)(e,q,q)=\begin{cases}\left(x.f^{-1}_{n-q,q}.e\theta^{n-q}.f_{n-q,q},m,n+q-q\right) & \text{if } n\geq q\\ \left(f^{-1}_{q-n,m}.x\theta^{q-n}.f_{q-n,n}.e,m+q-n,q\right) & \text{if } n\leq q\end{cases}$$

So we deduce that $q\leq n$ and so $q=n$.

Conversely, let $n=q$. For any arbitrary elements $(v,i,j),(w,l,k)\,\epsilon\,GBR^*(M,\theta)$,

$$(x,m,n)(v,i,j)=(x,m,n)(w,l,k).$$

Suppose $n\geq i$ and $n\geq l$. Then

$$\left(x.f^{-1}_{n-i,i}.v\theta^{n-i}.f_{n-i,j},m,n+j-i\right)=\left(x.f^{-1}_{n-l,l}.w\theta^{n-l}.f_{n-l,k},m,n+k-l\right).$$

Comparing the first and the third coordinates gives

$x.f^{-1}_{n-i,i}.v\theta^{n-i}.f_{n-i,j}=x.f^{-1}_{n-l,l}.w\theta^{n-l}.f_{n-l,k}$ and $n+j-i=n+k-l$.

This implies

$$y.f^{-1}_{n-i,i}.v\theta^{n-i}.f_{n-i,j}=y.\ f^{-1}_{n-l,l}.w\theta^{n-l}.f_{n-l,k} \quad \text{and} \quad n+j-i=n+k-l.$$

Hence, $(y,p,n)(v,i,j)=(y,p,n)(w,l,k)$.

(ii) The proof is similar to the proof of (i).

(iii) Let $(x,m,n)\,\epsilon\,E\left(GBR^*(M,\theta)\right)$. Then

$$(x,m,n)=(x,m,n)(x,m,n)=$$
$$\begin{cases}\left(x.f^{-1}_{n-m,m}.x\theta^{n-m}.f_{n-m,n},m,n+n-m\right) & \text{if } n\geq m\\ \left(f^{-1}_{m-n,m}.x\theta^{m-n}.f_{m-n,n}.x,m+m-n,n\right) & \text{if } n\leq m\end{cases}$$

thus $m=n$ and $x^2=x$.

Conversely, let $m=n$ and $x\,\epsilon\,E(M)$. Then certainly $(x,m,n)(x,m,n)=(x,m,n)$. From which it follows that $(x,m,n)\,\epsilon\,E\left(GBR^*(M,\theta)\right)$.

(iv) The proof is clear.

(v) We only prove that $GBR^*(M,\theta)$ is right type A, as the proof that $GBR^*(M,\theta)$ is left type A is dual.

Let $(e,m,m),(e,n,n)\,\epsilon\,E\left(GBR^*(M,\theta)\right)$. Suppose that $m>n$. Then

$$\begin{aligned}(e,m,m),(e,n,n)&=(e.f^{-1}_{m-n,n}e\theta^{m-n}f_{m-n,n},m)\\&=(f^{-1}_{m-n,n}.e\theta^{m-n}.f_{m-n,n}.e,m,m)\\&=(e,n,n)(e,m,m).\end{aligned}$$

Thus the idempotents of $GBR^*(M,\theta)$ commute. So every $\mathcal{L}^*$ -class of $GBR^*(M,\theta)$ contain an idempotent.

Let $(x,p,q)\,\epsilon\,GBR^*(M,\theta)$. Suppose $m\geq p$. Then

$$(x,p,q)[(e,m,m)(x,p,q)]^* = (x,p,q)(e.f_{m-p,p}^{-1}.x\theta^{m-p}.f_{m-p,q}, m, m+q-p)^*$$
$$= (x,p,q)(e.f_{m-p,p}^{-1}.x\theta^{m-p}f_{m-p,q}, m+q-p, m+q-p)$$
$$= (e,m,m)(x,p,q).$$

Theorem 2.2 (Structure theorem)

Let $S = GBR^*(M,\theta)$ be the generalized Bruck-Reilly *-extensions of M determined by θ. Then S is a *-bisimple type A I-semigroup. Conversely, every *-bisimple type A I-semigroup is isomorphic to $GBR^*(M,\theta)$.

Proof. It is known that $S = GBR^*(M,\theta)$ is a type A semigroup. That S is *-bisimple follows from Lemma 2.1 (i) & (ii).

Next, let $e_m = (e,m,m)$ and $e_n = (e,n,n) \epsilon E(S)$. Then for $m \geq n$.

$$e_m e_n = (e,m,m)(e,n,n) = (e.f_{m-n,n}^{-1}.e\theta^{m-n}.f_{m-n,n}, m, m+n-n)$$
$$= (e,m,m) = e_m$$
$$= (e,n,n)(e,m,m) = e_n e_m$$

Thus $e_m \leq e_n$ if and only if $m \geq n$, which shows that $E(S)$ is a chain

$$\ldots > (e,-2,-2) > (e,-1,-1) > (e,0,0) > (e,1,1) > (e,2,2) > \cdots$$

Hence S is a *-bisimple type A I-semigroup. The converse of the proof is a routine check.

From Lemma 2.1(iv), we have the following result

Corollary 2.3. Let M be a monoid. Then $S = GBR^*(M,\theta)$ is regular if and only if M is regular.

The following results show some other properties of $S = GBR^*(M,\theta)$.

Proposition 2.4. Let $S = GBR^*(M,\theta)$. Then S is unit regular if and only if M is unit regular.

Proof. Let $S = GBR^*(M,\theta)$ be unit regular. Then for any $(x,m,n) \epsilon S$, there exists an element $(y,n,m) \epsilon G$ (where G is the group of units of $GBR^*(M,\theta)$) such that

$$(x,m,n)(y,n,m)(x,m,n) = (x,m,n).$$

By considering left-hand side of the equation, we get

$$(x,m,n)(y,n,m)(x,m,n) = \big((x,m,n)(y,n,m)\big)(x,m,n)$$
$$= \big(x.f_{n-n,n}^{-1}.y\theta^{n-n}.f_{n-n,m}, m, n+m-n\big)(x,m,n)$$
$$= (xy,m,m)(x,m,n) = (xyx,m,n).$$

Therefore we obtain $x = xyx$. Consequently, M is unit regular.

Conversely, let us suppose that M is unit regular. Then for $x \epsilon M$, there exists an element $x \epsilon G_M$ (where G_M is the group of units of M) such that obtain $x = xyx$. Now we need to show that for any $(x,m,n) \epsilon GBR^*(M,\theta)$, there exist an element $(y,p,q) \epsilon G_M$ such that

$$(x,m,n) = (x,m,n)(y,p,q)\ (x,m,n).$$

Here we take $p = n$, $q = m$, then we have $(x, m, n)\ (y, n, m)(x, m, n) = (xyx, m, n)$. Since we have $x = xyx$, for any $x \epsilon M, y \epsilon G_M$, we obtain $(x, m, n)(y, p, q)(x, m, n) = (x, m, n)$. Thus S is unit regular.

Proposition 2.5 Let M be a monoid. Then $M' = \{(x, m, m) \mid x \epsilon M, m \epsilon \mathbb{N}^0\} \leq GBR^*(M, \theta)$ is coregular if and only if M is coregular.

Proof. Let $M' \leq GBR^*(M, \theta)$ be coregular. Then for $(x, 0,0) \epsilon GBR^*(M, \theta)$, there exists an element $(y, n, n) \epsilon GBR^*(M, \theta)$ such that

$$\big((x, 0,0)(y, n, n)\big)(x, 0,0) = (xyx, n, n) = (x, 0,0) \qquad (1)$$

$$\big((y, n, n)(x, 0,0)\big)(y, n, n) = (yxy, n, n) = (x, 0,0) \qquad (2)$$

From (1) and (2), we have that $n = 0$, $xyx = x$ and $yxy = x$. Thus M is coregular.

Conversely, let M be coregular. Then there exists $y \epsilon M$, with $xyx = x$ and $yxy = x$. Thus for $(x, m, n) \epsilon GBR^*(M, \theta)$, we have

$$\begin{aligned}\big((x, m, n)(y, m, m)\big)(x, m, m) &= (xy, m, m)(x, m, m)\\ &= (xyx, m, m)\\ &= (x, m, m).\end{aligned}$$

$$\begin{aligned}\big((y, m, m)(x, m, m)\big)(y, m, m) &= (yx, m, m)(y, m, m)\\ &= (yxy, m, m)\\ &= (x, m, m).\end{aligned}$$

Therefore, $M' = \{(x, m, m) \mid x \epsilon M, m \epsilon \mathbb{N}^0\} \leq GBR^*(M, \theta)$ is coregular.

It is important to note that not all regular semigroups are coregular. This is shown in the example below.

Example 2.6. Let X and Y be non-empty sets and set $T = X \times Y$ with the binary operation

$$(x, y)(u, v) = (x, v), \text{ for all } x, u \epsilon X,\ y, v \epsilon Y.$$

It can be easily seen that T is a semigroup. This semigroup is called a rectangular band. T is also regular, since for $(x, y), (u, v) \epsilon T$ we have $(x, y)(u, v)(x, y) = (x, y)$.

To show that T is not coregular, let $(x, y), (u, v) \epsilon T$, then we have

$$(x, y)(u, v)(x, y) = (x, y),$$

$$(u, v)(x, y)(u, v) = (u, v).$$

So $(x, y) \neq (u, v)$. Thus T is not coregular.

In the next theorem, we consider the orthodox property of $GBR^*(M, \theta)$

Theorem 2.7. Let $S = GBR^*(M, \theta)$. Then S is orthodox if and only if M is orthodox.

Proof. Let $GBR^*(M, \theta)$ be orthodox. By Corollary 2.3, we know that M is regular. Then it remains to show that $E(M)$ is a subsemigroup of M. In particular for each $e, e' \epsilon E(M)$,

$$(e,m,m)(e',m,m) = \left(e.f_{m-m,m}^{-1}.e'\theta^{m-m}.f_{m-m,m},m,m+m-m\right)$$
$$= (ee',m,m)$$

is an idempotent of $GBR^*(M,\theta)$ and so $(ee')^2 = ee'$. Hence M is orthodox.

Conversely, let M be orthodox. Then M is regular, and $E(M)$ is a subsemigroup of M. By Corollary 2.3, we know that $GBR^*(M,\theta)$ is regular.

Next, we show that $(e,m,m)(e',n,n) \in E\left(GBR^*(M,\theta)\right)$. From the multiplication $(e,m,m)(e',n,n)$, we have the following cases:

Case (1): If $m \geq n$, we have

$$(e,m,m)(e',n,n) = \left(\left(e.f_{m-n,n}^{-1}\right).\left(e'\theta^{m-n}.f_{m-n,n}\right),m,m+n-n\right)$$
$$= \left(\left(e.f_{m-n,n}^{-1}\right).\left(e'\theta^{m-n}.f_{m-n,n}\right),m,m\right).$$

Since $e,e' \in E(M)$, we deduce that $e.f_{m-n,n}^{-1}, e'\theta^{m-n}.f_{m-n,n} \in E(M)$. But the idempotents in M are commutative, consequently

$$\left(e.f_{m-n,n}^{-1}\right).\left(e'\theta^{m-n}.f_{m-n,n}\right) = \left(e'\theta^{m-n}.f_{m-n,n}\right).\left(e.f_{m-n,n}^{-1}\right).$$

So $\left(e'\theta^{m-n}.f_{m-n,n}\right),\left(e.f_{m-n,n}^{-1}\right) \in E\left(GBR^*(M,\theta)\right)$. Therefore $E\left(GBR^*(M,\theta)\right)$ is a subsemigroup of $GBR^*(M,\theta)$.

Case (2): If $m \leq n$, we have

$$(e,m,m)(e',n,n) = \left(\left(f_{n-m,n}^{-1}.e\theta^{n-m}\right).\left(f_{n-m,m}.e'\right),m+n-m,n\right)$$
$$= \left(\left(f_{n-m,m}^{-1}.e\theta^{n-m}\right).\left(f_{n-m,m}.e'\right),n,n\right).$$

From here, since $\left(f_{n-m,m}^{-1}.e\theta^{n-m}\right),\left(f_{n-m,m}.e'\right) \in E(M)$ and the idempotents in M are commutative, we deduce that $E\left(GBR^*(M,\theta)\right)$ is a subsemigroup of $GBR^*(M,\theta)$.

The connection between the Green's *-relations and congruences lies on the fact that $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. It can be easily verified that $\mathcal{H}^*$ is a congruence on $S = GBR^*(M,\theta)$. In our next section, we shall characterize the congruences on $S = GBR^*(M,\theta)$.

3. Idempotent-separating congruences

The following terms adopted from [8] will be used in the description of congruences on *-bisimple type A *I*-semigroups.

Definition 3.1. Let $S = GBR^*(M,\theta)$ be a *-bisimple type A I-semigroup where $\theta : M \to \mathcal{H}_1^*$. Let $\mathcal{H}^* = \rho$ be a congruence on *S*. Let us use $\rho(M)$ to denote the congruence on *M* induced by ρ, via the restriction of ρ to the monoid $\{(x,0,0) : x \in M\}$.

Definition 3.2. A congruence γ on M is said to be θ-admissible if $x\,\gamma\,y$ implies $x\theta\,\gamma\,y\theta$, for any $x,y \in M$.

A typical idempotent-separating congruence on $S = GBR^*(M,\theta)$ is characterized as follows:

Theorem 3.3. Let $S = GBR^*(M,\theta)$ be a *-bisimple type A I-semigroup and let ρ be a congruence on $S = GBR^*(M,\theta)$. Then $\rho(M)$ is θ-admissible. Conversely, if γ is any θ-admissible congruence on M, then the relation on S defined by

$$[(x,m,n)(y,p,q)] \in \gamma(S) \text{ if and only if } m = p, n = q \text{ and } (x,y) \in \gamma$$

is an idempotent-separating congruence.

Proof. Suppose $x\,\rho(M)\,y$. Then we have that $(x,0,0)\,\rho\,(y,0,0)$.
Consequently,

$$(x,0,0)(e,1,1)\rho\,(y,0,0)(e,1,1).$$

But $(x,0,0)(e,1,1) = \left(f_{1,0}^{-1}.x\theta f_{1,0}.e,1,1\right)$ and $(y,0,0)(e,1,1) = \left(f_{1,0}^{-1}.y\theta f_{1,0}.e,1,1\right)$.
Thus $\left(f_{1,0}^{-1}.x\theta f_{1,0}.e,1,1\right)\rho\left(f_{1,0}^{-1}.y\theta f_{1,0}.e,1,1\right) = (x\theta,1,1)\,\rho\,(y\theta,1,1)$.
Since $(x\theta,1,1)\,\rho\,(y\theta,1,1)$, then $(x\theta,1,1) = (y\theta,1,1)$.
Also we have $(e,0,1)(x\theta,1,1)(e,1,0)\,\rho\,(e,0,1)(y\theta,1,1)(e,1,0)$.
But $(e,0,1)(x\theta,1,1)(e,1,0) = (x\theta,0,0)$ and $(e,0,1)(y\theta,1,1)(e,1,0) = (y\theta,0,0)$.
Thus $(x\theta,0,0)\,\rho\,(y\theta,0,0)$. Since $(x\theta,0,0)\,\rho\,(y\theta,0,0)$, then $x\theta\,\rho(M)\,y\theta$.
Conversely, let γ be a θ-admissible congruence on M. We first show that $\gamma(S)$ is an equivalence relation.
$[(x,m,n)(x,m,n)] \in \gamma(S)$ since $(x,x) \in \gamma$. Thus $\gamma(S)$ is reflexive. By definition, $\gamma(S)$ is symmetric.
To show transitivity, let $(x,m,n)\,\gamma(S)\,(y,p,q)$ and $(y,p,q)\,\gamma(S)\,(z,i,j)$ for all $(x,m,n),(y,p,q),(z,i,j) \in S$. Then we have $m = p$, $n = q$, $(x,y) \in \gamma$ and $p = i$, $q = j$, $(y,z) \in \gamma$.
Consequently, $m = i$, $n = j$. Hence $(x,z) \in \gamma$, which means that $\gamma(S)$ is transitive.
Next is to show that $\gamma(S)$ is a congruence. Now let $a = (x,m,n)$, $b = (y,p,q)$. That $\gamma(S)$ is a congruence entails showing that

$$a\ \gamma(S)\ b \text{ implies } ax\ \gamma(S)\ bx \quad \text{(for right congruence)}$$
$$a\ \gamma(S)\ b \text{ implies } xa\ \gamma(S)\ xb \quad \text{(for left congruence)}$$

$\forall\ x = (z,k,l) \in S = GBR^*(M,\theta)$.
Consequently,

$$ax = (x,m,n)(z,k,l) = \begin{cases} \left(x.f_{n-k,k}^{-1}.z\theta^{n-k}.f_{n-k,l},m,n+l-k\right) & \text{if } n \geq k \\ \left(f_{k-n,m}^{-1}.x\theta^{k-n}.f_{k-n,n}.z,m+k-n,l\right) & \text{if } n \leq k \end{cases}$$

$$bx = (y,p,q)(z,k,l) = \begin{cases} \left(y.f_{q-k,k}^{-1}.z\theta^{q-k}.f_{q-k,l},p,q+l-k\right) & \text{if } q \geq k \\ \left(f_{k-q,p}^{-1}.y\theta^{k-q}.f_{k-q,q}.z,p+k-q,l\right) & \text{if } q \leq k \end{cases}$$

So if $(x,m,n)\,\gamma(S)\,(y,p,q)$, then

$$(x,m,n)(z,k,l)\,\gamma(S)\,(y,p,q)(z,k,l) =$$

$$\begin{cases} (x.f_{n-k,k}^{-1}.z\theta^{n-k}.f_{n-k,l}, m, n+l-k) & \text{if } n \geq k \\ (f_{k-n,m}^{-1}.x\theta^{k-n}.f_{k-n,n}.z, m+k-n, l) & \text{if } n \leq k \end{cases}$$

$$\gamma(S) \begin{cases} (y.f_{q-k,k}^{-1}.z\theta^{q-k}.f_{q-k,l}, p, q+l-k) & \text{if } q \geq k \\ (f_{k-q,p}^{-1}.y\theta^{k-q}.f_{k-q,q}.z, p+k-q, l) & \text{if } q \leq k \end{cases}$$

But $(x,m,n)\,\gamma(S)\,(y,p,q)$ if and only if $m = p, n = q$ and $x\,\gamma\,y$.

Thus, we have that

$$\begin{cases} (x.f_{n-k,k}^{-1}.z\theta^{n-k}.f_{n-k,l}, m, n+l-k) & \text{if } n \geq k \\ (f_{k-n,m}^{-1}.x\theta^{k-n}.f_{k-n,n}.z, m+k-n, l) & \text{if } n \leq k \end{cases}$$

$$\gamma(S) \begin{cases} (y.f_{n-k,k}^{-1}.z\theta^{n-k}.f_{n-k,l}, m, n+l-k) & \text{if } n \geq k \\ (f_{k-n,m}^{-1}.y\theta^{k-n}.f_{k-n,n}.z, m+k-n, l) & \text{if } n \leq k \end{cases}$$

Hence $\gamma(S)$ is a right congruence.
That $\gamma(S)$ is a left congruence follows similarly. Thus $\gamma(S)$ is a congruence.
Futhermore, $(e,m,m)\,\gamma(S)\,(e,n,n) \Longrightarrow m = n$ which implies that $(e,m,m) = (e,n,n)$. Thus $\gamma(S)$ is an idempotent-separating congruence.

Remark 3.4. $\mathcal{H}^*$ is an idempotent-separating congruence on $S = GBR^*(M,\theta)$ and $\gamma(S) \subseteq \mathcal{H}^*$.

4. Minimum cancellative monoid congruence

The idea of the minimum cancellative monoid congruence is to obtain a congruence σ on S, a type A semigroup with respect to which S/σ is cancellative.
Here we will determine the minimum cancellative monoid congruence on $S = GBR^*(M,\theta)$, as follows:

Now let $(h,m,n), (x,i,j) \in S = GBR^*(M,\theta)$.Define a relation σ on $S = GBR^*(M,\theta)$ by the rule

$$(h,m,n)\,\sigma\,(x,i,j) \text{ if and only if } m-n = i-j,\ h\theta^i = x\theta^m \text{ and } x\theta^i = h\theta^m.$$

Lemma 4.1. σ is a congruence on S.

Proof. That σ is symmetric and reflexive is known. To show that σ is transitive, let $(h,m,n)\,\sigma\,(x,i,j)$ and $(x,i,j)\,\sigma\,(y,p,q)$ for $(h,m,n), (x,i,j), (y,p,q) \in S$. Then $m-n = i-j$ and $i-j = p-q$ and so $m-n = p-q$.
Consequently, $x\theta^i = h\theta^m$ and $y\theta^p = x\theta^i$ implies $y\theta^p = h\theta^m$.

Also $h\theta^i = x\theta^m$ and $x\theta^p = y\theta^i$ implies that $h\theta^i = (y\theta^{i-p})\theta^m = y\theta^{i-p+m}$. Then $h\theta^{i+p} = y\theta^{i-p+m+p} = y\theta^{i+m}$. Hence $h\theta^p = y\theta^m$ which shows that σ is transitive.

Next we show that σ is a congruence. Now let $u = (h, m, n)$, $v = (x, i, j)$. That σ is a congruence we show that σ is both a left and right congruence. That is

$$\forall\ z \in S, \qquad u\,\sigma\,v \Longrightarrow uz\,\sigma\,vz \qquad \text{(for right congruence)}$$

and

$$\forall\ z \in S \qquad u\,\sigma\,v \Longrightarrow zu\,\sigma\,zv \qquad \text{(for left congruence).}$$

Let $z = (y, p, q) \in S$. Then

$$uz = (h,m,n)(y,p,q) = \begin{cases} \left(h.f_{n-p,p}^{-1}.y\theta^{n-p}.f_{n-p,q}, m, n+q-p\right) & \text{if } n \geq p \\ \left(f_{p-n,n}^{-1}.h\theta^{p-n}.f_{p-n,n}.y, m+p-n, q\right) & \text{if } n \leq p \end{cases}$$

and

$$vz = (x,i,j)(y,p,q) = \begin{cases} \left(x.f_{j-p,p}^{-1}.y\theta^{j-p}.f_{j-p,q}, i, j+q-p\right) & \text{if } j \geq p \\ \left(f_{p-j,j}^{-1}.x\theta^{p-j}.f_{p-j,j}.y, i+p-j, q\right) & \text{if } j \leq p \end{cases}$$

Evidently if $(h,m,n)\,\sigma\,(x,i,j)$, we have

$$m-(n+q-p) = (m-n)+(p-q) \quad \text{and} \quad i-(j+q-p) = (i-j)+(p-q)$$

$$m+p-n-q = (m-n)+(p-q) \quad \text{and} \quad i+p-j-q = (i-j)+(p-q).$$

But $m-n = i-j$ and so $(m-n)+(p-q) = (i-j)+(p-q)$.

For the first outer part, we know from definition that $h\theta^i = x\theta^m$ and $h\theta^n = x\theta^j$. It suffices to show that $(h\theta^{p-n}.y)\theta^{i+p-j} = \left(x\theta^{p-j}.y\right)\theta^{m+p-n}$.

Considering the left hand side of the equation we have

$$\begin{aligned}(h\theta^{p-n}.y)\theta^{i+p-j} &= h\theta^{p+p+i-n-j}.y\theta^{p-j+i} \\ &= h\theta^{i+(p+p)-j-n}.y\theta^{i+p-j} \\ &= \left(h\theta^i\right)\theta^{p-j-n+p}.y\theta^{p+(i-j)}\end{aligned}$$

But $i-j = m-n$ and $h\theta^i = x\theta^m$.

$$\begin{aligned}\text{Therefore,} \quad \left(h\theta^i\right)\theta^{p-j-n+p}.y\theta^{p+(i-j)} &= (x\theta^m)\theta^{p-j-n+p}.y\theta^{p+(m-n)} \\ &= x\theta^{m+p+p-j-n}.y\theta^{p+m-n} \\ &= \left(x\theta^{p-j}.y\right)\theta^{m+p-n}\end{aligned}$$

as required.

Hence σ is a right congruence. That σ is a left congruence follows similarly. Consequently σ is a congruence.

Lemma 4.2. σ is a cancellative monoid.

Proof. Since $(e,m,m)\ \sigma\ (e,n,n)$ for $m,n \in I$, it follows that the class of σ containing the idempotents is the identity element for S/σ. Thus $(1,m,n)\sigma\ (y,p,q)\sigma = (y,p,q)\sigma$ and hence S/σ is a monoid.

Next is to show that S/σ is cancellative. Now let $u = (h,m,n), v = (x,i,j)$.

That S/σ is cancellative entails showing that for all $z \in S$,

$$u\sigma\ z\sigma = v\sigma\ z\sigma \Longrightarrow u\sigma = v\sigma \qquad \text{(for right cancellative)}$$

and

$$z\sigma\ u\sigma = z\sigma\ v\sigma \Longrightarrow u\sigma = v\sigma \qquad \text{(for left cancellative).}$$

Let $z = (y,p,q) \in S$. Then

$$u\sigma\ z\sigma = (h,m,n)\sigma\ (y,p,q)\sigma = (x,i,j)\sigma\ (y,p,q)\sigma$$
$$= v\sigma\ z\sigma\ .$$

Consequently,

$$(h,m,n)\sigma\ (y,p,q)\sigma = (x,i,j)\sigma\ (y,p,q)\sigma$$
$$\Leftrightarrow [(h,m,n)(y,p,q)]\sigma = [(x,i,j)(y,p,q)]\sigma$$
$$\Leftrightarrow \begin{cases} \left(h.f_{n-p,p}^{-1}.y\theta^{n-p}.f_{n-p,q}, m, n+q-p\right) & \text{if } n \geq p \\ \left(f_{p-n,n}^{-1}.h\theta^{p-n}.f_{p-n,n}.y, m+p-n, q\right) & \text{if } n \leq p \end{cases} \times\sigma$$
$$= \begin{cases} \left(x.f_{j-p,p}^{-1}.y\theta^{j-p}.f_{j-p,q}, i, j+q-p\right) & \text{if } j \geq p \\ \left(f_{p-j,i}^{-1}.x\theta^{p-j}.f_{p-j,j}.y, i+p-j, q\right) & \text{if } j \leq p \end{cases} \times\sigma$$
$$\Leftrightarrow\ m-(n+q-p) = i-(y+q-p),\ (m+p-n)-q = (i+p-j)-q$$

and

$$(h\theta^{p-n}.y)\theta^{i+p-j} = \left(x\theta^{p-j}.y\right)\theta^{m+p-n}$$
$$\Leftrightarrow\ (m-n)+(p-q) = (i-j)+(p-q)$$

and

$$h\theta^{p-n+(i-j)+p}.y\theta^{p+(i-j)} = x\theta^{p-j+(m-n)+p}.y\theta^{p+(m-n)}$$
$$\Leftrightarrow\ \ m-n = i-j \text{ and } (h\theta^{i})^{p+p-n-j} = (x\theta^{m})\theta^{p+p-n-j}$$
$$\Leftrightarrow\ \ m-n = i-j \text{ and } h\theta^{i} = x\theta^{m}$$
$$\Leftrightarrow\ \ (h,m,n)\ \sigma\ (x,i,j)$$

which shows that S/σ is right cancellative. That S/σ is left cancellative follows similarly, and we conclude that S/σ is cancellative.

Lemma 4.3 σ is a minimum congruence.

Proof. Let Γ be any other cancellative monoid congruence. Then $(1,n,n)\ \Gamma\ (1,0,0)$ for all $n \in I$. Suppose $(h,m,n)\ \sigma\ (x,i,j)$. Then we have from $(h,m,n)(1,p,p) = (x,i,j)(1,p,p)$ for some $p \in I$,

$$\Rightarrow \begin{cases} (h. f_{n-p,p}^{-1}. 1\theta^{n-p}. f_{n-p,p}, m, n+p-p) & \text{if } n \geq p \\ (f_{p-n,m}^{-1}. h\theta^{p-n}. f_{p-n,n}. 1, m+p-n, p) & \text{if } n \leq p \end{cases}$$

$$= \begin{cases} (x. f_{j-p,p}^{-1}. 1\theta^{j-p}. f_{j-p,p}, i, j+p-p) & \text{if } j \geq p \\ (f_{p-j,i}^{-1}. x\theta^{p-j}. f_{p-j,j}. 1, i+p-j, p) & \text{if } j \leq p \end{cases}$$

$$\Rightarrow \begin{cases} (h, m, n) & \text{if } n \geq p \\ (h\theta^{p-n}, m+p-n, p) & \text{if } n \leq p \end{cases} = \begin{cases} (x, i, j) & \text{if } j \geq p \\ (x\theta^{p-j}, i+p-j, p) & \text{if } j \leq p \end{cases}$$

But $(1, n, n)\ \Gamma\ (1,0,0)$, so $(h, m, n)(1, p, p)\ \Gamma\ (h, m, n)$.
Also, $(x, i, j)(1, p, p)\ \Gamma\ (x, i, j)$. Therefore $(h, m, n)\ \Gamma\ (x, i, j)$. Thus $\sigma \subseteq \Gamma$.

Combining Lemma 4.1 to Lemma 4.3, we have proved the following theorem:

Theorem 4.4. Let $S = GBR^*(M, \theta)$ be a *-bisimple type A I-semigroup and let σ be defined on S by the rule that $(h, m, n)\ \sigma\ (x, i, j)$ if and only if $m - n = i - j$, $h\theta^i = x\theta^m$ and $x\theta^i = h\theta^m$. Then σ is the minimum cancellative monoid congruence on S.

5. The congruence μ

Here we will determine the maximum congruence μ on $S = GBR^*(M, \theta)$ contained in $\mathcal{H}^*$ by utilizing the approach of El-Qallali and Fountain [2].

Now let (e, m, m) and (e, n, n) be the idempotents in the $\mathcal{R}^*$-class and $\mathcal{L}^*$-class respectively. We define the relations μ_R and μ_L on $S = GBR^*(M, \theta)$ as follows:

$(x, m, n)\ \mu_L\ (y, p, q)$ if and only if $(e, n, n)(x, m, n)\ \mathcal{L}^*\ (e, n, n)(y, p, q)$, $m - n = p - q$, $x\theta^{n-m} = y\theta^{n-p}$ and $e\theta^{m-n}. x = e\theta^{p-n}. y$.

$(x, m, n)\ \mu_R\ (y, p, q)$ if and only if $(x, m, n)(e, m, m)\ \mathcal{R}^*\ (y, p, q)(e, m, m)$, $m - n = p - q$, $x\theta^{m-n} = y\theta^{m-q}$ and $x. e\theta^{n-m} = y. e\theta^{q-m}$.

Consequently,

$$\mu = \mu_L \cap \mu_R .$$

With the above relation, we obtain the following results

Proposition 5.1. Let $S = GBR^*(M, \theta)$. Then μ_L is the maximum congruence on S contained in $\mathcal{L}^*$.

Proof. Obviously, μ_L is an equivalence on S. Since $\mathcal{L}^*$ is a right congruence on S, μ_L is right compatible under the semigroup multiplication.

Next is to show that μ_L is also left compatible under the semigroup multiplication. Now let $(x, m, n), (y, p, q), (e, 0,0) \epsilon\ S$. That μ_L is left compatible entails showing that

$(e, n, n)(x, m, n)\ \mathcal{L}^*\ (e, n, n)(y, p, q)$ implies $(e, 0,0)(e, n, n)(x, m, n)\ \mathcal{L}^*\ (e, 0,0)(e, n, n)(y, p, q)$.

Thus we have

$$(e,0,0)(e,n,n)(x,m,n) = (e\theta^n,n,n)(x,m,n),$$

and

$$(e,0,0)(e,n,n)(y,p,q) = (e\theta^n,n,n)(y,p,q).$$

Consequently,

$$(e\theta^n,n,n)(x,m,n) = \begin{cases} (e\theta^n.x\theta^{n-m},n,n+n-m) & \text{if } n \geq m \\ (e\theta^m.x,m,n) & \text{if } n \leq m \end{cases}$$

$$(e\theta^n,n,n)(y,p,q) = \begin{cases} (e\theta^n.y\theta^{n-p},n,n+q-p) & \text{if } n \geq p \\ (e\theta^m.y,p,q) & \text{if } n \leq p \end{cases}$$

From $(e\theta^n,n,n)(x,m,n)$ and $(e\theta^n,n,n)(y,p,q)$, it follows that

$$n-(n+n-m) = m-n \text{ and } n-(n+q-p) = p-q.$$

It follows from definition that $m-n = p-q$.
For the first outer part of $(e\theta^n,n,n)(x,m,n)$ and $(e\theta^n,n,n)(y,p,q)$, we have

$$e\theta^n.x\theta^{n-m} = e\theta^n.y\theta^{n-p} \quad \text{(since from definition, } x\theta^{n-m} = y\theta^{n-p}\text{)}$$

$$e\theta^m.x = e\theta^p.y \quad \text{(since from definition, } e\theta^{m-n}.x = e\theta^{p-n}.y\text{)}.$$

Thus $(x,m,n)\ \mu_L\ (y,p,q)$ implies $(e,0,0)(x,m,n)\ \mu_L\ (e,0,0)(y,p,q)$.
To show that $\mu \subseteq \mathcal{L}^*$, we now consider the elements $(x,m,n),(y,p,q) \in GBR^*(M,\theta)$ such that $(x,m,n)\ \mu_L\ (y,p,q)$. But $(x,m,n)^* = (y,p,q)^*$ which implies that $(x,m,n)\ \mathcal{L}^*\ (y,p,q)$.
Now let ρ be a congruence on $GBR^*(M,\theta)$ such that $\rho \subseteq \mathcal{L}^*$. If $(x,m,n)\ \rho\ (y,p,q)$, then for any $(e,n,n) \in S$, $(e,n,n)(x,m,n)\ \rho\ (e,n,n)(y,p,q)$ so that $(e,n,n)(x,m,n)\ \mathcal{L}^*\ (e,n,n)(y,p,q)$, that is $(x,m,n)\ \mu_L\ (y,p,q)$ and whence $\rho \subseteq \mu_L$.

Proposition 5.2. Let $S = GBR^*(M,\theta)$. Then μ_R is the maximum congruence on S contained in $\mathcal{R}^*$.

Proof. The proof is similar to the proof of Proposition 3.1.

An immediate consequence of Proposition 3.1 and Proposition 3.2 is the following

Theorem 5.3. Let S be a *-bisimple type A I-semigroup. Then μ is the maximum congruence on S contained in $\mathcal{H}^*$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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