ON PAIRWISE C-CLOSED SPACE IN BITOPOLOGICAL SPACE

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Abstract. In this paper, we will obtain several results concerning the properties of pairwise C-closed spaces and to study the relations of pairwise C-closed spaces with some related pairwise topological properties like pairwise compactness, sequential spaces, pairwise quasi-k spaces and pairwise C-sequential spaces.

Keywords: pairwise C-closed spaces; pairwice k-spaces; pairwise quasi-k-spaces; pairwise tightness; sequential; pairwise C-sequential.

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1. INTRODUCTION

The study of bitopological spaces was first initiated by J. C. Kelly [1] in 1963 and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. In this paper, we study the notion of pairwise C-closed spaces in bitopological spaces and their relation with other bitopological concepts. we will show that pairwise countably compact C-closed space has countable tightness and we will introduce characterization of pairwise sequential compact hausdorff spaces . We use $R$ to denote the set of all real and $P$- to denote pairwise, $Cl$ to denote the closure of a set, and $t(X)$ to denote the tightness of $X$.

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2. Pairwise C-Closed Space

Definition 2.1: [6] A cover $V$ of the bitopological space $(X, \tau_1, \tau_2)$ is called pairwise open cover if $V \in \tau_1 \lor \tau_2$.

Definition 2.2: A bitopological space $(x, \tau_1, \tau_2)$ is said to be pairwise countably compact if every countably pairwise open cover of $X$ has finite subcover.

Definition 2.3: [1] A bitopological space $(x, \tau_1, \tau_2)$ is called pairwise hausdorff if for any tow distinct points $x, y \in X$, there exist disjoint $V_1 \in \tau_1$ and $V_2 \in \tau_2$ with $x \in V_1$ and $y \in V_2$.

Definition 2.4: [4] In a space $(x, \tau_1, \tau_2)$, $\tau_1$ is said to be regular with respect to $\tau_2$ if, for each point $x \in X$ and each $\tau_1$-closed subset $F$ s.t $x \notin F$, there are $\tau_1$-open set $U$ and $\tau_2$-open set $V$ s.t $x \in U$ and $F \subset V$ and $U \cap V = \emptyset$. $(x, \tau_1, \tau_2)$ is p- regular if $\tau_1$ regular with respect to $\tau_2$ and vise versa.

Reilly [5] proves the following proposition:

Proposition 2.5: If $(X, \tau_1, \tau_2)$ is a bitopological space, the following are equivalent:

a) $\tau_1$ is regular with respect to $\tau_2$

b) For each point $x \in X$ and $\tau_1$-open set $U$ containing $X$, there is a $\tau_1$-open set $V$ such that $X \in V \subset \tau_2$-cl $V \subset U$

Definition 2.6: A bitopological space $(x, \tau_1, \tau_2)$ is called pairwise C- closed if every $\tau_1$- countably compact subset of $X$ is $\tau_2$- closed in $X$ and every $\tau_2$- countably compact subset of $X$ is $\tau_1$- closed in $X$.

Definition 2.7: Let $(x, \tau_1, \tau_2)$ be bitopological space, $A \subset X$, we say that $x \in X$ is a $\tau_i$- cluster point for $A$, if for every $\tau_i$-open set $U$ containing $x$, $U \cap A / \{x\} \neq \emptyset$ i=1,2.

Definition 2.8: A bitopological space $(X, \tau_1, \tau_2)$ is called pairwise C-closed if every non $\tau_1$- closed subset $A$ of $X$ contains a sequence which has no $\tau_2$-cluster point in $A$, and every non $\tau_2$-closed subset $B$ of $X$ contains a sequence which has no $\tau_1$-cluster point in $B$.

From definition of pairwise C-closed we have:

Corollary 2.9: Every subspace of pairwise c-closed is pairwise C-closed.

Definition 2.10: A bitopological space $(x, \tau_1, \tau_2)$ is said to be sequential if both $(x, \tau_1)$ and $(x, \tau_2)$ sequential, i.e every non $\tau_1$- closed subset $A$ of $X$ contains a sequence converting to a
point in $X \setminus A$ and every non $\tau_2$-closed subset $B$ of $X$ contains a sequence converting to a point in $X \setminus B$.

Theorem 2.11: Let $(X, \tau_1, \tau_2)$ be pairwise Hausdorff space, let $(x_n)$ be a convergent sequence in $X$, then $(x_n)$ has exactly one limit point.

Proof: Suppose the contrary. Then $X_n \to x$ and $X_n \to y$ for some $x \neq y$, there exist disjoint $U \in \tau_1$ and $V \in \tau_2$ with $x \in U$ and $y \in V$. Therefore, there exist $N_U \in \mathbb{N}$ such that $x_n \in U$ for every $n > N_U$ and $N_V \in \mathbb{N}$ such that $x_n \in V$ for every $n > N_V$. Choose $N = \max\{N_U, N_V\}$. Thus, there exist $N \in \mathbb{N}$ such that $x_n \in U, x_n \in V$ for every $n > N$.

But $U \cap V = \emptyset$, which is the contradiction.

Proposition 2.12: Every pairwise Hausdorff sequential space is pairwise C-closed.

Proof: let $A$ be non $\tau_1$-closed subset of $X$, since $X$ is sequential, there exist a sequence $(x_n)$ converting to a point in $X \setminus A$ say $x$. By uniqueness of limit point of the sequence in pairwise hausdorff space, we conclude that $(X_n)$ has no $\tau_2$-cluster point in $A$, similarly we can proof that every non $\tau_2$-closed subset $B$ of $X$ contain sequence has no $\tau_1$-cluster point in $B$.

Hence the result.

Proposition 2.13: If $X$ is pairwise Hausdorff and every pairwise countably compact subset of $X$ is sequential then $X$ is pairwise C-closed.

Proof: let $A$ be $\tau_1$-countably compact subset of $X$ and suppose that $A$ is not $\tau_2$-closed in $X$, then there exist $x \in \tau_2$-$Cl A \setminus A$, let $B = A \cup \{x\}$, then $B$ is also $\tau_1$-countably compact, now $A$ is not $\tau_2$-closed in $B$, Since $B$ is sequential then there exist sequence $x_n$ in $A$ s.t $x_n \to B \setminus A = \{x\}$. Therefore there exist seq $x_n$ in $A$ has no $\tau_1$-cluster point in $A$, this is contradiction.

Note that every pairwise countably compact subset of a bitopological space $X$ may be sequential and $X$ may still be not sequential, such is, the following example:

Example 2.14: The space of all continuous real valued function on the interval $[0,1]$ and generalize example [7] by letting $\tau_1 = \tau_2 =$ the point wise convergence topology.

Definition 2.15: [2] A map $f : X \to Y$ from bitopological space $(X, \tau_1, \tau_2)$ to another bitopological space $(Y, \sigma_1, \sigma_2)$ is called pairwise continous if $f$ is continous both as a map from $(X, \tau_1)$ to $(Y, \sigma_1)$ and as a map from $(X, \tau_2)$ to $(Y, \sigma_2)$. 
Proposition 2.16: Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a pairwise continuous one-to-one function, if \( (X, \tau_1, \tau_2) \) is pairwise Hausdorff space and \( (Y, \sigma_1, \sigma_2) \) is pairwise C-closed, then \( (X, \tau_1, \tau_2) \) is pairwise C-closed.

Proof: Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be pairwise continuous and one-to-one map, then \( f: (X, \tau_1) \rightarrow (Y, \sigma_1) \) and \( f: (X, \tau_2) \rightarrow (Y, \sigma_2) \) are both continuous, let \( A \) be \( \tau_1 \)-countably compact subset of \( X \), then \( f(A) \) is \( \sigma_1 \)-countably compact subset of \( Y \), but \( Y \) is pairwise C-closed, thus \( f(A) \) is \( \sigma_2 \)-closed subset of \( Y \), since \( f \) is pairwise continuous and one-to-one map, we get \( f^{-1}(f(A)) = A \) is \( \tau_2 \)-closed subset of \( X \). Similarly we can prove that if \( A \) is \( \tau_2 \)-countably compact subset of \( X \) then \( A \) is \( \tau_1 \)-closed subset of \( X \), this completes the proof.

Corollary 2.17: In a bitopological space \( (x, \tau_1, \tau_2) \), if \( X \) has a weaker bitopological space which is pairwise C-closed, then \( X \) is is pairwise C-closed.

Proposition 2.18: Let \( X \) be a pairwise regular space and every point has a pairwise C-closed neighbourhood, then \( X \) is pairwise C-closed.

Proof: Let \( A \) be \( \tau_1 \)-countably cooompact subset of \( X \) and \( x \in \tau_2\text{-cl}(A) \), want to show that \( x \in A \), let \( U \) be a \( \tau_2 \)-open set containing \( x \) and \( U \) is pairwise C-closed, then by \( p \)-regularity there is a \( \tau_2 \)-open set \( V \) such that \( x \in V \subset \tau_1\text{-cl}(V) \subset U \). Since \( A \) is \( \tau_1 \)-countably compact, then \( \tau_1\text{-cl}(V) \cap A \) is also \( \tau_1 \)-countably compact subset of \( U \), hence it is \( \tau_2 \)-closed subset of \( U \). But \( x \in \tau_2\text{-cl}(\tau_1\text{-cl}(V) \cap A) = \tau_1\text{-cl}(V) \cap A \), hence \( x \in A \), therefore \( A \) is \( \tau_2 \)-closed subset of \( X \). Similarly, we can prove that if \( A \) is \( \tau_2 \)-countably compact subset of \( X \), then \( A \) is \( \tau_1 \)-closed subset of \( A \), this completes the proof.

Definition 2.18: [3] The tightness of \( x \) \( \text{t}(X) \) denoted by the smallest cardinal numbers \( \Gamma \)'s, that whenever \( A \subset X \) and \( x \in \overline{A} \), then there is a subset \( B \) of \( A \) so that \( |B| \leq \Gamma \) and \( x \in \overline{B} \).

Definition 2.19: A bitopological space \( (x, \tau_1, \tau_2) \) is said to have a pairwise countable tightness property if it has \( \tau_1 \)-countable tightness and \( \tau_2 \)-countable tightness property.

Definition 2.20: A subset \( A \) of bitopological space \( (x, \tau_1, \tau_2) \) is called pairwise \( k \)-closed if for every pairwise compact subset \( K \) of \( X \), \( A \cap K \) is \( \tau_1 \)-closed \( (\tau_2 \)-closed) in \( K \).

Definition 2.21: A subset \( A \) of bitopological space is called pairwise quasi \( k \)-closed if for every pairwise countably compact subset \( K \) of \( X \), \( A \cap K \) is \( \tau_1 \)-closed \( (\tau_2 \)-closed) in \( K \).
Definition 2.22: A bitopological space \((x, \tau_1, \tau_2)\) is said to be pairwise k-space if every \(\tau_1\)-\(k\)-closed (\(\tau_2\)-\(k\)-closed) subset of \(X\) is \(\tau_1\)-closed (\(\tau_2\)-closed) in \(X\).

Example 2.23: Consider \((R, \tau_1, \tau_2)\) where \(\tau_1\) is the discrete topology and \(\tau_2 = \{U \subset R : 0 \notin R\} \cup \{R\}\), then \((R, \tau_1, \tau_2)\) is a a pairwise-k space.

Definition 2.24: A bitopological space \((x, \tau_1, \tau_2)\) is said to be pairwise quasi-k-space if every \(\tau_1\)-quasi-\(k\)-closed (\(\tau_2\)-quasi-\(k\)-closed) subset of \(X\) is \(\tau_1\)-closed (\(\tau_2\)-closed) in \(X\).

Proposition 2.25: If \(X\) is a pairwise hausdorff, pairwise quasi-\(k\) and (in particular pairwise countably compact or pairwise \(k\)) and pairwise C-closed space, then \(t(X) \leq \omega_0\).

Proof: Let \(A \subset X\), \(Y = \bigcup \{\tau_1\text{-cl}(B) : B \subset A \text{ and } |B| \leq \omega_0\}\) and \(Z = \bigcup \{\tau_2\text{-cl}(F) : F \subset A \text{ and } |F| \leq \omega_0\}\). We need to show that \(\tau_1\text{-cl}(A) = Y\) and \(\tau_2\text{-cl}(B) = Z\). Now \(A \subset Y \subset \tau_1\text{-cl}(A)\) and \(A \subset Z \subset \tau_2\text{-cl}(A)\), we need to show that \(Y\) is \(\tau_1\)-closed in \(X\) and \(Z\) is \(\tau_2\)-closed in \(X\). Assume the contrary that \(Y\) is not \(\tau_1\)-closed in \(X\) or \(Z\) is not \(\tau_2\)-closed in \(X\). If \(Y\) is not \(\tau_1\)-closed in \(X\), then \(Y\) is not quasi-\(k\)-closed in \(X\), i.e. there is a pairwise countably compact subset \(K\) of \(X\) s.t. \(K \cap Y\) is not \(\tau_1\)-closed in \(K\). Since \(K\) is pairwise C-closed, then \(K \cap Y\) is not \(\tau_2\)-countably compact, i.e. there is a sequence \(x_n\) in \(K \cap Y\) which has no cluster point in \(K \cap Y\), but \(K\) is pairwise countably compact, hence \(x_n\) must have a cluster point in \(K\) say \(x\), therefore \(x \notin Y\). Now for every \(n\) choose \(B_n \subset A\) s.t. \(B_n\) is countable and \(x_n \in \tau_1\text{-cl}(B_n)\) and let \(B = \bigcup_{n=1}^{\infty} B_n\), then \(x \in \tau_1\text{-cl}(B)\), but \(\tau_1\text{-cl}(B) \subset Y\), thus \(x \in Y\), this is a contradiction.

The assumption of quasi-\(k\) space in the above proposition is very important to get the result, the following example shows this:

Example 2.26: Let \((X, \tau_1, \tau_2)\) be topological space, where \(X = Y \cup \{x\}\), where \(\tau_1\) consist of \(Y\) which is discrete space of cardinality \(\omega_1\) and \(x\) has countable neighborhoods and \(\tau_2\) has discrete topology, then every \(\tau_1\)-countably compact subset of \(X\) is finite, therefore it is \(\tau_2\)-closed, and every \(\tau_2\)-countably compact subset of \(X\) is finite and hence it is \(\tau_1\)-closed subset of \(X\), therefore \(X\) is pairwise C-closed space, but \(t(X) = \omega_1\).

Definition 2.27: Let \((x, \tau_1, \tau_2)\) be a bitopological space, let \(A \subset X\), then \(x\) is called \(\tau_i\)-isolated point of \(A\) if there exist open set \(U \in \tau_i\) s.t. \(U \cap A = \{x\}\), \(i = 1, 2\).
Definition 2.28: A bitopological space \((x, \tau_1, \tau_2)\) is said to be \((C\text{-sequential})\) if for every \(\tau_1\)-closed \((\tau_2\text{-closed})\) subset \(A\) of \(X\) and for every non \(\tau_1\)-isolated \((\text{non } \tau_2\text{-isolated})\) point \(x\) of \(A\), there is a sequence \(x_n\) in \(A \setminus \{x\}\) converging to \(x\).

Proposition 2.29: If \(X\) is pairwise Hausdorff, pairwise quasi-\(k\) and pairwise \(C\)-closed, then \(X\) is pairwise \(C\)-sequential.

Proof: since every \(P\)-closed subset of \(X\) is pairwise quasi-\(K\) and pairwise \(C\)-closed, it is enough to show that if \(x\) is not \(\tau_1\)-isolated \((\text{not } \tau_2\text{-isolated})\) point in \(X\), then there is a sequence \(x_n\) in \(X \setminus \{x\}\) converging to \(x\). if \(x\) is not \(\tau_1\)-isolated point of \(X\), then \(U \cap A \neq \{x\}\) for every \(U \in \tau_1\) and hence \(X \setminus \{x\}\) is not \(\tau_1\)-closed in \(X\). Similarly, if \(x\) is not \(\tau_2\)-isolated point in \(X\), then \(V \cap A \neq \{x\}\) for every \(V \in \tau_2\) and hence \(X \setminus \{x\}\) is not \(\tau_2\)-closed in \(X\). If \(X \setminus \{x\}\) is not \(\tau_1\)-closed in \(X\), then there is \(\tau_1\)-countably compact subset \(K\) of \(X\) that \(K \setminus \{x\}\) is not \(\tau_1\)-closed in \(K\). Since \(K\) is \(C\)-closed, \(K \setminus \{x\}\) is not \(\tau_1\)-closed in \(K\), then there is a sequence \(x_n\) in \(K \setminus \{x\}\) which has no \(\tau_2\)-cluster point in \(K \setminus \{x\}\). Therefore \(x_n \to x\). Similarly, if \(X \setminus \{x\}\) is not \(\tau_2\)-closed in \(X\), we get \(x_n \to x\). This completes the proof.

Definition 2.30: A bitopological space \((x, \tau_1, \tau_2)\) is said to be sequentially compact with respect to \(\tau_i\) if every infinite sequence has convergent subsequence with respect to \(\tau_i\), i.e., for every sequence \(\{x_n: n \in \mathbb{N}\}\) and for every \(\tau_i\)-open neighborhood \(U\) of \(x\) such that \(x_n \in U\) whenever \(n \geq m\) for some \(m\), there exists a subsequence \(\{x_{n_k}: k \in \mathbb{N}\}\) of \(x_n\) such that \(x_{n_k} \in U\) whenever \(k \geq m\). \(i=1,2\).

Definition 2.31: A bitopological space space \((x, \tau_1, \tau_2)\) is said to be pairwise sequentially compact if it is sequentially compact with respect to \(\tau_1\) and sequentially compact with respect to \(\tau_2\).

Proposition 2.32: A pairwise sequentially compact Hausdorff space \(X\) is pairwise sequential iff it is pairwise \(C\)-closed.

Proof: \((\Rightarrow)\) it is obvious from Corollary 2.12 \((\Leftarrow)\) let \(A\) be non \(\tau_1\)-closed subset of \(X\), then there is a sequence \(x_n\) in \(A\) which has no \(\tau_2\)-cluster point in \(A\), but \(X\) is pairwise sequentially compact, thus \(x_n\) has convergent subsequence \(x_{n_k}\) with respect to \(\tau_2\) say to \(x \in X\), since \(x_{n_k}\) has no \(\tau_2\)-cluster point in \(A\), then \(x \in X \setminus A\), therefore there is a sequence in \(A\) converging to a point in \(X/A\). We get \((X, \tau_1)\) is sequential. \((1)\) similarly, if \(B\) is non \(\tau_2\)-closed subset of \(X\), then there is a sequence \(x_m\) in \(B\) which has no \(\tau_1\)-cluster point in \(B\), since \(X\) is pairwise sequentially compact, thus \(x_m\) has convergent subsequence \(x_{m_k}\) with respect to \(\tau_1\) say to \(x \in X\), since \(x_{m_k}\) has no \(\tau_1\)-cluster point in \(B\), then \(x \in X \setminus B\), therefore there is a sequence in \(B\) converging to a point in \(X/B\). We get \((X, \tau_2)\) is sequential.
compact, $x_m$ has convergent subsequence $x_m^L$ with respect to $\tau_1$ say to $y \in X$, but $x_m^L$ has no $\tau_1$-cluster point in $A$, hence $y \in X /B$ and $(X, \tau_2)$ is sequential. (2) from 1 and 2, we get $X$ is pairwise sequential.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**