CONGRUENCES ON *-BISIMPLE TYPE A I-SEMIGROUPS

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Abstract. This paper studies congruences on a *-bisimple type A I-semigroup in the light of known results in the areas of inverse semigroups and type A ω-semigroups. It turns out that for a *-bisimple type A I-semigroup, we have the idempotent-separating congruence and the minimum cancellative monoid congruence.

Keywords: type A I-semigroups; idempotent-separating; cancellative monoid congruence; generalized Bruck-Reilly *-extension.

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1. Introduction and Summary

Let S be a semigroup and let $E(S)$ denote the set of its idempotents. As well known, $E(S)$ is partially ordered in the sense that: if $e, f \in E(S), e \leq f$ if and only if $ef = fe = e$. Let $I$ denote the set of all integers and let $\mathbb{N}^0$ denote the set of nonnegative integers. A semigroup $S$ is called an I-semigroup if and only if $E(S)$ is order isomorphic to $I$ under the reverse of the partial order. The *-bisimple type A I-semigroup have been classified by Shang and Wang in [9]. The case in which $D^* = \bar{D}$ was shown to be the generalized Bruck-Reilly *-extension of a cancellative monoid.

The main purpose of this paper is to present an explicit description of the congruences on *-bisimple type A I-semigroups.

This work is divided into 5 sections; section 2 contains some preliminaries and results concerning *-bisimple type A I-semigroups. The content of section 3 is the characterization of the idempotent-separating congruences on *-bisimple type A I-semigroups. A description of the minimum cancellative monoid congruence on *-bisimple type A I-semigroup is the subject of section 4 while the maximum idempotent-separating congruence is treated in section 5.

Now we recall some definitions which will be useful in the study. Terms not given here can be found in [4], [6] and [9], for more detailed knowledge.

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A semigroup $S$ is said to be

- regular if all its elements are regular. Let $S$ be a semigroup. An element $x \in S$ is said to be regular if there exists $y \in S$ such that $xyx = x$.
- unit regular if for each $x \in S$ there exists a unit $y$ of $S$ for which $x = xyx$.
- An element $x \in S$ is said to be coregular and $y$ its coinverse if $x = xyx = yxy$. $S$ is coregular if all its elements are coregular.
- orthodox if it is regular and the set $E(S)$ of idempotent elements of the semigroup $S$ forms a subsemigroup.

Let $S$ be a semigroup and $a, b \in S$. The elements $a$ and $b$ in $S$ are said to be $\mathcal{R}^*$-related written $a \mathcal{R}^* b$ if and only if $a$ and $b$ are related in $\mathcal{R}$ in some oversemigroup of $S$. Dually, we can define the relation $\mathcal{L}^*$. The following Lemma gives an alternative characterization of $\mathcal{R}^*$, the dual for the relation $\mathcal{L}^*$.

**Lemma 1.1** [4]. Let $S$ be a semigroup and $a, b \in S$. Then $a \mathcal{R}^* b$ if and only if for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$.

As an easy but useful consequence of Lemma 1.1, we have

**Lemma 1.2** [4]. Let $S$ be a semigroup and $a, e^2 = e \in S$. Then $a \mathcal{R}^* e$ if and only if for any $x, y \in S^1$, $xa = ya$ implies $xe = ye$.

The join of the equivalence relations $\mathcal{R}^*$ and $\mathcal{L}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. Thus $a \mathcal{H}^* b$ if and only if $a \mathcal{R}^* b$ and $a \mathcal{L}^* b$. In general $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$ (see [4]). Basically, $a \mathcal{D}^* b$ if and only if there exist elements $x_1, x_2, \ldots, x_{2n-1}$ in $S$ such that $a \mathcal{L}^* x_1 \mathcal{R}^* x_2 \ldots x_{2n-1} \mathcal{R}^* b$.

Following Fountain [5] a semigroup is an abundant semigroup if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class in $S$ contain idempotents. An abundant semigroup $S$ is an adequate [5] if $E(S)$ forms a semilattice. In an adequate semigroup every $\mathcal{L}^*$-class $\mathcal{R}^*$-class contains a unique idempotent.

Let $a$ be an element of an adequate semigroup $S$, and $a^* (a^\dagger)$ denotes the unique idempotent in the $\mathcal{L}^*$-class $L_a^*$ ($\mathcal{R}^*$-class $R_a^*$) containing $a$.

Fountain in [3] introduced the concept of right type A semigroup as special type of right PP monoids which is $e$-cancellable for an idempotent. He followed it in [4] with introduction of type $a$ as an adequate semigroup satisfying certain internal conditions. An adequate semigroup $S$ is a type A semigroup if $ea = a(ea)^*$ and $ae = (ae)^\dagger a$ for all $a \in S$ and $e \in E(S)$. If a type A semigroup $S$ contain precisely one $\mathcal{D}^*$-class it is said to be a $^*$-bisimple type A semigroup. $^*$-bisimple type A semigroup has been studied in [1].

2. The $^*$-Bisimple Type A I-Semigroup
In [9], Yu Shang and Limin Wang considered a similar construction of the one given earlier by Warne [10]. They used this construction to give the structure theorem for \(^*\)-bisimple type A I-semigroups. We now introduce the construction.

Let \( M \) be a monoid with \( \mathcal{H}_1^* \) as the \( \mathcal{H}^* \)-class which contain the identity element 1 of \( M \). Let \( S = M \times I \times I \) (where I denotes the set of all integers) with multiplication defined by the rule

\[
(x, m, n)(y, p, q) = \begin{cases} 
(x, f_n^{-1}, p, y\theta^{n-p} \cdot f_{n-p, q}, m, n + q - p) & \text{if } n \geq p \\
(f_{p-n, m} \cdot x\theta^{p-n}, f_p - n, y, m + p - n, q) & \text{if } n \leq p
\end{cases}
\]

where \( \theta \) is an endomorphism of \( M \) with images in \( \mathcal{H}_1^* \). \( \theta^0 \) denotes the identity automorphism of \( M \), and for \( m \in \mathbb{N}^0, n \in I, f_{0,n} = 1 \), the identity of \( M \), and for \( m > 0, f_{m,n} = u_{n+1} \theta^{m-1} \cdot u_{n+2} \theta^{m-2} \ldots u_{n+(m-1)} \theta \cdot u_{n+m} \) and \( f_{m, n}^{-1} = u^{-1}_{n+1} \cdot u^{-1}_{n+2} \theta^{-m-2} \cdot u^{-1}_{n+m} \theta^{-m-1} \), where \( \{ u_n \mid n \in I \} \) is a collection of elements of \( H_1 \) with \( u_n = 1 \) if \( n > 0 \).

Under the above multiplication, \( S = M \times I \times I \) is a semigroup (see [9]) and this semigroup is called the generalized Bruck-Reilly \( * \)-extensions of \( M \) determined by \( \theta \) and it is usually denoted by \( S = \text{GBR}^*(M, \theta) \).

The following results are proved in [9]. We give a sketch proof of (i), (iii) and (v)

**Lemma 2.1.** Let \((x, m, n), (y, p, q) \in \text{GBR}^*(M, \theta)\). Then

(i) \((x, m, n) \mathcal{L}^* (y, p, q) \text{ if and only if } n = q \text{ and } x \mathcal{L}^*(M) y\).

(ii) \((x, m, n) \mathcal{R}^* (y, p, q) \text{ if and only if } m = p \text{ and } x \mathcal{R}^*(M) y\).

(iii) \((x, m, n) \in E(\text{GBR}^*(M, \theta)) \text{ if and only if } m = n \text{ and } x \in E(M)\).

(iv) \((x, m, n) \text{ has an inverse } (y, p, q) \in S \text{ if and only if } p = n, q = m \text{ and } x \text{ is the inverse of } y \in M\).

(v) \( \text{GBR}^*(M, \theta) \) is a type A semigroup if and only if \( M \) is a type A semigroup.

**Proof.** (i) Let \((x, m, n) \mathcal{L}^* (y, p, q)\). For \((e, 0, 0), (e, n, n) \in \text{GBR}^*(M, \theta)\) we have

\[
(x, m, n)(e, 0, 0) = (x, m, n)(e, n, n),
\]

\[
(y, p, q)(e, 0, 0) = (y, p, q)(e, n, n).
\]

Consequently,

\[
(y, p, q) = (y, p, q)(e, n, n). \text{ If } q < n, \text{ this gives}
\]

\[
(y, p, q) = (f_{n-q, p} \cdot y\theta^{n-q} \cdot f_{n-q, q} \cdot e, p + n - q, n).
\]

Comparing the third coordinates gives \( q = n \), which is a contradiction. Thus \( q \geq n \).

Similarly, using the idempotent \((e, q, q)\) we have
\[ (x, m, n)(e, q, q) = \begin{cases} (x, f^{-1}_{n-\theta \cdot q, \theta} \cdot e^{n-q} \cdot f_{n-q,\theta} \cdot m, n + q - q) & \text{if } n \geq q \\ (f^{-1}_{n-\theta \cdot m, \theta} \cdot x^{q-n} \cdot f_{q-n,\theta}, e, m + q - n, q) & \text{if } n \leq q \end{cases} \]

So we deduce that \( q \leq n \) and so \( q = n \).

Conversely, let \( n = q \). For any arbitrary elements \((v, i, j), (w, l, k) \in GBR^*(M, \theta)\), 

\[ (x, m, n)(v, i, j) = (x, m, n)(w, l, k). \]

Suppose \( n \geq i \) and \( n \geq l \). Then 

\[ (x, f^{-1}_{n-\theta \cdot i, \theta} \cdot v^{n-i} \cdot f_{n-\theta \cdot i, \theta} \cdot m, n + j - i) = (x, f^{-1}_{n-\theta \cdot l, \theta} \cdot w^{n-l} \cdot f_{n-\theta \cdot l, \theta} \cdot m, n + k - l). \]

Comparing the first and the third coordinates gives 

\[ x \cdot f^{-1}_{n-\theta \cdot i, \theta} \cdot v^{n-i} \cdot f_{n-\theta \cdot i, \theta} = x \cdot f^{-1}_{n-\theta \cdot l, \theta} \cdot w^{n-l} \cdot f_{n-\theta \cdot l, \theta} \quad \text{and} \quad n + j - i = n + k - l. \]

This implies 

\[ y \cdot f^{-1}_{n-\theta \cdot i, \theta} \cdot v^{n-i} \cdot f_{n-\theta \cdot i, \theta} = y \cdot f^{-1}_{n-\theta \cdot l, \theta} \cdot w^{n-l} \cdot f_{n-\theta \cdot l, \theta} \quad \text{and} \quad n + j - i = n + k - l. \]

Hence, \((y, p, n)(v, i, j) = (y, p, n)(w, l, k)\).

(ii) The proof is similar to the proof of (i).

(iii) Let \((x, m, n) \in E(GBR^*(M, \theta))\). Then 

\[ (x, m, n) = (x, m, n)(x, m, n) = \begin{cases} (x, f^{-1}_{n-\theta \cdot m, \theta} \cdot x^{n-m} \cdot f_{n-\theta \cdot m, \theta} \cdot m, n + n - m) & \text{if } n \geq m \\ (f^{-1}_{n-\theta \cdot m, \theta} \cdot x^{n-m} \cdot f_{n-\theta \cdot m, \theta}, x, m + m - n, n) & \text{if } n \leq m \end{cases} \]

thus \( m = n \) and \( x^2 = x \).

Conversely, let \( m = n \) and \( x \in E(M) \). Then certainly \((x, m, n)(x, m, n) = (x, m, n)\). From which it follows that \((x, m, n) \in E(GBR^*(M, \theta))\).

(iv) The proof is clear.

(v) We only prove that \( GBR^*(M, \theta) \) is right type A, as the proof that \( GBR^*(M, \theta) \) is left type A is dual.

Let \((e, m, m), (e, n, n) \in E(GBR^*(M, \theta))\). Suppose that \( m > n \). Then 

\[ (e, m, m), (e, n, n) = (e, f^{-1}_{m-\theta \cdot n, \theta} \cdot e^{m-n} \cdot f_{m-\theta \cdot n, \theta} \cdot m) \]

\[ = (f^{-1}_{m-\theta \cdot n, \theta} \cdot e^{m-n} \cdot f_{m-\theta \cdot n, \theta} \cdot e, m, m) \]

\[ = (e, n, n)(e, m, m). \]

Thus the idempotents of \( GBR^*(M, \theta) \) commute. So every \( L^* \)-class of \( GBR^*(M, \theta) \) contain an idempotent.

Let \((x, p, q) \in GBR^*(M, \theta)\). Suppose \( m \geq p \). Then
(x, p, q)[(e, m, m)(x, p, q)]^* = (x, p, q)(e \cdot f_{m-p,p}^{-1} \cdot x \theta^{m-p} \cdot f_{m-p,q}, m, m + q - p)^* \\
= (x, p, q)(e \cdot f_{m-p,p}^{-1} \cdot x \theta^{m-p} \cdot f_{m-p,q}, m + q - p, m + q - p) \\
= (e, m, m)(x, p, q).

**Theorem 2.2** (Structure theorem)

Let $S = GBR^*(M, \theta)$ be the generalized Bruck-Reilly *-extensions of $M$ determined by $\theta$. Then $S$ is a *-bisimple type A I-semigroup. Conversely, every *-bisimple type A I-semigroup is isomorphic to $GBR^*(M, \theta)$.

**Proof.** It is known that $S = GBR^*(M, \theta)$ is a type A semigroup. That $S$ is *-bisimple follows from Lemma 2.1 (i) & (ii).

Next, let $e_m = (e, m, m)$ and $e_n = (e, n, n) \in E(S)$. Then for $m \geq n$.

\[
\begin{align*}
    e_m e_n &= (e, m, m)(e, n, n) = (e \cdot f_{m-n,n}^{-1} \cdot e \theta^{m-n} \cdot f_{m-n,n}, m, m + n - n) \\
    &= (e, m, m) = e_m \\
    &= (e, n, n)(e, m, m) = e_n e_m
\end{align*}
\]

Thus $e_m \leq e_n$ if and only if $m \geq n$, which shows that $E(S)$ is a chain

\[
\ldots > (e, -2, -2) > (e, -1, -1) > (e, 0, 0) > (e, 1, 1) > (e, 2, 2) > \ldots
\]

Hence $S$ is a *-bisimple type A I-semigroup. The converse of the proof is a routine check.

From Lemma 2.1(iv), we have the following result

**Corollary 2.3.** Let $M$ be a monoid. Then $S = GBR^*(M, \theta)$ is regular if and only if $M$ is regular.

The following results show some other properties of $S = GBR^*(M, \theta)$.

**Proposition 2.4.** Let $S = GBR^*(M, \theta)$. Then $S$ is unit regular if and only if $M$ is unit regular.

**Proof.** Let $S = GBR^*(M, \theta)$ be unit regular. Then for any $(x, m, n) \in S$, there exists an element $(y, n, m) \in G$ (where $G$ is the group of units of $GBR^*(M, \theta)$) such that

\[(x, m, n)(y, n, m)(x, m, n) = (x, m, n).
\]

By considering left-hand side of the equation, we get

\[
\begin{align*}
    (x, m, n)(y, n, m)(x, m, n) &= ((x, m, n)(y, n, m))(x, m, n) \\
    &= (x \cdot f_{n-n,n}^{-1} \cdot y \theta^{n-n} \cdot f_{n-n,m}, m, n + m - n)(x, m, n) \\
    &= (xy, m, m)(x, m, n) = (xyx, m, n).
\end{align*}
\]

Therefore we obtain $x = xyx$. Consequently, $M$ is unit regular.

Conversely, let us suppose that $M$ is unit regular. Then for $x \in M$, there exists an element $x \in G_M$ (where $G_M$ is the group of units of $M$) such that obtain $x = xyx$. Now we need to show that for any $(x, m, n) \in GBR^*(M, \theta)$, there exist an element $(y, p, q) \in G_M$ such that

\[(x, m, n) = (x, m, n)(y, p, q)(x, m, n).
\]
Here we take \( p = n \), \( q = m \), then we have \((x, m, n)(y, n, m)(x, m, n) = (xy, m, n)\). Since we have \( x = xyx \), for any \( x \in M \), \( y \in G_M \), we obtain \((x, m, n)(y, p, q)(x, m, n) = (x, m, n)\). Thus \( S \) is \( \theta \) unit regular.

**Proposition 2.5** Let \( M \) be a monoid. Then \( M' = \{ (x, m, m) \mid x \in M, m \in \mathbb{N}^0 \} \) \( \leq \mathit{GBR}^*(M, \theta) \) is coregular if and only if \( M \) is coregular.

**Proof.** Let \( M' \) \( \leq \mathit{GBR}^*(M, \theta) \) be coregular. Then for \((x, 0, 0) \in \mathit{GBR}^*(M, \theta)\), there exists an element \((y, n, n) \in \mathit{GBR}^*(M, \theta)\) such that

\[
(x, 0, 0)(y, n, n)(x, 0, 0) = (xyx, n, n) = (x, 0, 0)
\]

(1)

\[
(y, n, n)(x, 0, 0)(y, n, n) = (yxy, n, n) = (x, 0, 0)
\]

(2)

From (1) and (2), we have that \( n = 0 \), \( xyx = x \) and \( yxy = x \). Thus \( M \) is coregular.

Conversely, let \( M \) be coregular. Then there exists \( y \in M \), with \( xyx = x \) and \( yxy = x \). Thus for \((x, m, n) \in \mathit{GBR}^*(M, \theta)\), we have

\[
((x, m, n)(y, m, m))(x, m, m) = (xy, m, m)(x, m, m)
\]

\[
= (xyx, m, m)
\]

\[
= (x, m, m).
\]

\[
((y, m, m)(x, m, m))(y, m, m) = (yx, m, m)(y, m, m)
\]

\[
= (yxy, m, m)
\]

\[
= (x, m, m).
\]

Therefore, \( M' = \{ (x, m, m) \mid x \in M, m \in \mathbb{N}^0 \} \) \( \leq \mathit{GBR}^*(M, \theta) \) is coregular.

It is important to note that not all regular semigroups are coregular. This is shown in the example below.

**Example 2.6.** Let \( X \) and \( Y \) be non-empty sets and set \( T = X \times Y \) with the binary operation

\[
(x, y)(u, v) = (x, v), \text{ for all } x, u \in X, \ y, v \in Y.
\]

It can be easily seen that \( T \) is a semigroup. This semigroup is called a rectangular band. \( T \) is also regular, since for \((x, y), (u, v) \in T\) we have \((x, y)(u, v)(x, y) = (x, y)\).

To show that \( T \) is not coregular, let \((x, y), (u, v) \in T\), then we have

\[
(x, y)(u, v)(x, y) = (x, y),
\]

\[
(u, v)(x, y)(u, v) = (u, v).
\]

So \((x, y) \neq (u, v)\). Thus \( T \) is not coregular.

In the next theorem, we consider the orthodox property of \( \mathit{GBR}^*(M, \theta) \).

**Theorem 2.7.** Let \( S = \mathit{GBR}^*(M, \theta) \). Then \( S \) is orthodox if and only if \( M \) is orthodox.

**Proof.** Let \( \mathit{GBR}^*(M, \theta) \) be orthodox. By Corollary 2.3, we know that \( M \) is regular. Then it remains to show that \( E(M) \) is a subsemigroup of \( M \). In particular for each \( e, e' \in E(M) \),
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\[(e, m, m)(e', m, m) = (e \cdot f_{m-1,m,m}^{-1} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m}, m + m - m) = (ee', m, m)\]

is an idempotent of \(\text{GBR}^∗(M, \theta)\) and so \((ee')^2 = ee'\). Hence \(M\) is orthodox.

Conversely, let \(M\) be orthodox. Then \(M\) is regular, and \(E(M)\) is a subsemigroup of \(M\). By Corollary 2.3, we know that \(\text{GBR}^∗(M, \theta)\) is regular.

Next, we show that \((e, m, m)(e', n, n) \in E(\text{GBR}^∗(M, \theta))\). From the multiplication \((e, m, m)(e', n, n)\), we have the following cases:

Case (1): If \(m \geq n\), we have

\[(e, m, m)(e', n, n) = \left( (e \cdot f_{m-1,m,m}^{-1} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m}, m, m + n - n \right) = \left( (e \cdot f_{m-1,m,m}^{-1} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m}, m \right).

Since \(e, e' \in E(M)\), we deduce that \(e \cdot f_{m-1,m,m}^{-1} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \in E(M)\). But the idempotents in \(M\) are commutative, consequently

\( (e \cdot f_{m-1,m,m}^{-1} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m}) = (e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \cdot e' \cdot \theta^{m-n} \cdot f_{m-1,m,m}). \)

So \((e' \cdot \theta^{m-n} \cdot f_{m-1,m,m} \cdot e', e' \cdot \theta^{m-n} \cdot f_{m-1,m,m}) \in E(\text{GBR}^∗(M, \theta))\). Therefore \(E(\text{GBR}^∗(M, \theta))\) is a subsemigroup of \(\text{GBR}^∗(M, \theta)\).

Case (2): If \(m \leq n\), we have

\[(e, m, m)(e', n, n) = \left( (f_{n-1,m,m}^{-1} \cdot e \cdot \theta^{n-m} \cdot f_{n-1,m,m} \cdot e', m + n - m, n \right) = \left( (f_{n-1,m,m}^{-1} \cdot e \cdot \theta^{n-m} \cdot f_{n-1,m,m} \cdot e', n, n \right).

From here, since \((f_{n-1,m,m}^{-1} \cdot e \cdot \theta^{n-m} \cdot f_{n-1,m,m} \cdot e', f_{n-1,m,m}^{-1} \cdot e \cdot \theta^{n-m} \cdot f_{n-1,m,m} \cdot e') \in E(M)\) and the idempotents in \(M\) are commutative, we deduce that \(E(\text{GBR}^∗(M, \theta))\) is a subsemigroup of \(\text{GBR}^∗(M, \theta)\).

The connection between the Green’s ∗-relations and congruences lies on the fact that \(L^∗\) is a right congruence and \(R^∗\) is a left congruence. It can be easily verified that \(H^∗\) is a congruence on \(S = \text{GBR}^∗(M, \theta)\). In our next section, we shall characterize the congruences on \(S = \text{GBR}^∗(M, \theta)\).

3. Idempotent-separating congruences

The following terms adopted from [8] will be used in the description of congruences on ∗-bisimple type A I-semigroups.

**Definition 3.1.** Let \(S = \text{GBR}^∗(M, \theta)\) be a ∗-bisimple type A I-semigroup where \(\theta : M \rightarrow H^∗_1\). Let \(H^∗ = \rho\) be a congruence on \(S\). Let us use \(\rho(M)\) to denote the congruence on \(M\) induced by \(\rho\), via the restriction of \(\rho\) to the monoid \(\{(x, 0, 0) : x \in M\}\).

**Definition 3.2.** A congruence \(\gamma\) on \(M\) is said to be \(\theta\)-admissible if \(x \gamma y\) implies \(x \theta y \gamma \theta\), for any \(x, y \in M\).
A typical idempotent-separating congruence on \( S = GBR^*(M, \theta) \) is characterized as follows:

**Theorem 3.3.** Let \( S = GBR^*(M, \theta) \) be a \(^*\)-bisimple type A I-semigroup and let \( \rho \) be a congruence on \( S = GBR^*(M, \theta) \). Then \( \rho(M) \) is \( \theta \)-admissible. Conversely, if \( \gamma \) is any \( \theta \)-admissible congruence on \( M \), then the relation on \( S \) defined by

\[
[(x, m, n)(y, p, q)] \in \gamma(S) \text{ if and only if } m = p, n = q \text{ and } (x, y) \in \gamma
\]

is an idempotent-separating congruence.

**Proof.** Suppose \( x \rho(M) y \). Then we have that \( (x, 0, 0) \rho (y, 0, 0) \).

Consequently,

\[
(x, 0, 0)(e, 1, 1) \rho (y, 0, 0)(e, 1, 1).
\]

But \( (x, 0, 0)(e, 1, 1) = (f_{1,0}^{-1} \cdot x\theta f_{1,0} \cdot e, 1, 1) \) and \( (y, 0, 0)(e, 1, 1) = (f_{1,0}^{-1} \cdot y\theta f_{1,0} \cdot e, 1, 1) \).

Thus \( (f_{1,0}^{-1} \cdot x\theta f_{1,0} \cdot e, 1, 1) \rho (f_{1,0}^{-1} \cdot y\theta f_{1,0} \cdot e, 1, 1) = (x\theta, 1, 1) \rho (y\theta, 1, 1) \).

Since \( (x\theta, 1, 1) \rho (y\theta, 1, 1) \), then \( (x\theta, 1, 1) = (y\theta, 1, 1) \).

Also we have \( (e, 0, 1)(x\theta, 1, 1)(e, 1, 0) \rho (e, 0, 1)(y\theta, 1, 1)(e, 1, 0) \).

But \( (e, 0, 1)(x\theta, 1, 1)(e, 1, 0) = (x\theta, 0, 0) \) and \( (e, 0, 1)(y\theta, 1, 1)(e, 1, 0) = (y\theta, 0, 0) \).

Thus \( (x\theta, 0, 0) \rho (y\theta, 0, 0) \). Since \( (x\theta, 0, 0) \rho (y\theta, 0, 0) \), then \( x\theta \rho(M) y\theta \).

Conversely, let \( \gamma \) be a \( \theta \)-admissible congruence on \( M \). We first show that \( \gamma(S) \) is an equivalence relation.

\[
[(x, m, n)(x, m, n)] \in \gamma(S) \text{ since } (x, x) \in \gamma. \text{ Thus } \gamma(S) \text{ is reflexive. By definition, } \gamma(S) \text{ is symmetric. To show transitivity, let } (x, m, n) \gamma(S) (y, p, q) \text{ and } (y, p, q) \gamma(S) (z, i, j) \text{ for all } (x, m, n), (y, p, q), (z, i, j) \in S. \text{ Then we have } m = p, n = q, \ (x, y) \in \gamma \text{ and } p = i, q = j, \ (y, z) \in \gamma.
\]

Consequently, \( m = i, n = j \). Hence \( (x, z) \in \gamma \), which means that \( \gamma(S) \) is transitive.

Next is to show that \( \gamma(S) \) is a congruence. Now let \( a = (x, m, n), b = (y, p, q) \). That \( \gamma(S) \) is a congruence entails showing that

\[
\forall \ x = (z, k, l) \in S = GBR^*(M, \theta).
\]

Consequently,

\[
ax = (x, m, n)(z, k, l) = \begin{cases} (x, f_{n-k,k}^{-1} \cdot z\theta^{n-k} \cdot f_{n-k,k}, m, n + l - k) & \text{if } n \geq k \\ (f_{k-n,m}^{-1} \cdot x\theta^{k-n} \cdot f_{k-n,n}, z, m + k - n, l) & \text{if } n < k \end{cases}
\]

\[
bx = (y, p, q)(z, k, l) = \begin{cases} (y, f_{q-k,k}^{-1} \cdot z\theta^{q-k} \cdot f_{q-k,k}, p, q + l - k) & \text{if } q \geq k \\ (f_{k-q,p}^{-1} \cdot y\theta^{k-q} \cdot f_{k-q,q}, z, p + k - q, l) & \text{if } q < k \end{cases}
\]

So if \( (x, m, n) \gamma(S) (y, p, q) \), then

\[
(x, m, n)(z, k, l) \gamma(S) (y, p, q)(z, k, l) =
\]
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\[
\begin{align*}
&\{ (x.f_{n-k,k}.z\theta^{n-k}.f_{n-k,l},m,n+l-k) \text{ if } n \geq k \\
&\{ (f_{k-n,m}.x\theta^{k-n}.f_{k-n,n},z,m+k-n,l) \text{ if } n \leq k \\
&\gamma(S) \{ (y.f_{q-k,k}.z\theta^{q-k}.f_{q-k,l},p,q+l-k) \text{ if } q \geq k \\
&\{ (f_{k-q,p}.y\theta^{p-q}.f_{k-q,q},z,p+k-q,l) \text{ if } q \leq k
\end{align*}
\]

But \((x,m,n)\gamma(S)\ (y,p,q)\) if and only if \(m = p, n = q\) and \(x \gamma y\).

Thus, we have that

\[
\begin{align*}
&\{ (x.f_{n-k,k}.z\theta^{n-k}.f_{n-k,l},m,n+l-k) \text{ if } n \geq k \\
&\{ (f_{k-n,m}.x\theta^{k-n}.f_{k-n,n},z,m+k-n,l) \text{ if } n \leq k \\
&\gamma(S) \{ (y.f_{q-k,k}.z\theta^{q-k}.f_{q-k,l},m,n+l-k) \text{ if } n \geq k \\
&\{ (f_{k-q,p}.y\theta^{p-q}.f_{k-q,q},z,m+k-n,l) \text{ if } n \leq k
\end{align*}
\]

Hence \(\gamma(S)\) is a right congruence.

That \(\gamma(S)\) is a left congruence follows similarly. Thus \(\gamma(S)\) is a congruence.

Futhermore, \((e,m,m)\gamma(S)(e,n,n) \Rightarrow m = n\) which implies that \((e,m,m) = (e,n,n)\). Thus \(\gamma(S)\) is an idempotent-separating congruence.

**Remark 3.4.** \(\mathcal{H}^*\) is an idempotent-separating congruence on \(\text{GBR}^*(M,\theta)\) and \(\gamma(S) \subseteq \mathcal{H}^*\).

4. Minimum cancellative monoid congruence

The idea of the minimum cancellative monoid congruence is to obtain a congruence \(\sigma\) on \(S\), a type A semigroup with respect to which \(S/\sigma\) is cancellative.

Here we will determine the minimum cancellative monoid congruence on \(S = \text{GBR}^*(M,\theta)\), as follows:

Now let \((h,m,n),(x,i,j) \in S = \text{GBR}^*(M,\theta)\). Define a relation \(\sigma\) on \(S = \text{GBR}^*(M,\theta)\) by the rule

\((h,m,n)\sigma(x,i,j)\) if and only if \(m - n = i - j\), \(h\theta^i = x\theta^m\) and \(x\theta^i = h\theta^m\).

**Lemma 4.1.** \(\sigma\) is a congruence on \(S\).

**Proof.** That \(\sigma\) is symmetric and reflexive is known. To show that \(\sigma\) is transitive, let \((h,m,n)\sigma(x,i,j)\) and \((x,i,j)\sigma(y,p,q)\) for \((h,m,n),(x,i,j),(y,p,q) \in S\). Then \(m - n = i - j\) and \(i - j = p - q\) and so \(m - n = p - q\).

Consequently, \(x\theta^i = h\theta^m\) and \(y\theta^p = x\theta^i\) implies \(y\theta^p = h\theta^m\).
Also \( h\theta^i = x\theta^m \) and \( x\theta^p = y\theta^i \) implies that \( h\theta^i = (y\theta^{i-p})\theta^m = y\theta^{i-p+m} \). Then \( h\theta^{i+p} = y\theta^{i-p+m+p} = y\theta^{i+m} \). Hence \( h\theta^p = y\theta^m \) which shows that \( \sigma \) is transitive.

Next we show that \( \sigma \) is a congruence. Now let \( u = (h, m, n) \), \( v = (x, i, j) \). That \( \sigma \) is a congruence we show that \( \sigma \) is both a left and right congruence. That is

\[
\forall z \in S, \quad u \sigma v \implies uz \sigma vz \quad \text{(for right congruence)}
\]

and

\[
\forall z \in S \quad u \sigma v \implies zu \sigma zv \quad \text{(for left congruence)}.
\]

Let \( z = (y, p, q) \in S \). Then

\[
uz = (h, m, n)(y, p, q) = \begin{cases} 
(h, f_{n-p, p, q}^{-1} \cdot y\theta^{n-p} \cdot f_{n-p, q}^{-1}, m, n + q - p) & \text{if } n \geq p \\
(f_{p-n, n, q}^{-1} \cdot h\theta^{-n} \cdot f_{p-n, q}^{-1}, y, m + p - n, q) & \text{if } n \leq p
\end{cases}
\]

and

\[
vz = (x, i, j)(y, p, q) = \begin{cases} 
(x, f_{j-p, q}^{-1} \cdot y\theta^{-p} \cdot f_{j-p, q}^{-1}, i, j + q - p) & \text{if } j \geq p \\
(f_{p-j, i}^{-1} \cdot x\theta^{-p} \cdot f_{p-j, i}^{-1}, i + p - j, q) & \text{if } j \leq p
\end{cases}
\]

Evidently if \( (h, m, n) \sigma (x, i, j) \), we have

\[
m - (n + q - p) = (m - n) + (p - q) \quad \text{and} \quad i - (j + q - p) = (i - j) + (p - q)
\]

\[
m + p - n - q = (m - n) + (p - q) \quad \text{and} \quad i + p - j - q = (i - j) + (p - q).
\]

But \( m - n = i - j \) and so \( (m - n) + (p - q) = (i - j) + (p - q) \).

For the first outer part, we know from definition that \( h\theta^i = x\theta^m \) and \( h\theta^n = x\theta^j \). It suffices to show that \( (h\theta^{p-n}, y)\theta^{i+p-j} = (x\theta^{p-j}, y)\theta^{m+p-n} \).

Considering the left hand side of the equation we have

\[
(h\theta^{p-n}, y)\theta^{i+p-j} = h\theta^{p+i-n-j} \cdot y\theta^{p-j+i} = h\theta^{i+p-p-j} \cdot y\theta^{p-j-i} = (h\theta^i)\theta^{p-j-n+p} \cdot y\theta^{p+i-j}
\]

But \( i - j = m - n \) and \( h\theta^i = x\theta^m \).

Therefore,

\[
(h\theta^i)\theta^{p-j-n+p} \cdot y\theta^{p+i-j} = (x\theta^{m})\theta^{p-j-n+p} \cdot y\theta^{p+i-(m-n)} = x\theta^{m+p-j-n} \cdot y\theta^{p+m-n} = (x\theta^{p-j}, y)\theta^{m+p-n}
\]

as required.

Hence \( \sigma \) is a right congruence. That \( \sigma \) is a left congruence follows similarly. Consequently \( \sigma \) is a congruence.

**Lemma 4.2.** \( \sigma \) is a cancellative monoid.
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Proof. Since \((e, m, m) \sigma (e, n, n)\) for \(m, n \in I\), it follows that the class of \(\sigma\) containing the idempotents is the identity element for \(S/\sigma\). Thus \((1, m, n) \sigma (y, p, q) \sigma = (y, p, q) \sigma\) and hence \(S/\sigma\) is a monoid.

Next is to show that \(S/\sigma\) is cancellative. Now let \(u = (h, m, n), v = (x, i, j)\).

That \(S/\sigma\) is cancellative entails showing that for all \(z \in S\),

\[ u \sigma z \sigma = v \sigma z \sigma \Rightarrow u \sigma = v \sigma \quad \text{(for right cancellative)} \]

and

\[ z \sigma u \sigma = z \sigma v \sigma \Rightarrow u \sigma = v \sigma \quad \text{(for left cancellative)} . \]

Let \(z = (y, p, q) \in S\). Then

\[ u \sigma z \sigma = (h, m, n) \sigma (y, p, q) \sigma = (x, i, j) \sigma (y, p, q) \sigma = v \sigma z \sigma . \]

Consequently,

\[
(h, m, n) \sigma (y, p, q) \sigma = (x, i, j) \sigma (y, p, q) \sigma \\
\iff [(h, m, n)(y, p, q)] \sigma = [(x, i, j)(y, p, q)] \sigma \\
\iff \left\{ \\
\begin{array}{ll}
(h_f^{-1} y \theta^{n-p}, f_{n-p,q}, m, n + q - p) & \text{if } n \geq p \\
(f_p^{-1} h \theta^{p-n}, f_{p-n,n} y, m + p - n, q) & \text{if } n \leq p
\end{array} \right. \times \sigma
\]

\[
= \left\{ \\
\begin{array}{ll}
(x f_{j-p,q} y \theta^{j-p}, f_{j-p,q}, i, j + q - p) & \text{if } j \geq p \\
(f_{p-j,i} x \theta^{p-j}, f_{p-j,i} y, i + p - j, q) & \text{if } j \leq p
\end{array} \right. \times \sigma
\]

\[
\iff m - (n + q - p) = i - (y + q - p), \ (m + p - n) - q = (i + p - j) - q
\]

and

\[
(h \theta^{p-n} y \theta^{i+p-j} = (x \theta^{p-j} y) \theta^{m+p-n}
\iff (m - n) + (p - q) = (i - j) + (p - q)
\]

and

\[
h \theta^{p-n+(i-j)+p} y \theta^{p+(i-j)} = x \theta^{p-j+(m-n)+p} y \theta^{p+(m-n)}
\iff m - n = i - j \ \text{and} \ \ (h \theta^{i+p-n-j} = (x \theta^{m}) \theta^{p-n-j}
\iff m - n = i - j \ \text{and} \ \ h \theta^{i} = x \theta^{m}
\iff (h, m, n) \sigma (x, i, j)
\]

which shows that \(S/\sigma\) is right cancellative. That \(S/\sigma\) is left cancellative follows similarly, and we conclude that \(S/\sigma\) is cancellative.

Lemma 4.3 \(\sigma\) is a minimum congruence.

Proof. Let \(\Gamma\) be any other cancellative monoid congruence. Then \((1, n, n) \Gamma (1, 0, 0)\) for all \(n \in I\).

Suppose \((h, m, n) \sigma (x, i, j)\). Then we have from \((h, m, n)(1, p, p) = (x, i, j)(1, p, p)\) for some \(p \in I\),
\[\begin{align*}
1 \cap \mu &= \leq n + p - p \\
(\theta_{p-n,m}) &= p \theta_{p-n,m} \leq n, i + j + p - p \\
&= (x_{j-p,i} \cdot \theta_{j-p,i} = i \cdot \theta_{j-p,i} \cdot 1 + i \cdot \theta_{p-j,p} \cdot x \theta_{p-j} \\
&= (h, m) \text{ if } n \geq p \\
(h \theta_{p-n,m}, m + p - n, p) &\text{ if } n \leq p \\
&= (x, i, j) \text{ if } j \geq p \\
&= (x \theta_{p-j}, i + j - p, p) \text{ if } j \leq p
\end{align*}\]

But \((1, n, n) \Gamma (1, 0, 0), (h, m, n) \Gamma (1, p, p) \Gamma (h, m, n)\).
Also, \((x, i, j)(1, p, p) \Gamma (x, i, j)\). Therefore \((h, m, n) \Gamma (x, i, j)\). Thus \(\sigma \subseteq \Gamma\).

Combining Lemma 4.1 to Lemma 4.3, we have proved the following theorem:

**Theorem 4.4.** Let \(S = GBR^*(M, \theta)\) be an \(\ast\)-bisimple type A 1-semigroup and let \(\sigma\) be defined on \(S\) by the rule that \((h, m, n) \sigma (x, i, j)\) if and only if \(m - n = i - j, \ h \theta = x \theta, \) and \(x \theta = h \theta\). Then \(\sigma\) is the minimum cancellative monoid congruence on \(S\).

### 5. The congruence \(\mu\)

Here we will determine the maximum congruence \(\mu\) on \(S = GBR^*(M, \theta)\) contained in \(H^\ast\) by utilizing the approach of El-Qallali and Fountain [2].

Now let \((e, m, m)\) and \((e, n, n)\) be the idempotents in the \(R^\ast\)-class and \(L^\ast\)-class respectively. We define the relations \(\mu_R\) and \(\mu_L\) on \(S = GBR^*(M, \theta)\) as follows:

\((x, m, n) \mu_L (y, p, q)\) if and only if \((e, n, n)(x, m, n) \in L^\ast (e, n, n)(y, p, q), m - n = p - q, x \theta = y \theta = e \theta\).

\((x, m, n) \mu_R (y, p, q)\) if and only if \((x, m, n)(e, m, m) \in R^\ast (y, p, q)(e, m, m), m - n = p - q, x \theta = y \theta = e \theta\).

Consequently,

\[\mu = \mu_L \cap \mu_R.\]

With the above relation, we obtain the following results

**Proposition 5.1.** Let \(S = GBR^*(M, \theta)\). Then \(\mu_L\) is the maximum congruence on \(S\) contained in \(L^\ast\).

**Proof.** Obviously, \(\mu_L\) is an equivalence on \(S\). Since \(L^\ast\) is a right congruence on \(S, \mu_L\) is right compatible under the semigroup multiplication.

Next is to show that \(\mu_L\) is also left compatible under the semigroup multiplication. Now let \((x, m, n), (y, p, q), (e, 0, 0) \in S\). That \(\mu_L\) is left compatible entails showing that \((e, n, n)(x, m, n) \in L^\ast (e, n, n)(y, p, q)\) implies \((e, 0, 0)(e, n, n)(x, m, n) \in L^\ast (e, 0, 0)(e, n, n)(y, p, q)\).
Thus we have
\[(e, 0, 0)(e, n, n)(x, m, n) = (e\theta^n, n, n)(x, m, n),\]

and
\[(e, 0, 0)(e, n, n)(y, p, q) = (e\theta^n, n, n)(y, p, q).\]

Consequently,
\[(e\theta^n, n, n)(x, m, n) = \begin{cases} (e\theta^n.x\theta^{n-m}, n, n + n - m) & \text{if } n \geq m \\ (e\theta^m.x, m, n) & \text{if } n \leq m \end{cases}\]

\[(e\theta^n, n, n)(y, p, q) = \begin{cases} (e\theta^n.y\theta^{n-p}, n, n + q - p) & \text{if } n \geq p \\ (e\theta^m.y, p, q) & \text{if } n \leq p \end{cases}\]

From \((e\theta^n, n, n)(x, m, n)\) and \((e\theta^n, n, n)(y, p, q)\), it follows that
\[n - (n + n - m) = m - n \text{ and } n - (n + q - p) = p - q.\]

It follows from definition that \(m - n = p - q\).

For the first outer part of \((e\theta^n, n, n)(x, m, n)\) and \((e\theta^n, n, n)(y, p, q)\), we have
\[e\theta^n.x\theta^{n-m} = e\theta^n.y\theta^{n-p} \quad \text{(since from definition, } x\theta^{n-m} = y\theta^{n-p})\]
\[e\theta^m.x = e\theta^p.y \quad \text{(since from definition, } e\theta^{m-n}.x = e\theta^{p-n}.y).\]

Thus \((x, m, n) \mu_L (y, p, q)\) implies \((e, 0, 0)(x, m, n) \mu_L (e, 0, 0)(y, p, q)\).

To show that \(\mu \subseteq \mathcal{L}^*\), we now consider the elements \((x, m, n), (y, p, q) \in GBR^*(M, \theta)\) such that \((x, m, n) \mu_L (y, p, q)\). But \((x, m, n)^* = (y, p, q)^*\) which implies that \((x, m, n) \mathcal{L}^* (y, p, q)\).

Now let \(\rho\) be a congruence on \(GBR^*(M, \theta)\) such that \(\rho \subseteq \mathcal{L}^*\). If \((x, m, n) \rho (y, p, q)\), then for any \((e, n, n) \in S, (e, n, n)(x, m, n) \rho (e, n, n)(y, p, q)\) so that \((e, n, n)(x, m, n) \mathcal{L}^* (e, n, n)(y, p, q)\). That is \((x, m, n) \mu_L (y, p, q)\) and whence \(\rho \subseteq \mu_L\).

**Proposition 5.2.** Let \(S = GBR^*(M, \theta)\). Then \(\mu_R\) is the maximum congruence on \(S\) contained in \(\mathcal{R}^*\).

**Proof.** The proof is similar to the proof of Proposition 3.1.

An immediate consequence of Proposition 3.1 and Proposition 3.2 is the following

**Theorem 5.3.** Let \(S\) be a *-bisimple type A I-semigroup. Then \(\mu\) is the maximum congruence on \(S\) contained in \(\mathcal{H}^*\).

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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