ON SOME SATURATED NUMERICAL SEMIGROUPS WITH MULTIPLICITY EIGHT

AHMET ÇELİK* AND SEDAT İLHAN

University of Dicle, Department of Mathematics, 21280 Diyarbakır / TURKEY

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Abstract: In this paper, we will investigate saturated numerical semigroups with multiplicity 8 and conductor $C$. Also, we will give formulas for Frobenius number, determiner number and genus of these semigroups.

Keywords: saturated numerical semigroups; Frobenius number; genus.

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1. Introduction

We consider that $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$. Let $\mathbb{Z}$ be integer set. The subset $S \subseteq \mathbb{N}$ is a numerical semigroup if

i. $x + y \in S$, for $x, y \in S$

ii. $\gcd(S) = 1$

iii. $0 \in S$

*Corresponding author
E-mail address: celk_ahmet@hotmail.com
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(Here, $\gcd(S)$ = greatest common divisor the elements of $S$).

A numerical semigroup $S$ can be written that

$$S = \langle x_1, x_2, ..., x_n \rangle = \left\{ \sum_{i=1}^{n} a_i x_i : a_i \in \mathbb{N} \right\}.$$  

$T \subseteq \mathbb{N}$ is minimal system of generators of $S$ if $\langle T \rangle = S$ and there isn’t any subset $M \subseteq T$ such that $\langle M \rangle = S$. Also, $\mu(S) = \min \{x \in S : x > 0\}$ is called as multiplicity of $S$ (see [3]).

Let $S$ be a numerical semigroup, then $F(S) = \max (\mathbb{Z} \setminus S)$ is called as Frobenius number of $S$.

Also, $C$ is conductor of $S$ if $C = F(S) + 1$, and $n(S) = \text{Card} \left( \{0,1,2,...,F(S)\} \cap S \right)$ is called as the determiner number of $S$.

If $S$ is a numerical semigroup such that $S = \langle x_1, x_2, ..., x_n \rangle$, then we observe that

$$S = \langle x_1, x_2, ..., x_n \rangle = \left\{ s_0 = 0, s_1, s_2, ..., s_{n-1}, s_n = F(S) + 1, \rightarrow \right\},$$

where $s_i < s_{i+1}$, $n = n(S)$ and the arrow means that every integer greater than $F(S) + 1$ belongs to $S$ for $i = 1, 2, ..., n = n(S)$.

If $y \in \mathbb{N}$ and $y \not\in S$, then $y$ is called gap of $S$. We denote the set of gaps of $S$, by $H(S)$, i.e, $H(S) = \mathbb{N} \setminus S$. The $G(S) = \#(H(S))$ is called the genus of $S$. It known that $G(S) = F(S) + 1 - n(S)$ (see [3]).

A numerical semigroup $S$ is Arf if $x_1 + x_2 - x_3 \in S$, for all $x_1, x_2, x_3 \in S$ such that $x_1 \geq x_2 \geq x_3$. Also, a numerical semigroup $S$ is saturated if $s + d_1 s_1 + d_2 s_2 + ... + d_m s_m \in S$, where $s, s_i \in S$ and $d_i \in \mathbb{Z}$ such that $d_1 s_1 + d_2 s_2 + ... + d_m s_m \geq 0$ and $s_i \leq s$ for $i = 1, 2, ..., m$. A saturated numerical is Arf, but an Arf numerical semigroup need not be saturated. For example, $S = \{8,13,17,18,19,20,22,23\} = \{0,8,13,16,\rightarrow \}$ is Arf numerical semigroup but it is not saturated. Many researchs have studied on saturated numerical semigroups.
Especially, saturated numerical semigroups with multiplicity 3, 4, 5, 6 and 7 have studied by Ilhan et al. (for details, see [1], [4], [5], [6], [7], [8]). In this paper, we will give some saturated numerical semigroups multiplicity 8 and conductor $C$. Also, we will obtain formulas for Frobenius number, determiner number and genus of these saturated numerical semigroups.

2. Main results

Proposition 2.1. ([3]) Let $S$ be a numerical semigroup. Then following conditions are equivalent:

1) $S$ is a saturated numerical semigroup.

2) $y + d_S(y) \in S$ for all $y \in S$, $y > 0$ where $d_S(y) = \gcd\{x \in S : x \leq y\}$.

3) $y + md_S(y) \in S$ for all $y \in S$, $y > 0$ and $m \in \mathbb{N}$.

Now, we give our first result in the following theorem.

Theorem 2.2. Let $C \neq 8q + 1$ ($q \in \mathbb{N}$, $q \geq 1$) be an integer and $S$ a numerical semigroup with multiplicity 8 and conductor $C \geq 8$. Then

1) The semigroup $S = \langle 8, C + 1, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle$ is saturated numerical semigroup, where $C \equiv 0 \pmod{8}$,

2) The semigroup $S = \langle 8, C, C + 1, C + 2, C + 3, C + 4, C + 5, C + 7 \rangle$ is saturated numerical semigroup, where $C \equiv 2 \pmod{8}$,

3) The semigroup $S = \langle 8, C, C + 1, C + 2, C + 3, C + 4, C + 6, C + 7 \rangle$ is saturated numerical semigroup, where $C \equiv 3 \pmod{8}$,

4) The semigroup $S = \langle 8, C, C + 1, C + 2, C + 3, C + 5, C + 6, C + 7 \rangle$ is saturated numerical semigroup, where $C \equiv 4 \pmod{8}$,
5) The semigroup $S = \langle 8, C + 1, C + 2, C + 4, C + 5, C + 6, C + 7 \rangle$ is saturated numerical semigroup, where $C \equiv 5 \pmod{8}$.

6) The semigroup $S = \langle 8, C + 1, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle$ is saturated numerical semigroup, where $C \equiv 6 \pmod{8}$.

7) The semigroup $S = \langle 8, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle$ is saturated numerical semigroup, where $C \equiv 7 \pmod{8}$.

**Proof.** We will prove only one case. Other cases can be proved in a similar way.

Let prove case (1).

Let $C = 8q \ (q \in \mathbb{N}, \ q \geq 1)$ be an integer. Then we have

$$S = \langle 8, C + 1, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle = \langle 8q + 1, 8q + 2, 8q + 3, 8q + 4, 8q + 5, 8q + 6, 8q + 7 \rangle .$$

$$= \{0, 8, 16, 24, \ldots, 8(q - 1), 8q, \ldots \}.$$

In this case,

i. if $s > C$ then $s + d_\delta(s) = s + 1 \in S$, since $d_\delta(s) = 1$ and $s \in S, s > 0$. Thus, we obtain that $S$ is saturated numerical semigroup by Proposition 2.1.

ii. if $s \leq C$ then $s + d_\delta(s) = s + 8 \in S$, since $d_\delta(s) = 8$ and $s \in S, s > 0$. Thus, we obtain that $S$ is saturated numerical semigroup by Proposition 2.1.

**Theorem 2.3.** Let $C = 8q \ (q \in \mathbb{N}, \ q \geq 1)$ be an integer and $S = \langle 8, C + 1, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle$ is saturated numerical semigroup with multiplicity 8 and conductor $C$. Then, we have

a) $F(S) = 8q - 1$,

b) $n(S) = q$,

c) $G(S) = 7q$. 
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**Proof.** Let \( C = 8q \ (q \in \mathbb{N}, \ q \geq 1) \) be an integer and
\[ S = \langle 8, C + 1, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle \]
is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then we write that

a) \( F(S) = 8q - 1 \) since \( C = F(S) + 1 = 8q \).

b) Since \( C = 8q \ (q \in \mathbb{N}, \ q \geq 1) \), \( S \) is
\[ S = \langle 8, C + 1, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle = \langle 8q + 1, 8q + 2, 8q + 3, 8q + 4, 8q + 5, 8q + 6, 8q + 7 \rangle = \{0, 8, 16, 24, \ldots, 8(q - 1), 8q, \ldots\}. \]
So, we have
\[ n(S) = \#(\{0, 1, 2, \ldots, 8q - 8, \ldots, 8q - 2, 8q - 1\} \cap S) = \#(\{0, 8, 16, 24, \ldots, 8(q - 1)\}) = q. \]

c) \( G(S) = F(S) + 1 - n(S) = 8q - 1 + 1 - q = 7q. \)

**Theorem 2.4.** Let \( C = 8q + 2 \ (q \in \mathbb{N}, \ q \geq 1) \) be an integer and
\[ S = \langle 8, C, C + 1, C + 2, C + 3, C + 4, C + 5, C + 7 \rangle \]
is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then, we have

a) \( F(S) = 8q + 1, \)

b) \( n(S) = q + 1, \)

c) \( G(S) = 7q + 1. \)

**Proof.** Let \( C = 8q + 2 \ (q \in \mathbb{N}, \ q \geq 1) \) be an integer and
\[ S = \langle 8, C, C + 1, C + 2, C + 3, C + 4, C + 5, C + 7 \rangle \]
is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then,

a) It is trivial \( F(S) = 8q + 1 \) from \( C = F(S) + 1. \)

b) If \( S = \langle 8, C, C + 1, C + 2, C + 3, C + 4, C + 5, C + 7 \rangle \) is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then we write
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\[ S = \langle 8, C, C+1, C+2, C+3, C+4, C+5, C+7 \rangle \]
\[ = \langle 8,8q+2,8q+3,8q+4,8q+5,8q+6,8q+7,8q+9 \rangle \]
\[ = \{0,8,16,24,\ldots,8(q-1),8q,8q+2,\ldots\}. \]

In this case, \( n(S) = \#(\{0,1,2,\ldots,8q-8,\ldots,8q-2,8q-1,8q,8q+1,8q+2\} \cap S) \]
\[ = \#(\{0,8,16,24,\ldots,8(q-1),8q\}) = q+1. \]

c) \( G(S) = F(S) + 1 - n(S) = 8q + 1 + 1 - (q + 1) = 7q + 1. \)

The following theorems will be given without their proofs. Anyone can be proved by similar ways in Theorem 2.3 and Theorem 2.4.

**Theorem 2.5.** Let \( C = 8q + 3 (q \in \mathbb{N}, q \geq 1) \) be an integer and \( S = \langle 8, C, C+1, C+2, C+3, C+4, C+6, C+7 \rangle \) is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then, we have

a) \( F(S) = 8q + 2, \)

b) \( n(S) = q + 1, \)

c) \( G(S) = 7q + 2. \)

**Theorem 2.6.** Let \( C = 8q + 4 (q \in \mathbb{N}, q \geq 1) \) be an integer and \( S = \langle 8, C, C+1, C+2, C+3, C+5, C+6, C+7 \rangle \) is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then, we have

a) \( F(S) = 8q + 3, \)

b) \( n(S) = q + 1, \)

c) \( G(S) = 7q + 3. \)

**Theorem 2.7.** Let \( C = 8q + 5 (q \in \mathbb{N}, q \geq 1) \) be an integer and \( S = \langle 8, C, C+1, C+2, C+4, C+5, C+6, C+7 \rangle \) is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then, we have
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\[ a) \quad F(S) = 8q + 4, \]
\[ b) \quad n(S) = q + 1, \]
\[ c) \quad G(S) = 7q + 4. \]

**Theorem 2.8.** Let \( C = 8q + 6 (q \in \mathbb{N}, q \geq 1) \) be an integer and \( S = \langle 8, C, C + 1, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle \) is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then, we have

\[ a) \quad F(S) = 8q + 5, \]
\[ b) \quad n(S) = q + 1, \]
\[ c) \quad G(S) = 7q + 5. \]

**Theorem 2.9.** Let \( C = 8q + 7 (q \in \mathbb{N}, q \geq 1) \) be an integer and \( S = \langle 8, C, C + 1, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle \) is saturated numerical semigroup with multiplicity 8 and conductor \( C \). Then, we have

\[ a) \quad F(S) = 8q + 6, \]
\[ b) \quad n(S) = q + 1, \]
\[ c) \quad G(S) = 7q + 6. \]

**Example 2.10.** If we take \( C = 15 \) (for \( q = 1 \)) in Theorem 2.9, then we write

\[ S = \langle 8, 15, 16, 17, 18, 19, 20, 21, 22 \rangle = \{ 0, 8, 15, \ldots \}. \]

In this case, we find that \( F(S) = 8q + 6 = 14 \), \( n(S) = q + 1 = 2 \) and \( G(S) = 7q + 6 = 13 \).

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Conflict of Interests

The authors declare that there is no conflict of interests.

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