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RANK TWO SOLUTIONS OF THE ABSTRACT CAUCHY PROBLEM

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Abstract. In this paper we discuss solutions of the Abstract Cauchy Problem

$$\begin{aligned} Bu^{(\alpha)}(t) &= Au(t)+f(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (E) \\ u(0) &= x_0 \end{aligned}$$

that are of the form $u = u_1 \otimes \delta_1 + u_2 \otimes \delta_2$, where u and f are functions defined on $[0,a]$ with values in the Hilbert space ℓ^2.

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1. Introduction

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One of the most important differential equations in Banach spaces, which is involved in many sectors of sciences, is the so-called the Abstract Cauchy Problem, and it has the form:

$$\begin{aligned} Bu'(t) &= Au(t)+f(t) \dots\dots\dots\dots\dots\dots\dots\dots (E) \\ u(0) &= x_0 \end{aligned}$$

where u is continuously differentiable vector valued function and f is continuous vector valued function on I ; I is equal to $[0,1]$ or $[0,\infty)$, and both u and f have values in a Banach space X. In Problem (E), A and B are densely defined linear operators on X. If B is not invertible, then Problem (E) is called degenerate problem, otherwise it is called non-degenerate. If $f=0$, then Problem (E) is homogenous, otherwise it is called non-homogenous.

Many researchers were and are interested in this Problem and studied it using a variety of methods. In the mid-seventies of the last century, Carroll,R.W. and Showalter,R.E. , released an article about degenerate Abstract Cauchy Problem[4]. In 1996, Thaller, B. and Thaller, S. used the Factorization of degenerate Abstract Cauchy Problem in a Hilbert space[7]. Favini, A. and Yagi, A. ,(1999), produced an interesting material on the degenerate Abstract Cauchy Problem by unifying the methods of semigroups and operational approaches to treat the solvability of such problem[5]. In 2002, the inverse form of Problem (E) was studied by Al Horani, M. , where some conditions were put on operators A and B to convert Problem (E) to a non-degenerate problem[1]. In 2004, Al Horani, M. and Favini, A. discussed the inverse problem when it is of the second order[2]. In 2010, Ziqan, A.M., Al Horani, M. and Khalil, R. used tensor product technique to find a unique solution for the Abstract Cauchy Problem under certain conditions on the operators A and B [8], [9]. Khalil, R. and Abdullah, L. (2010), used the same technique and found an atomic solution for certain degenerate and non-degenerate inverse problems [6].

In this paper we studied the non-homogeneous Abstract Cauchy Problem using the tensor product technique when the non-homogeneous part of the problem is a two rank function. In other words f in Problem (E) is equal to $f_1 \otimes \delta_1 + f_2 \otimes \delta_2$, where f_1, f_1 are real-valued continuous functions on I and δ_1, δ_1 are two orthogonal unit vectors in ℓ^2, the Banach space consisting of the square summable sequences. We did prove the existence of a unique solution for this type of problems when u has the form $u_1 \otimes \delta_1 + u_2 \otimes \delta_2$, where u_1, u_2 are real-valued continuously differentiable functions on I, and with some conditions on the operators A and B. Also, we found an atomic solution for non-degenerate problem of this type of the non-homogeneous Abstract Cauchy Problem.

Throughout the paper, the homogeneous Abstract Cauchy Problem is

$$Bu'(t) = Au(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E_1)$$

$$u(0) = x_0$$

and the nonhomogeneous Abstract Cauchy Problem is

$$Bu'(t) = Au(t) + f(t) \ldots\ldots\ldots\ldots\ldots (E_2)$$

$$u(0) = x_0,$$

where A and B are densely defined linear operators on the Banach space X , $u \in C^1(I, X)$, $f \in C(I, X)$ and $x_0 \in X$.

In this paper we study the non-homogeneous $\alpha-$Abstract Cauchy Problem using the tensor product technique when the non-homogeneous part of the problem is a two rank function. In other words f in Problem (E) is equal to $f_1 \otimes \delta_1 + f_2 \otimes \delta_2$, where f_1, f_1 are real-valued continuous functions on I and δ_1, δ_1 are two orthogonal unit vectors in ℓ^2, the Banach space consisting of the square summable sequences. We did prove the existence of a unique solution for this type of problems when u has the form $u_1 \otimes \delta_1 + u_2 \otimes \delta_2$, where u_1, u_2 are real-valued continuously $\alpha-$differentiable functions on I, and with some conditions on the operators A and B.

2. Basic Facts On Conformable Fractional Derivatives

There are many definitions available in the literature for fractional derivatives. The main ones are the Riemann Liouville definition and the Caputo definition, see [8] .

(i) Riemann - Liouville Definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

(ii) Caputo Definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

Such definitions have many setbacks such as

(i) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ ($D_a^{\alpha}(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^{\alpha}(fg) = fD_a^{\alpha}(g) + gD_a^{\alpha}(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^{\alpha}(f/g) = \frac{gD_a^{\alpha}(f) - fD_a^{\alpha}(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^{\alpha}(f \circ g)(t) = f^{(\alpha)}(g(t))\, g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f$, in general.

(vi) All fractional derivatives, specially Caputo definition, assumes that the function f is differentiable.

We refer the reader to [3] for more results on Caputo and Riemann - Liouville Definitions.

Recently, the authors in [2], gave a new definition of fractional derivative which is a natural extension to the usual first derivative. So many papers since then were written, and many equations were solved using such definition. The definition goes as follows:

Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. Then for all $t > 0, \quad \alpha \in (0,1)$, let

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

T_α is called the conformable fractional derivative of f of order α.
Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$.

If f is $\alpha-$differentiable in some $(0,b)$, $b > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

According to this definition, we have the following properties, [ ],

1. $T_\alpha(1) = 0$,

2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,

3. $T_\alpha(\sin at) = at^{1-\alpha} \cos at, \quad a \in \mathbb{R}$,

4. $T_\alpha(\cos at) = -at^{1-\alpha} \sin at, \quad a \in \mathbb{R}$

5. $T_\alpha(e^{at}) = at^{1-\alpha} e^{at}, \quad a \in \mathbb{R}$.

Further, many functions behave as in the usual derivative. Here are some formulas

$$\begin{aligned}
&T_\alpha(\frac{1}{\alpha}t^\alpha) = 1 \\
&T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}, \\
&T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos(\frac{1}{\alpha}t^\alpha), \\
&T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin(\frac{1}{\alpha}t^\alpha).
\end{aligned}$$

We refere to [ ] for more on fractional derivative.

3. Two Rank Solution

Let u be an $\alpha-$ differentiable function on $[0,b]$ with values in the Hilbert space ℓ^2

Consider the problem

$$\begin{aligned} Bu^{(\alpha)}(t) &= Au(t)+f(t)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(E) \\ u(0) &= x_0 \end{aligned}$$

where A and B are densely defined linear operators on the Banach space X , $u \in C^{(\alpha)}(I,X)$, $f \in C(I,X)$ and $x_0 \in X$.

In this section, we study the problem

$$\begin{aligned} Bu^{(\alpha)}(t) &= Au(t)+f(t) \quad \ldots\ldots\ldots\ldots(E_2) \\ u(0) &= x_0 \end{aligned}$$

Where we assume that $u(t) = u_1(t)\delta_1 + u_2(t)\delta_2$, and $f(t) = f_1(t)\delta_1 + f_2(t)\delta_2$ and we assume A and B are densely defined closed operators on ℓ^2.

One of our main results is the following:

Theorem 3.1 In Problem (E_2), let $B = I$, and $u(t) = u_1(t)\delta_1 + u_2(t)\delta_2$, with $u_1(t)$ and $u_2(t)$ are continuously $\alpha-$differentiable functions on $[0,\infty)$. Assume further that $f_1(t)$ and $f_2(t)$ are continuous on $[0,\infty)$.Then Problem (E_2) has a unique solution.

Proof : Since $u^{(\alpha)}(t) = u_1^{(\alpha)}(t)\delta_1 + u_2^{(\alpha)}(t)\delta_2$, we get

$$u_1^{(\alpha)}(t)\delta_1 + u_2^{(\alpha)}(t)\delta_2 = u_1(t)A\delta_1 + u_2(t)A\delta_2 + f_1(t)\delta_1 + f_2(t)\delta_2\ldots\ldots\ldots(1)$$

If $[\delta_1,\delta_2]$ is an invariant subspace of A, then the restriction of A to $[\delta_1,\delta_2]$ has a matrix representation $\widehat{A} = [a_{ij}]_{2\times 2}$, where $a_{ij} = \langle A\delta_j, \delta_i\rangle, i,j = 1,2.$

Taking the inner product of δ_1 and δ_2 to both sides of (1), we get

$$\begin{aligned} u_1^{(\alpha)}(t)\langle\delta_1,\delta_1\rangle + u_2^{(\alpha)}(t)\langle\delta_2,\delta_1\rangle &= u_1(t)\langle A\delta_1,\delta_1\rangle + u_2(t)\langle A\delta_2,\delta_1\rangle \\ &\quad + f_1(t)\langle\delta_1,\delta_1\rangle + f_2(t)\langle\delta_2,\delta_1\rangle\ldots\ldots\ldots(2) \end{aligned}$$

And

$$\begin{aligned} u_1^{(\alpha)}(t)\langle\delta_1,\delta_2\rangle + u_2^{(\alpha)}(t)\langle\delta_2,\delta_2\rangle &= u_1(t)\langle A\delta_1,\delta_2\rangle + u_2(t)\langle A\delta_2,\delta_2\rangle \\ &\quad + f_1(t)\langle\delta_1,\delta_2\rangle + f_2(t)\langle\delta_2,\delta_2\rangle \ldots\ldots\ldots..(3) \end{aligned}$$

Since $\{\delta_1,\delta_2\}$ is an orthonormal set, we get from equations (2) and (3) :

$$u_1^{(\alpha)}(t) = u_1(t)a_{11} + u_2(t)a_{12} + f_1(t) \ldots\ldots\ldots..(4)$$

And

$$u_2^{(\alpha)}(t) = u_1(t)a_{21} + u_2(t)a_{22} + f_2(t) \ldots\ldots\ldots..(5)$$

Now, equations (4) and (5) represent a non-homogeneous system of two linear $\alpha-$differential equations

$$U^{(\alpha)}(t) = \widehat{A}U(t) + F(t) \ldots\ldots\ldots..(6)$$

where $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and $F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$.

The general solution of system (6) U_g is the sum of the homogeneous solution and a particular solution. The details are as follows:

Now, the corresponding homogeneous system of (6) is

$$U^{(\alpha)}(t) = \widehat{A}U(t) \ldots\ldots\ldots..(7)$$

For such $\widehat{A}$ we have the following cases:

Case 1: $\widehat{A}$ has distinct real eigenvalues (i.e $\lambda_1 \neq \lambda_2$). Then the general solution of system (7) is of the form

$$U_h(t) = c_1 e^{\frac{\lambda_1}{\alpha}t^{\alpha}}\xi_1 + c_2 e^{\frac{\lambda_2}{\alpha}t^{\alpha}}\xi_2$$

where ξ_1 and ξ_2 are the corresponding eigenvectors of λ_1 and λ_2, respectively.

Case 2: $\widehat{A}$ has equal eigenvalues (i.e $\lambda_1 = \lambda_2 = \lambda$), then we have the following sub-cases:

Case 2.1: λ has two linearly independent eigenvectors ξ_1 and ξ_2. Then the general solution of system (7) is given by

$$U_h(t) = (c_1\xi_1 + c_2\xi_2)\, e^{\frac{\lambda}{\alpha}t^{\alpha}}$$

Case 2.2: λ has a single linearly independent eigenvector ξ. Then the general solution of system (7) is given by

$$U_h(t) = (c_1\xi + c_2(t\xi + \eta))\, e^{\frac{\lambda}{\alpha}t^\alpha}$$

where η satisfies equation $\left(\widehat{A} - \lambda I\right)\eta = \xi$ and I is the identity matrix.

Case 3: $\widehat{A}$ has complex conjugate eigenvalues, $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. Let $\xi_1 + \xi_2 i$ and $\xi_1 - \xi_2 i$ are the corresponding eigenvectors of λ_1 and λ_2, respectively, where ξ_1 and ξ_2 are vectors.

Then the general solution of system (7) is given by

$$U_h(t) = e^{\frac{a}{\alpha}t^\alpha}\left[c_1\left(\xi_1\cos\left(\frac{b}{\alpha}t^\alpha\right) - \xi_2\sin\left(\frac{b}{\alpha}t^\alpha\right)\right) + c_2\left(\xi_1\sin\left(\frac{b}{\alpha}t^\alpha\right) + \xi_2\cos\left(\frac{b}{\alpha}t^\alpha\right)\right)\right]$$

Now, to get a particular solution, form the matrix $\Psi(t) = (e^{\lambda_1 t}\xi_1 : e^{\lambda_2 t}\xi_2)$ which is known as the fundamental matrix of the system. From [10] it is known that the inverse of Ψ exists and a particular solution to system (6) is given by the formula:

$$U_p(t) = \Psi(t)\, J_\alpha(\Psi^{-1}(t)\, F(t))$$

where $J_\alpha(f)$ is defined by $J_\alpha(f) = \int_a^t \frac{f(x)}{x^{(1-\alpha)}}dx$, and the integral is the usual Rimann improper integral, and $\alpha \in (0,1)$.

Furthermore, for any of the above cases the general solution of system (6) is of the form

$$U_g(t) = U_h(t) + U_p(t)$$

Where $U_h(t)$ is the general solution of the corresponding homogeneous system and $U_p(t)$ is the particular solution of the system.

By the initial condition $u(0) = x_0$, we have

$$(\xi_1 : \xi_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(\delta_1\ \delta_2) = x_0$$

Since ξ_1 and ξ_2 are linearly independent eigenvectors, then the matrix $(\xi_1 : \xi_2)$ is invertible. Multiplying $(\xi_1 : \xi_2)^{-1}$ to both sides, we get

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(\delta_1\ \delta_2) = (\xi_1 : \xi_2)^{-1}x_0$$

Taking the inner product of δ_1 and δ_2 to both sides, we get

$$c_i = \left\langle (\xi_1 : \xi_2)^{-1} x_0, \delta_i \right\rangle,\ i = 1,2.$$

So, the Problem has a unique solution.

Now, if $[\delta_1, \delta_2]$ is not an invariant subspace of A, then $A\delta_1 = a_{11}\delta_1 + a_{12}\delta_2 + a_{13}\theta_1$ and $A\delta_2 = a_{21}\delta_1 + a_{22}\delta_2 + a_{23}\theta_2$, where θ_1, θ_2 are orthogonal with $\{\delta_1, \delta_2\}$, and a_{13}, a_{23} not both are equal to zero.

So, the Problem becomes

$$\begin{aligned} u_1^{(\alpha)}(t)\,\delta_1 + u_2^{(\alpha)}(t)\,\delta_2 &= a_{11}u_1(t)\,\delta_1 + a_{12}u_1(t)\,\delta_2 + a_{13}u_1(t)\,\theta_1 \\ &\quad + a_{21}u_2(t)\,\delta_1 + a_{22}u_2(t)\,\delta_2 + a_{23}u_2(t)\,\theta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2 \ldots\ldots (8) \end{aligned}$$

By equating the coefficients of $\delta_1, \delta_2, \theta_1$ and θ_2 in both sides, we have

$$a_{13}u_1(t) = 0 \text{ and } a_{23}u_2(t) = 0$$

So, we have the following cases:

Case (i) : $a_{13} = 0$ and $a_{23} = 0$. This case contradicts the assumption on a_{13} and a_{23}.

Case (ii) : If $u_1(t) = 0$ and $u_2(t) = 0$, then $u(t) = 0$, and hence the Problem has the trivial unique solution.

Case (iii) : If $(a_{13} = 0 \text{ and } u_2(t) = 0)$ or $(a_{23} = 0 \text{ and } u_1(t) = 0)$, then $u(t) = u_i(t)\,\delta_i$, for some $i = 1,2$.

Then equation (6) becomes

$$u_i^{(\alpha)}(t)\,\delta_i = a_{i1}u_i(t)\,\delta_1 + a_{i2}u_i(t)\,\delta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2 \ldots\ldots\ldots\ldots (9)$$

By taking the inner product of δ_i to both sides of (9), we have

$$u_i^{(\alpha)}(t) = a_{ii}u_i(t) + f_i(t) \ldots\ldots\ldots\ldots (10)$$

Equation (10) is first order linear $\alpha-$differential equation, and has a general solution of the form

$$u_i(t) = e^{\frac{a_{ii}}{\alpha}t}\left(J_\alpha(e^{-\frac{a_{ii}}{\alpha}t^\alpha} f_i(t)) + c\right)$$

And by the initial condition $u(0) = x_0$, we have

$$u_i(0) = c$$

Then

$$u(0) = u_i(0)\,\delta_i = c\ \delta_i = x_0$$

Taking the inner product of δ_i to both sides, we have

$$c = \langle x_0, \delta_i \rangle$$

And hence the Problem has a unique solution. □

Theorem 3.2 Consider Problem (E_2). Let $B_2 = B\mid_{[\delta_1,\delta_2]}$be orthogonally diagonalizable linear operator with respect to the orthonormal basis $\{\theta_1, \theta_2\}$ and corresponding eigenvalues λ_1, λ_2 such that $\langle A\theta_j, \delta_i \rangle \neq 0$ for some $i, j \in \{1,2\}$. Then Problem (E_2) has a unique solution.

Proof : Let $D = diag(\lambda_1, \lambda_2)$ be the matrix representation of B_2 with respect to $\{\theta_1, \theta_2\}$.

Now, if $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then B_2 is invertible and we can use Theorem 5.1, so the problem has a unique solution.

Assume $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Let $u(t) = v_1(t)\,\theta_1 + v_2(t)\,\theta_2$. Then $u^{(\alpha)}(t) = v_1^{(\alpha)}(t)\,\theta_1 + v_2^{(\alpha)}(t)\,\theta_2$.

Hence,

$$v_1^{(\alpha)}(t)\,B\theta_1 + v_2^{(\alpha)}(t)\,B\theta_2 = v_1(t)\,A\theta_1 + v_2(t)\,A\theta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2.$$

Since $B\theta_1 = \lambda_1\theta_1$ and $B\theta_2 = 0$, we have

$$\lambda_1 v_1^{(\alpha)}(t)\,\theta_1 = v_1(t)\,A\theta_1 + v_2(t)\,A\theta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2 \text{...........} (11)$$

Taking the inner product of θ_1 and θ_2 with both sides of (5.11), we get

$$\lambda_1 v_1^{(\alpha)}(t) = v_1(t)\,\langle A\theta_1, \theta_1\rangle + v_2(t)\,\langle A\theta_2, \theta_1\rangle + f_1(t)\,\langle \delta_1, \theta_1\rangle + f_2(t)\,\langle \delta_2, \theta_1\rangle \text{...........} (12)$$

and

$$0 = v_1(t)\,\langle A\theta_1, \theta_2\rangle + v_2(t)\,\langle A\theta_2, \theta_2\rangle + f_1(t)\,\langle \delta_1, \theta_2\rangle + f_2(t)\,\langle \delta_2, \theta_2\rangle \text{...........} (13)$$

Now putting $\alpha_{ji} = \langle A\theta_i, \theta_j \rangle$ and $\beta_{ji} = \langle \delta_i, \theta_j \rangle, i, j = 1,2$, then equation (13) gives the following cases:

Case $(\mathbf{i})$: If $\alpha_{22} \neq 0$, then

$$v_2(t) = \frac{-(\alpha_{21} v_1(t) + \beta_{21} f_1(t) + \beta_{22} f_2(t))}{\alpha_{22}} \text{..........} (14)$$

Substituting (14) in (12), we get

$$v_1^{(\alpha)}(t) = K_1 v_1(t) + K_2 f_1(t) + K_3 f_2(t) \text{..........} (15)$$

where $K_1 = \frac{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}{\lambda_1 \alpha_{22}}, K_2 = \frac{\alpha_{22}\beta_{11} - \alpha_{12}\beta_{21}}{\lambda_1 \alpha_{22}}$ and $K_3 = \frac{\alpha_{22}\beta_{12} - \alpha_{12}\beta_{22}}{\lambda_1 \alpha_{22}}$.

Then (15) is a first order linear $\alpha-$differential equation and it has a general solution of the form

$$v_1(t) = e^{\frac{K_1}{\alpha} t^\alpha} J_\alpha \left(e^{-\frac{K_1}{\alpha} t^\alpha} (K_2 f_1(t) + K_3 f_2(t)) + c \right) \text{..........} (16)$$

where the integral is the usual Rimann improper integral, and $\alpha \in (0,1)$.

Substituting (16) in (14), we get

$$\begin{aligned} v_2(t) &= -\frac{1}{\alpha_{22}} (\alpha_{21} e^{\frac{K_1}{\alpha} t^\alpha} J_\alpha \left(e^{-\frac{K_1}{\alpha} t^\alpha} (K_2 f_1(t) + K_3 f_2(t)) + c \right) \\ &\quad + \beta_{21} f_1(t) + \beta_{22} f_2(t)) \text{..........} (17) \end{aligned}$$

From (16), (17) and the initial condition $u(0) = x_0$, we can determine constant c uniquely as follows

$$\begin{aligned} u(0) &= x_0 = v_1(0)\theta_1 + v_2(0)\theta_2 \\ &= c\theta_1 - \frac{1}{\alpha_{22}} (\alpha_{21} c + \beta_{21} f_1(0) + \beta_{22} f_2(0)) \theta_2 \end{aligned}$$

By taking the inner product of θ_1 with both sides, we get

$$c = \langle x_0, \theta_1 \rangle$$

Thus, Problem (E_2) has a unique solution.

Case $(\mathbf{ii})$: If $\alpha_{22} = 0$, then we have the following sub-cases:

Case $(\mathbf{ii.1})$: If $\alpha_{12} \neq 0$ and $\alpha_{21} \neq 0$, then from equations (12) and (13)

$$v_1(t) = -\frac{1}{\alpha_{21}} (\beta_{21} f_1(t) + \beta_{22} f_2(t)) \text{..........} (18)$$

And

$$v_2(t) = \frac{1}{\alpha_{12}}\left(\lambda_1 v_1^{(\alpha)}(t) - \alpha_{11}v_1(t) - \beta_{11}f_1(t) - \beta_{12}f_2(t)\right) \text{..........} (19)$$

Substituting equation (18) in (19), we get a unique solution for Problem (E_2).

Case $(\mathbf{ii.2})$: If $\alpha_{12} \neq 0$ and $\alpha_{21} = 0$, then in this case we have one equation

$$\lambda_1 v_1^{(\alpha)}(t) = \alpha_{11}v_1(t) + \alpha_{12}v_2(t) + \beta_{11}f_1(t) + \beta_{12}f_2(t) \text{..........} (20)$$

So, we need another equation to find $v_1(t)$ and $v_2(t)$.

Now, by the assumption on A, without loss of generality, we can assume that $\langle A\theta_2, \delta_1\rangle \neq 0$.

Taking the inner product of δ_1 to both sides of equation (11), we get

$$\lambda_1\beta_{11}v_1^{(\alpha)}(t) = v_1(t)\langle A\theta_1, \delta_1\rangle + v_2(t)\langle A\theta_2, \delta_1\rangle + f_1(t)$$

If $\gamma_{1i} = \langle A\theta_i, \delta_1\rangle, i = 1, 2$, then the above equation becomes

$$\lambda_1\beta_{11}v_1^{(\alpha)}(t) = \gamma_{11}v_1(t) + \gamma_{12}v_2(t) + f_1(t) \text{..........} (21)$$

From equations (20) and (21), we have

$$v_1^{(\alpha)}(t) = h_1v_1(t) + h_2f_1(t) + h_3f_2(t),$$

where $h_1 = \frac{\gamma_{12}\alpha_{11}-\gamma_{11}\alpha_{12}}{\lambda_1(\gamma_{12}-\alpha_{12}\beta_{11})}, h_2 = \frac{\gamma_{12}\beta_{11}-\alpha_{12}}{\lambda_1(\gamma_{12}-\alpha_{12}\beta_{11})}$ and $h_3 = \frac{\gamma_{12}\beta_{12}}{\lambda_1(\gamma_{12}-\alpha_{12}\beta_{11})}$.

This is a first order linear $\alpha-$differential equation, and has a general solution of the form

$$v_1(t) = e^{\frac{h_1}{\alpha}t^\alpha}\left(J_\alpha(e^{-\frac{h_1}{\alpha}t^\alpha}(h_2f_1(t) + h_3f_2(t)) + c\right) \text{..........} (22)$$

where the integral is the usual Rimann improper integral, and $\alpha \in (0,1)$.

By substituting equation (22) in (21), we determine $v_2(t)$ uniquely.

And again, in equation (22), $c = \langle x_0, \theta_1\rangle$ by the initial condition $u(0) = x_0$.

Case $(\mathbf{ii.3})$: If $\alpha_{12} = 0$ and $\alpha_{21} \neq 0$, then $v_1(t)$ determine uniquely by equation (18). And by substituting equation (18) in (21), we determine $v_2(t)$ uniquely.

Case $(\mathbf{ii.4})$: If $\alpha_{12} = 0$ and $\alpha_{21} = 0$, then $v_1(t)$ determine uniquely by substituting $\alpha_{12} = 0$ in equation (22). By substituting equation (22) in (21), we determine $v_2(t)$ uniquely.

Hence, the Problem has a unique solution. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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