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OPERATOR LIPSCHITZ ESTIMATE FUNCTIONS ON BANACH SPACES

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Abstract. In this paper, let X, Y be Banach spaces and let $\mathcal{L}(\mathrm{X}, \mathrm{Y})$ be the space of bounded linear sequence of operators from X toY. We develop the theory of double sequence of operators integrals on $\mathcal{L}(\mathrm{X}, \mathrm{Y})$ and apply this theory to obtain commutator series estimates, for a large class of functions f_j, where $A_j \in \mathcal{L}(X), \mathrm{B}_j \in \mathcal{L}(Y)$ are scalar type the sequence of operators and $S \in \mathcal{L}(X, Y)$. In particular, we establish this estimate for $f_j(1+\epsilon) := |1+\epsilon|$ and for diagonalizable estimates derive hold for diagonalizable matrices with a constant independent of the size of the sequence of operators on $X = \ell_{(1+\epsilon)}$ and $Y = \ell_{(1+\epsilon)}$, for $\epsilon = 0$, and $\mathrm{X} = \mathrm{Y} = \mathrm{c}_0$. Also, we obtain results for $\epsilon \geq 0$, studied the estimate above [1] in the setting of Banach ideals in $\mathcal{L}(X, Y)$.

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1. INTRODUCTION

Let X be a Banach space and let $\mathcal{L}(X)$ be the space of all bounded linear sequence of

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operatorson X. Let $A_j, \mathrm{B}_j \in \mathcal{L}(X)$ be scalar type sequence of operators on X. Let $f_j: sp(A_j) \cup sp(\mathrm{B}_j)) \rightarrow \boldsymbol{C}$ be a bounded Borel function, where $sp(A_j)$ is the spectrum of the sequence of operators A_j. Interested in Lipschitz type series estimates

$$\sum_j \left\| f_j\ (\mathrm{B}_j) - f_j(A_j) \right\|_{\mathcal{L}(\mathrm{X})} \leq \mathrm{const} \left\| \sum_j \mathrm{B}_j - A_j \right\|_{\mathcal{L}(\mathrm{X})'} \quad (1)$$

where $\|\cdot\|_{\mathcal{L}(X)}$ is the uniform operator norm on the space $\mathcal{L}(X)$, and more generally in commutator series estimates

$$\sum_j \left\| f_j\ (\mathrm{B}_j)\mathrm{S} - \mathrm{S}f_j(A_j) \right\|_{\mathcal{L}(\mathrm{X},\mathrm{Y})} \leq \mathrm{const} \left\| \sum_j \mathrm{B}_j\mathrm{S} - \mathrm{S}A_j \right\|_{\mathcal{L}(\mathrm{X},\mathrm{Y})} \quad (2)$$

for Banach spaces X and Y, scalar type sequence of operators $A_j \in \mathcal{L}(X)$ and $\mathrm{B}_j \in \mathcal{L}(Y)$, and $S \in \mathcal{L}(X, Y)$. This problem is well known in the special case where $X = Y$ is aseparable Hilbert space, such as ℓ_2, and A_j and B_j are normal sequence of operators on X. Study such estimates in the Banach space setting, and specifically for $X = \ell_{(1+\epsilon)}$ and $Y = \ell_{(1+\epsilon)}$ with $0 \leq \epsilon \leq \infty$.

In the special case where $A_j, \mathrm{B}_j$are self-adjoint bounded sequence of operators on a Hilbert space H the series estimate

$$\sum_j \left\| f_j(\mathrm{B}_j)\mathrm{S} - \mathrm{S}f_j(A_j) \right\|_{\mathcal{L}(\mathrm{H})} \leq \mathrm{const} \left\| \sum_j \mathrm{B}_j - A_j \right\|_{\mathcal{L}(\mathrm{H})} \quad (3)$$

was established by Peller [2] (see also [3]) for $f_j: \mathbb{R} \rightarrow \mathbb{R}$in the Besov class $\left(\mathrm{B}_j\right)^1_{\infty,1}(\mathbb{R})$, in which the theory of double operator integration was developed to study the difference $f_j(\mathrm{B}_j) - f_j(A_j)$ in [4] . This theory was revised and extended in various directions, including the Banach space setting, in[5]. However, the results in the general setting were much weaker than in the Hilbert space setting. Showed that by Jan Rozendaal, Fedor Sukochev, Anna Tomas [1] for scalar type sequence of operators on Banach spaces one can obtain results matching those on Hilbert spaces.

In Corollary 4.9 below prove that (1) holds when $A_j, \mathrm{B}_j \in \mathcal{L}(X)$ are scalar Type sequence of operators with real spectrum and $f_j \in \left(\dot{\mathrm{B}}_\mathrm{j}\right)^1_{\infty,1}(\mathbb{R})$. It is immediate from the definition of a scalar type sequence of operator that every normal operator on H is of scalar Type. Therefore, Corollary 4.9 extends (3) to the Banach space setting. More generally, (2) holds for $f_j \in$

$\left(\dot{\mathrm{B}}_{\mathrm{j}}\right)^{1}_{\infty,1}(\mathbb{R})$ and for all $S \in \mathcal{L}(X,Y)$ If f_j is the absolute value function then $f_j \notin \left(\dot{\mathrm{B}}_{\mathrm{j}}\right)^{1}_{\infty,1}(\mathbb{R})$ and the results mentioned above do not apply. Moreover, the techniques which used to obtain (1) for $f_j \in \left(\dot{\mathrm{B}}_{\mathrm{j}}\right)^{1}_{\infty,1}(\mathbb{R})$ cannot be applied to the absolute value function. However, the absolute value function is very important in the theory of matrix analysis and perturbation theory [6].

In the case where H is an infinite-dimensional Hilbert space, that the function $(1+\epsilon) \longmapsto |1+\epsilon|\,, 1+\epsilon \in \mathbb{R}$ does not satisfy (3). An earlier example showed the failure in general of the commutator estimate (2) for this function, in the case X = Y = H. Later, it was proved that for $0 \leq \epsilon \leq \infty$ and the Schatten von-Neumann ideal $(1+\epsilon)_{(1+\epsilon)}$ with the corresponding norm $\|\cdot\|_{(1+\epsilon)_{\frac{1+\epsilon}{\epsilon}}}$, the series estimates

$$\sum_j \left\| |\mathrm{B}_j| - |A_j| \right\|_{(1+\epsilon)_{(1+\epsilon)}} \leq const \left\| \sum_j \mathrm{B}_j - A_j \right\|_{(1+\epsilon)_{(1+\epsilon)}}$$

holds for all $A_j, \mathrm{B}_j \in (1+\epsilon)_{(1+\epsilon)}$ if and only if $0 < \epsilon < \infty$. Commutator estimates forthe absolute value function and different Banach ideals in $\mathcal{L}(H)$ have also been studied in [7]. The proofs in [5, 7] are based on Macaev's celebrated theorem or on the UMD-property of the reflexive Schatten von-Neumann ideals.

However, the spaces $\mathcal{L}(X,Y)$ are not UMD-spaces, and therefore the techniques used do not apply to them. To study (2) for $X = \ell_{(1+\epsilon)}$and $Y = \ell_{(1+\epsilon)}$, completely different methods are used. Instead, we use the theory of Schur multipliers on the space $L(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$.

Let $0 \leq \epsilon \leq \infty$ with $\epsilon = 0$ or $\epsilon = \infty$. It is shown that, for diagonalizable sequence of operators and $B_j \in \mathcal{L}(\ell_{(1+\epsilon)})$, and for the absolute value functions f_j,

$$\sum_j \left\| f_j(S(B_j)_{(1+\epsilon)}) S f_j(A_j) \right\|_{\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})} \leq \mathrm{const} \left\| \sum_j B_j S - S A_j \right\|_{\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})} \tag{4}$$

holds for all $S \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$. In particular,

$$\sum_j \left\| f_j(B_j) - f_j(A_j) \right\|_{\mathcal{L}(\ell_{(1+\epsilon)})} \leq \; const \; \left\| \sum_j B_j - A_j \right\|_{\mathcal{L}(\ell_1)} \tag{5}$$

and

$$\sum_j \left\|f_j(B_j) - f_j(A_j)\right\|_{\mathcal{L}(c_0)} \leq const \left\|\sum_j B_j - A_j\right\|_{\mathcal{L}(c_0)} \tag{6}$$

for diagonalizable operators on ℓ_1 respectively c_0. Therefore, it is shown that, even though (4) fails for $\epsilon = 1$, and in particular (1) fails for $X = \ell_2$ and f_j, the absolute value function, one can obtain commutator estimates and Lipschitz estimates (1) for $X = \ell_1$ or $X = c_0$.

Also results for $\epsilon \geq 0$ are obtained. In particular, for $\epsilon = 2$, it is proved that for each $0 < \epsilon \leq 1$ there exists a constant $C \geq 0$ such that the following holds. Let $A_j, B_j \in \mathcal{L}(\ell_2)$ be compact self-adjoint sequence of operators, and let $U, V \in \mathcal{L}(\ell_2)$ be unitaries such that

$$\sum_j UA_jU^{-1} = \sum_j \sum_{j=1}^{\infty} \left(\lambda_j\right)_j \mathcal{P}_j \ and\ VBV^{-1} = \sum_j \sum_{j=1}^{\infty} \left(\mu_j\right)_j \mathcal{P}_j\,,$$

where$\{\left(\lambda_j\right)_j\}_{j=1}^{\infty}$ and $\{\left(\mu_j\right)_j\}_{j=1}^{\infty}$ are sequences of real numbers and the $\mathcal{P}_j \in \mathcal{L}(\ell_2)$, for $j \in \mathbb{N}$, are the basis projections corresponding to the standard basis of.ℓ_2. Then

$$\sum_j \left\| |B_j| - |A_j| \right\|_{\mathcal{L}(\ell_2)} \leq$$

$$(1+\epsilon)min\left(\left\|\sum_j V\left(B_j - A_j\right)U^{-1}\right\|_{\mathcal{L}(\ell_2,\ell_{2-\epsilon})}, \left\|V\left(B_j - A_j\right)U^{-1}\right\|_{\mathcal{L}(\ell_{2+\epsilon},\ell_2)},\right) \tag{7}$$

where let the right-hand side equal infinity if $V(B_j - A_j)U^{-1} \notin \mathcal{L}(\ell_2, \ell_{2-\epsilon}) \cup \mathcal{L}(\ell_2, \ell_{2+\epsilon}, \ell_2)$.

Note that the constants which appear in the results depend on the spectral constants of A_j and B_j, and in (4), (5) and (6) on the diagonalizability constants of A_j and B_j from (32). These quantities are independent of the norms of A_j and B_j, and to obtain constants which do not depend on A_j and B_j in any way one merely has to restrict to sequence of operators with a sufficiently bounded spectral or diagonalizability constant. This is done implicitly on Hilbert spaces by considering normal sequence of operators, for which the quantities involved are equal to 1.

For example, in (7) the constant C does not depend on A_j or B_j in any way. The results therefore truly extend the known estimates on Hilbert spaces, the main difference between Hilbert spaces and general Banach spaces being that on Hilbert spaces one has a large and easily identifiable class of sequence of operators with spectral constant1 or which are diagonalizable by an isometry.

We study the commutator estimate in (2) in the more general form

$$\sum_{j} \left\| f_j(B_j)S - Sf_j(A_j) \right\|_I \leq \text{const} \left\| \sum_{j} B_jS - SA_j \right\| \quad (8)$$

Where $\mathcal{T}^j$ are sequence of operator ideal in $\mathcal{L}(X,Y)$. For example, prove in Corollary 4.8 that (3) holds for a general Banach ideal $\mathcal{T}^j$in $\mathcal{L}(X)$ with the strong convex compactness property, with respect to the norm $\|\cdot\|_{\mathcal{T}^j}$.

Also, we present an example of a Banach ideal $\left(\mathcal{T}^j, \|\cdot\|_{\mathcal{T}^j}\right)in\mathcal{L}(\ell_{(1+\epsilon)^*}, \ell_{(1+\epsilon)})$, for $0 < \epsilon < \infty$, namely the ideal of $(1+\epsilon)$-summing sequence of operators, such that any Lipschitz series functions f_j satisfies (8).

We apply our results to finite-dimensional spaces, and obtain commutator estimates for diagonalizable matrices. Any diagonalizable matrix is a scalar type sequence of operator, hence estimates (4)-(8) hold for diagonalizable matrices A_j and $B_j$with a constant independent of the size of the matrix.

2. NOTATION AND TERMINOLOGY

All vector spaces are over the complex number field. Throughout, X and Y denoteBanach spaces, the space of bounded linear sequence of operators from X to $Y$is $\mathcal{L}(X,Y)$,and $\mathcal{L}(X):=\mathcal{L}(X,X)$. Identify the algebraic tensor product $X^* \otimes Y$ with thespace of finite rank of sequence of operators in $\mathcal{L}(X,Y)$ via $(x^* \otimes y)(x):=\langle x^*, x\rangle y$ for $x \in X$,$x^* \in X^*$ and $y \in Y$. The spectrums of $A_j \in \mathcal{L}(X)$ is $sp(A_j)$, and by $I_X \in \mathcal{L}(X)$ denote the identity of sequence of operator on X. Throughout the text use the abbreviations SOT and WOT for the strong and weak the sequence of operator topology, respectively.

The Borel σ-algebra on a Borelmeasurable subset $\sigma \subseteq \mathbb{C}$will be denoted by $\left(\mathfrak{B}_j\right)_\sigma$,and$\mathfrak{B}_j:=\left(\mathfrak{B}_j\right)_{\mathbb{C}}$. For measurable spaces $\left((\Omega_j)_1, (\Sigma_j)_1\right)$ and $\left((\Omega_j)_2, (\Sigma_j)_2\right)$ denote by $(\Sigma_j)_1 \otimes \left(\Sigma_j\right)_2$ the σ −algebra on $(\Omega_j)_1 \times (\Omega_j)_2$ generated by all measurable rectangles $\sigma_1 \times \sigma_2$ with $\sigma_1 \in (\Sigma_j)_1$and $\sigma_2 \in \left(\Sigma_j\right)_2$. If (Ω_j, Σ_j) are measurable space then $\mathfrak{B}_j\left(\Omega_j, \Sigma_j\right)$ is the space ofall bounded S-measurable complex-valued functions on Ω_j, a Banach algebra with the supremum series norms

$$\sum_j \|f_j\|_{\mathfrak{B}_j(\Omega_j,\Sigma_j)} := sup_{\omega^j \in \Omega_j} \left| \sum_j f_j\ (\omega^j) \right| \ (\ f_j \in B_j(\Omega_j, \Sigma_j)).$$

Simply write $\sum_j B_j(\Omega_j) := \sum_j B_j(\Omega_j, \Sigma_j)$ and $\sum_j \|f_j\|_\infty := \sum_j \|f_j\|_{B_j(\Omega_j,\Sigma_j)}$ when little confusioncan arise.

If μ_j are complex Borel measure on a measurable space (Ω_j,Σ_j) and X is a Banachspace, then functions$f_j: \Omega_j \to X$ is μ_j-measurable if there exists a sequence of X-
valued simple functions converging to $f_j\mu_j$-almost everywhere. For Banach spaces$X$ and Y and a functions $f_j: \Omega_j \to \mathcal{L}(X, Y)$, we say that f_j are strongly measurable if
$\omega^j \to f_j(\omega^j)x$are μ_j-measurable mapping $W^j \to Y$ for each $x \in X$.

If $\mu_j$is a positive measure on a measurable space (Ω_j,Σ_j) and $f_j: \Omega_j \to [0, \infty]$ is
function, let

$$\sum_j \overline{\int_{\Omega_j}} f_j(\omega^j) d\mu_j(w^j) :=\ inf \sum_j \int_{\Omega_j} g^j(\omega^j)\ d\mu_j\ ,$$

where the infimum is taken over all measurable $g^j: \Omega_j \to [0, \infty]$ such that $g^j(\omega^j) \geq f_j(\omega^j)$ for $\omega^j \in \Omega_j$.

The Hölder conjugate of $0 \leq \epsilon \leq \infty$ is denoted by $\frac{1+\epsilon}{\epsilon}$ and. The indicator function of a subset σ of a set Ω_j is denoted by $\mathbf{1}_\sigma$. Often identify functions defined on σ with their extensions to Ω_j by setting them equal to zero off σ.

3. PRELIMINARIES

Scalar type operators. Summarize some of the basics of scalar type operators.

Let X be a Banach space. A spectral measure on X is a map $E: \mathfrak{B}_j \to \mathcal{L}(X)$ suchthat the following hold:

- $E(\emptyset_j) = 0\ and\ E(\mathbb{C}) = I_X$;
- $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2) for\ all\ \sigma_1, \sigma_2 \in \mathfrak{B}_j$;
- $E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2) - E(\sigma_1)E(\sigma_2) for\ all\ \sigma_1, \sigma_2 \in \mathfrak{B}_j$;
- E is σ-additive in the strong sequence of operator topology.

Note that these conditions imply that E is projection-valued. Moreover, there exists a constant K such that

$$\|E(\sigma)\|_{\mathcal{L}(X)} \leq K\ (\sigma \in \mathfrak{B}_j)(9)$$

An operator $A_j\mathcal{L}(X)$ is a spectral sequence of operators if there exists a spectral measure Eon X such that

$A_jE(\sigma) = E(\sigma)A_j$ and $sp(A_j, E(\sigma)X) \subseteq \bar{\sigma}$for all $\sigma \in \mathfrak{B}_j$, where

$sp(A_jE(\sigma)X)$ denotes the sequence of spectrum of A_j in the space $E(\sigma)X$. For a spectral sequence of operators A_j, let $\nu^j(A_j$ denote the minimal constant K occurring in (9) and call $\nu^j(A_j$ the spectral constant of A_j. This is well-defined since the sequence of spectral measure E A_j associatedwith A_j is unique .Moreover, E is supported on$sp(A_j)$ in the sense that $Esp(A_j)) = I_X$. Hence definean integral with respect to E of bounded Borel measurable functions on $sp(A_j)$, asfollows. For $f_j = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{\sigma_j}$ a finite simple function with $\alpha_j \in \mathbb{C}$ and

$\sigma_j \subseteq sp(A_j)$ Borel for $1 \leq j \leq| n$, let

$$\sum_j \int_{sp(A_j)} f_j dE := \sum_j \sum_{j=1}^{n} \alpha_j \; E(\sigma_j). \tag{10}$$

This definition is independent of the representation off_j , and

$$\sum_j \left\| \int_{sp(A_j)} f_j dE \right\|_{\mathcal{L}(X)} = \sup_{\|x\|_X = \|x^*\|_{X^*} = 1} \left| \sum_{j=1}^{n} \alpha_j x^* E(\sigma_j)\, x \right|$$

$$\leq \sup_j \sum_j |\alpha_j| \sup_{\|x\|_X = \|x^*\|_{X^*} = 1} \sum_j \|x^* E(\cdot) x\|_{var}$$

$$\leq 4 \sum_j \|f_j\|_{sp(A_j))} \sup_{\|x\|_X = \|x^*\|_{X^*} = 1} \sup_{\sigma \subseteq sp(A_j)} \sum_j |x^* E(\sigma) x|$$

$$\leq 4\nu^j(A_j) \left\| \sum_j f_j \right\|_{(\mathcal{B}_j)(sp(A_j))} ,$$

Where $\|x^*E(\cdot)x\|_{var}$ is the variation norm of the measure $x^*E(\cdot)x$. Since the simplefunctions lie dense in $\mathcal{B}_j sp(A_j)$, for general $f_j \in \mathcal{B}_j sp(A_j$ define

$$\sum_j \int_{sp(A_j)} (f_j) dE := \sum_j \lim_{n\to\infty} \int_{sp(A_j)} (f_j)_n dE \in \mathcal{L}(X)$$

if $\{ (f_j)_n\}_{n=1}^{\infty} \subseteq \mathcal{B}_j(sp(A_j))$is a sequence of simple functions with

$\sum_j \left\| (f_j)_n - f_j \right\|_\infty \to 0asn \to \infty$. This definition is independent of the choice of approximating sequenceand

$$\sum_j \left\| \int_{sp(A_j)} (f_j) dE \right\|_{\mathcal{L}(X)} \leq 4\nu^j(A_j) \left\| \sum_j f_j \right\|_{\mathcal{B}_j(sp(A_j))} . \tag{11}$$

It is straightforward to check that

$$\sum_j \int_{sp(A_j)} (\alpha f_j + g^j) dE = \alpha \sum_j \int_{sp(A_j)} (f_j) dE + \int_{sp(A_j)} (g^j) dE,$$

$$\sum_j \int_{sp(A_j)} (f_j g^j) dE = \left(\sum_j \int_{sp(A_j)} (f_j) dE \right) \left(\int_{sp(A_j)} (g^j) dE \right)$$

for all $\alpha \in \mathbb{C}$and simple $f_j, g^j \mathcal{B}_j(sp(A_j))$, and approximation extends these identitiesto general $\sum_j f_j, g^j \in \sum_j \mathcal{B}_j\,(sp(A_j))$. Moreover, $\sum_j \int_{sp(A_j)} \mathbf{1}\, dE = E \sum_j (sp(A_j)) = I_X$. Hencethe map $\sum_j f_j \longmapsto \sum_j \int_{sp(A_j)} (f_j) dE$ is a continuous morphism $\sum_j \mathcal{B}_j\,(sp(A_j)) \to \mathcal{L}(X)$ of unitalBanach algebras. Since the spectrum of $A_j$is compact, the identity functions $\lambda_j \longmapsto \lambda_j$is bounded on $sp(A_j)$ and$\sum_j \int_{sp(A j)} \lambda_j dE\,(\lambda_j) \in \mathcal{L}(X)$ is well defined.

Definition3.1. A spectral sequence of operators$A_j \mathcal{L}(X)$ with spectral measure E is a scalartype sequence of operators if

$$\sum_j A_j = \sum_j \int_{sp(A_j)} \lambda_j dE\,(\lambda_j).$$

The class of scalar type sequence of operators on X is denoted by $\mathcal{L}_{1+\epsilon}(X)$.

For $A_j \in \mathcal{L}_{1+\epsilon}(X)$ with spectral measure E and $f_j \in \mathcal{B}_j sp(A_j))$define

$$\sum_j f_j(A_j) := \sum_j \int_{sp(A_j)} f_j dE . \tag{12}$$

Then, as remarked above, $f_j \longmapsto f_j(A_j)$ is a continuous morphism

$\mathcal{B}_j\left(sp(A_j)\right) \to \mathcal{L}(X)$of unital Banach algebras with norm bounded by $4\nu^j(A_j)$. Note also that

$$\sum_j \langle x^*, f_j\ (A_j)x\rangle = \sum_j \int_{sp(A_j)} \langle f_j(\lambda_j)\ dx^*, E(\lambda_j)x\rangle \qquad (13)$$

for all $f_j \in \mathcal{B}_j(sp(A_j)), x \in X$ and $x^* \in X^*$. Indeed, for simple functions this followsfrom (10), and by taking limits one obtains (13) for general $f_j \in \mathcal{B}_j(sp(A_j))$.

Finally, note that a normal sequence of operators A_j on a Hilbert space H is a scalar type sequence of operators with$\nu^j(A_j) = 1$, and in this case (11) improves to

$$\sum_j \left\| \int_{sp(A_j)} f_j dE \right\|_{\mathcal{L}(H)} \leq \left\| \sum_j f_j \right\|_{\mathcal{B}_j(sp(A_j))}, \qquad (14)$$

as known from the Borel functional calculus for normal sequence of operators.

Spaces of operators. Discuss some properties of spaces of sequence of operators that need later on.

First provide a lemma about approximation by finite rank sequence of operators. Recallthat a Banach space X has the bounded approximation property if there exists $M \geq 1$such that, for each $K \subseteq X$ compact and $\epsilon > 0$, there exists $S \ \in X^* \otimes X$ with
$\|S\|_{\mathcal{L}(X)} \leq M$and $sup_{x\in K}\|Sx - x\|_X < \epsilon$..

Lemma 3.2. Let X and Y be Banach spaces such that X is separable and either X or Y hasthe bounded approximation property. Then All$T^j \in \mathcal{L}(X,Y)$ is the SOT-limit of norms bounded sequence of finite rank sequence of operators.

Proof. Fixing $T^j \in \mathcal{L}(X,Y)$. , there exists a norm bounded Net_$\left\{(T^j)_j\right\}_{j\in J} \subseteq X^* \otimes Y$ having T^j as its SOT-limit. It is straightfroward to see that thestrong sequence of operator topology is metrizable on bounded subsets of $L(X,Y)$ by

$$d(S_1, S_2) := \sum_{k=1}^{\infty} 2^{-k}\ \|S_1x_k\ - S_2x_k\|_Y\ (S_1, S_2 \in \mathcal{L}(X,Y)),$$

where$\{x_k\}_{k\in\mathbb{N}} \subseteq X$ is a countable subset that is dense in the unit ball of X. Hencethere exists a subsequence of$\left\{(T^j)_j\right\}_{j\in J}$ with T^j as its SOT-limit.

Let X and Y be Banach spaces and let Z be a Banach space which is continuously embedded in $\mathcal{L}(X,Y)$ Following [8], say that Z has the strong convex compactness property if the following holds. For all finite measure space $(\Omega_j, \Sigma_j, \mu_j)$ and all strongly measurable bounded $f_j : \Omega_j \to Z$, the sequence of operators $T^j \in \mathcal{L}(X, Y)$ defined by

$$\sum_j T^j x := \sum_j \int_{\Omega_j} f_j(\omega^j)\, x d\mu_j(\omega^j) \quad (x \in X), \tag{15}$$

belongs to Z with $\sum_j \|T^j\|_Z \leq \sum_j \overline{\int_{\Omega_j}} \|f_j(\omega^j)\|_Z \, d\mu_j(\omega^j)$. By the Pettis Measurability Theorem, any separable Z has this property. Indeed, if Z is separable shows that any strongly measurable $f_j : \Omega_j \to Z$ is μ_j-measurable as a map to Z. If f_j are bounded as well, then (15) defines an element of Z with $\sum_j \|T^j\|_Z \leq \sum_j \int_{\Omega_j} \|f_j(\omega^j)\|_Z \; d\mu_j(\omega^j)$.

It is shown in [8] and [9] that the compact and weakly compact sequence of operators have the strong convex compactness property, but not all subspaces of $\mathcal{L}(X, Y)$ do.

Moreover, if $\mathcal{N}$ is a semifinite von Neumann algebra on a separable Hilbert space H, with faithful normal semifinite trace τ, and $\mathcal{F}$ is a rearrangement invariant Banach function space with the Fatou property, then $\mathcal{E} = \mathcal{N} \cap \mathcal{F}(N, \tau)$ has the strong convex compactness property .

Lemma 3.3. Let X and Y be separable Banach spaces and Z a Banach space continuously embedded in $\mathcal{L}(X, Y)$. If $(\mathrm{B_j})_Z := \{z \in Z \mid \|z\|_Z \leq 1\}$ is SOT-closed in $\mathcal{L}(X, Y)$, then Z has the strong convex compactness property.

Proof. The proof follows that of [10, Lemma 3.5]. First we show that $(\mathrm{B_j})_Z$ is a Polishspace in the strong sequence of operator topology. As in the proof of Lemma 3.2, bounded subsets of $\mathcal{L}(X, Y)$ are SOT-metrizable. The finite rank sequence of operators are SOT-dense in $\mathcal{L}(X, Y)$, hence $\mathcal{L}(X, Y)$ is SOT-separable. Therefore $(\mathrm{B_j})_Z$ is SOT-separable and metrizable. By assumption, $(\mathrm{B_j})_Z$is complete.

Let (Ω_j, μ_j) be a finite measure space and let$f_j : \Omega_j \to Z$ be bounded and strongly measurable. Without loss of generality, assume that $f_j(\Omega_j) \subseteq (\mathrm{B_j})_Z$and that $\mu_j$are probability measure. For each $y^* \in Y^*$and $x \in X$, the mapping$(\mathrm{B_j})_Z \to [0, \infty), \sum_j T^j \longmapsto \sum_j |\langle y^*, T^j x\rangle|$ are continuous. The collection of all these mappings, for $y^* \in Y^*$and $x \in X$, separates the points of $(\mathrm{B_j})_Z$. Moreover, $\sum_j \omega^j \longmapsto \sum_j \left|\langle y^*, f_j(\omega^j) x\rangle\right||$ are measurable mapping $\Omega_j \to [0, \infty)$ for each $y^* \in Y^*$and x$\in X$, f_j are the μ_j -almost everywhere SOT-limit of a sequence of $(\mathrm{B_j})_Z$ -valued simple functions$\{(f_j)_k\}_{k=1}^{\infty}$. Let $\sum_j \left(T^j\right)_n := \sum_j \int_{\Omega_j} (f_j)_n \, d\mu_j \in (\mathrm{B_j})_Z$. By the dominated convergence theorem, $\sum_{j}\left(T^{j}\right)_{n}(x) \rightarrow T^{j}(x):=\sum_{j}\int_{\Omega_j} f_j\,(\omega^j)x\,d\mu_j(\omega^j)$ as $n \rightarrow \infty$, for all $x \in X$. By assumption, $T^j \in (\mathrm{B_j})_Z$.

Let $g^j\colon \Omega_j \rightarrow [0,\infty)$ be measurable such that $1 \geq \sum_j g^j(\omega^j) \geq \left\|\sum_j f_j\,(\omega^j)\right\|_Z$ for $\omega^j \in \Omega_j$, and define $h\sum_j\left(T^j\right):=\sum_j \frac{f_j(\omega^j)}{g^j(\omega^j)}$ and $\sum_j d\nu^j\left(\omega^j\right):=\sum_j \frac{g^j(\omega^j)}{\int_{\Omega_j} g^j(\eta)d\mu_j(\eta)}d\mu_j\left(\omega^j\right)$ for $\omega^j \in \Omega_j$ .By what have shown above, $x \longmapsto \sum_j \int_{\Omega_j} h\left(\omega^j\right)\, x\, d\nu^j\left(\omega^j\right)$ defines an element of $(\mathrm{B_j})_Z$ .Since $\sum_j T^j x \;= \sum_j \int_{\Omega_j} f_j\left(\omega^j\right) x\, d\mu_j\left(\omega^j\right) = \sum_j \int_{\Omega_j} g^j(\omega^j)\, d\mu_j(\omega^j) \int_{\Omega_j} h(\omega^j)\, x\, d\nu^j(\omega^j)$, we obtain $\sum_j\left\|T^j\right\|_Z \leq \sum_j \int_{\Omega_j} g^j(\omega^j)\, d\mu_j(\omega^j)$, as remained to be shown.

Remark 3.4. Note that the converse implication does not hold. Indeed, if X is a Hilbert space then the finite rank sequence of operators of norm less than or equal to 1 are SOTdensein the unit ball of $\mathcal{L}(X)$. Therefore the compact sequence of operators of norm less than or equal to 1 are not SOT-closed in $\mathcal{L}(X)$ if X is infinite-dimensional. However, the space of compact sequence of operators on X has the strong convex compactness property.

Let X and Y be Banach spaces and I a Banach space which is continuously embedded in $\mathcal{L}(X,Y)$. We say that (I, k · ka) is a Banach ideal in $\mathcal{L}(X,Y)$ if for all $R \in \mathcal{L}(Y), S \in I$and $T^j \in \mathcal{L}(X)$, $RST \in I$ with $\sum_j\left\|RST^j\right\|_I \leq \|R\|_{\mathcal{L}(Y)}\|S\|_I\left\|\sum_j T^j\right\|_{\mathcal{L}(X)}$;• $X^* \otimes Y \subseteq I$ with $\|x^* \otimes y\|_I = \|y\|_Y$ $for\ all\ x^* \in X^*$and $y \in Y$. By Lemma 3.3 and [11, Proposition 17], for separable X and Y, any maximal Banach ideal in $\mathcal{L}(X,Y)$ has the strong convex compactness property.

This includes a large class of sequence of operator ideals, such as the ideal of absolutely p, (1+ϵ)-summing operators, the ideal of integral operators.

Algebras of functions. Discussing some algebras of functions that will be essential.

Let $\sigma_1, \sigma_2 \subseteq \mathbb{C}$be Borel measurable subsets and let $\mathfrak{A}(\sigma_1 \times \sigma_2)$ be the class ofBorel functions $\varphi_j\colon \sigma_1 \times \sigma_2 \rightarrow \mathbb{C}$such that

$$\sum_j \varphi_j\left((\lambda_j)_1, \left(\lambda_j\right)_2\right) = \sum_j \int_{\Omega_j} a_1\left(\lambda_j)_1, \omega^j\right) a_2\left(\lambda_j)_2, \omega^j\right) d\mu_j\left(\omega^j\right) \quad (16)$$

for all $(\lambda_j)_1, (\lambda_j)_2) \in \sigma_1 \times \sigma_2$, where $(\Omega_j, \Sigma_j, \mu_j)$ are finite measure space and

$$a_1 \in \mathcal{B}_j\left(\sigma_1 \times \Omega_j, \left(\mathfrak{B}_j\right)_{\sigma_1} \otimes \Sigma_j\right), a_2 \in \mathcal{B}_j\left(\sigma_2 \times \Sigma_j, \left(\mathfrak{B}_j\right)_{\sigma_2} \otimes \Sigma_j\right).$$

$for\ \varphi_j \in \mathfrak{A}(\sigma_1 \times \sigma_2)$. Let

$$\sum_j \left\|\varphi_j\right\|_{\mathfrak{A}(\sigma_1\times\sigma_2)} := inf \sum_j \int_{\Omega_j} \left\|a_1(\cdot,\omega^j)\right\|_{B_j(\sigma_1)} \left\|a_2(\cdot,\omega^j)\right\|_{B_j(\sigma_2)} d\mu_j(\omega^j),$$

where the infimum runs over all possible representations1 in (16) , it is straight forward to shown that the map $\sum_j \omega^j \ \mapsto \sum_j \left\|a_1(\cdot,\omega^j)\right\|_{B_j(\sigma_{1,})} \left\|a_2(\cdot,\omega^j)\right\|_{B_j(\sigma_2)}$ is measurable.

Lemma 3.5. For all $\sigma_1, \sigma_2 \subseteq \mathbb{C}$measurable, $\mathfrak{A}(\sigma_1 \times \sigma_2)$ is a unital Banach algebra whichis contractively included in $B_j(\sigma_1 \times \sigma_2)$.

Proof. That $\mathfrak{A}(\sigma_1 \times \sigma_2)$is a vector space is straightforward, and that it is normedalgebra is shown in [12, Lemma 3] for $\sigma_1 = \sigma_2 = \mathbb{R}$. The completeness of $\mathfrak{A}(\sigma_1 \times \sigma_2)$follows by showing that an absolutely convergent series of elements in $\mathfrak{A}(\sigma_1 \times \sigma_2)$ converges in $\mathfrak{A}(\sigma_1 \times \sigma_2)$.this is done by considering a direct sum of the measure spaces involved.

State sufficient conditions for a function to belong to $\mathfrak{A}$. The first will beused in the proof of Proposition 5.6. Let$(W^j)^{1,2}(\mathbb{R})$ be the space of all $g^j \in L^2(\mathbb{R})$ withweak derivative$\acute{g}^j \in L^2(\mathbb{R})$, endowed with the norm $\sum_j \left\|g^j\right\|_{(W^j)^{1,2}(\mathbb{R})}$:

$$= \sum_k \left\|g^j\right\|_{L^2(\mathbb{R})} + \sum_j \left\|\acute{g}^j\right\|_{L^2(\mathbb{R})}\ for\ g^j \in (W^j)^{1,2}(\mathbb{R}).$$

Lemma 3.6. Let $(W^j)^{1,2}(\mathbb{R})$ and let

$$\sum_j (\psi_j)_{g^j}\left((\lambda_j)_1,(\lambda_j)_2\right) := \begin{cases} (\sum_j g^j\left(log\left(\frac{(\lambda_j)_1}{(\lambda_j)_2}\right)\right) & if\ (\lambda_j)_1,(\lambda_j)_2 > 0 \\ 0 & other\ wise \end{cases} \tag{17}$$

Then $(\psi_j)_{g^j} \in \mathfrak{A}(\mathbb{R}^2)$ with$\left\|(\psi_j)_{g^j}\right\|_{\mathfrak{A}(\mathbb{R}^2)} \le \sqrt{2}\left\|g^j\right\|_{(W^j)^{1,2}(\mathbb{R})}$.

The second condition involves the Besov space $\dot{}(\dot{B}_j)^1_{\infty,1}(\mathbb{R})$. Following [13], let$\{(\psi_j)_k\}_{k\in\mathbb{Z}}$ be a sequence of Schwartz functions on $\mathbb{R}$ such that, for each $k \in \mathbb{Z}$,the Fourier transform $\mathcal{F}_{(\psi_j)_k}$of $(\psi_j)_k$are supported on $[2^{k-1}, 2^{k+1}]$ and $\mathcal{F}(\psi_j)_{k+1}(x) = \mathcal{F}(\psi_j)_k(2x)$ for all $x > 0$, and such that $\sum_{k=-\infty}^{\infty} \mathcal{F}(\psi_j)_k\,(x) = 1$ for all $x > 0$. Let $\mathcal{F}\left(\psi_j^*\right)_k$be defined by $\mathcal{F}\left(\psi_j^*\right)_k(x) = \mathcal{F}(\psi_j)_k(-x)$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. If f_j are distribution on $\mathbb{R}$ such that $\left\{2^k \sum_j \left\|f_j{}^*(\psi_j)_k\right\|_{L^2(\mathbb{R})}\}\right\}_{k\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$and $\left\{2^k \sum_j \left\|f_j{}^*\left(\psi_j^*\right)_k\right\|_{L^2(\mathbb{R})}\}\right\}_{k\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$, then f_j admits a representation

$$\sum_j f_j = \sum_{k\in\mathbb{Z}}\sum_j {f_j}^*\left(\psi_j\right)_k + \sum_{k\in\mathbb{Z}}\sum_j {f_j}^*\left(\psi_j^*\right)_k + P, (18)$$

where$P$ is a polynomial.

Let the homogeneous Besov space $^{\cdot}(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})$ consist of all distributions as abovefor which $P = 0$. Then $^{\cdot}(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})$ is a Banach space when equipped with the series norm

$$\sum_j \|f_j\|_{(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})} := \sum_j \sum_{k=-\infty} 2^k \left\|{f_j}^*\left(\psi_j\right)_k\right\|_{L^2(\mathbb{R})} + \sum_{k=-\infty} 2^k \left\|{f_j}^*\left(\psi_j^*\right)_k\right\|_{L^2(\mathbb{R})}$$

$\left(f_j \in (\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})\right)$

For $\left(f_j \in (\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})\right)$ define $(\psi_j)_{f_j}$: $\mathbb{C}^2 \longrightarrow \mathbb{C}$by

$$\sum_j (\psi_j)_{f_j}\left((\lambda_j)_1, (\lambda_j)_2\right):$$

$$= \begin{cases} \sum_j \frac{f_j\left((\lambda_j)_2\right) - f_j\left((\lambda_j)_1\right)}{(\lambda_j)_2 - (\lambda_j)_1} \ if \ \left((\lambda_j)_1, (\lambda_j)_2\right) \in \mathbb{R}^2 \ and \ (\lambda_j)_1 \neq (\lambda_j)_2 \\ \sum_j {f_j}'\left((\lambda_j)_1\right) if (\lambda_j)_1 = (\lambda_j)_2 \in \mathbb{R} \end{cases}.$$

.**Lemma 3.7.**There exists a constant $C \geq 0$ such that $(\psi_j)_{f_j} \in \mathfrak{A}(\mathbb{R}^2)$ for all $f_j \in {}^{\cdot}(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})$, with $\sum_j \left\|(\psi_j)_{f_j}\right\|_{\mathfrak{A}(\mathbb{R}^2)} \leq C \left\|\sum_j f_j\right\|_{\dot{B}^1_{\infty,1}(\mathbb{R})}$.

Proof. Let $f_j \in (\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})$. In [13, Theorem 2] it is shown that $(\psi_j)_{f_j}$has a representation

$$\sum_j (\psi_j)_{f_j}\left((\lambda_j)_1, (\lambda_j)_2\right) = \sum_j \int_{\Omega_j} a_1\left((\lambda_j)_1, \omega^j\right) a_2\left((\lambda_j)_2, \omega^j\right) d\mu_j\left(\omega^j\right)$$

for $(\lambda_j)_1, (\lambda_j)_2) \in \mathbb{R}^2$,where (Ω_j, μ_j) are measure space and a_1 and a_2 are measurable series functions on $\mathbb{R} \times \Omega_j$ such that

$$\sum_j \int_{\Omega_j} \left\|a_1(\cdot, \omega^j)\right\|_\infty \left\|a_2(\cdot, \omega^j)\right\|_\infty d|\mu_j|\left(\omega^j\right) \leq C \left\|\sum_j f_j\right\|_{(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})}.$$

for some constant $\epsilon \geq -1$ independent of f_j. The desired conclusion follows by replacing $a_1((\lambda_j)_1, \omega^j)$ and $a_2((\lambda_j)_2, \omega^j)$ by $\sum_j \frac{a_1((\lambda_j)_1,\omega^j)}{\|a_1(\cdot,\omega^j)\|_\infty}$ respectively $\sum_j \frac{a_2((\lambda_j)_2,\omega^j)}{\|a_2(\cdot,\omega^j)\|_\infty}$ and $d\mu_j(\omega)$by$\sum_j \|a_1(\cdot,\omega^j)\|_\infty \|a_2(\cdot,\omega^j)\|_\infty d\mu_j(\omega^j)$.

4. Double Operator Integrals and Lipschitz Estimates

Double operator integrals. Fix Banach spaces X and Y, scalar type sequence of operators $A_j(X) \in \mathcal{L}_{1+\epsilon}(X)$ and $\mathrm{B_j} \in \mathcal{L}_{1+\epsilon}(Y)$ with spectral measures E respectively F, and $\varphi_j \in \mathfrak{A}\left(sp(A_j) \times sp(\mathrm{B_j})\right)$. Let a representation (16) for $\varphi_j$be given, with corresponding(Ω_j, μ_j) and $a_1 \in \mathcal{B}_j(sp(A_j) \times \Omega_j), a_2 \in \mathcal{B}_j(sp(\mathrm{B_j}) \times \Omega_j)$. For $\omega^j \in \Omega_j$, let $a_1(A_j, \omega^j) := a_1(\cdot, \omega^j)(A_j) \in \mathcal{L}(X)$ and $a_2(\mathrm{B_j}, \omega^j) := a_2(\cdot, \omega^j)(\mathrm{B_j}) \in \mathcal{L}(Y)$ be defined by the functional calculus for A_j respectively $\mathrm{B_j}$.

Lemma 4.1. Let $S \in \mathcal{L}(X, Y)$ have separable range. Then, for each $x \in X, \omega^j \longmapsto a_2(\mathrm{B_j}, \omega^j) S a_1(A_j \omega^j) x$ is a weakly measurable map $\Omega_j \to Y$.

Proof. Fix $x \in X$. If $a_1 = \mathbf{1}_\sigma$for some $\sigma \subseteq sp(A_j) \times \Omega_j$ then it is straight forward to show that $\langle x^*, a_1(A_j,\cdot)x\rangle$ is measurable for each $x^* \in X^*$. As S has separablerange, $Sa_1(A_j,\cdot)x$ is μ_j-measurable. If a_2 is an indicator function as well, the same argument shows that $a_2(\mathrm{B_j},\cdot)y$ is weakly measurable for each $y \in Y$. General arguments, approximating $Sa_1(A_j,\cdot)x$ by simple functions, show that $a_2(\mathrm{B_j},\cdot)Sa_1(A_j,\cdot)x$is weakly measurable. By linearity this extends to simple $a_1$and a_2, and for generala_1 and a_2 let $\{(f_j)_k\}_{k\in\mathbb{N}}, \{(g^j)_k\}_{k\in\mathbb{N}}$be sequences of simple functions such that $a_1 = \lim_{k\to\infty} (f_j)_k$ and $a_2 = \lim_{k\to\infty} (g^j)_k$ uniformly. Then $a_1(A_j, \omega^j) = \lim_{k\to\infty} (f_j)_k (A_j)$ and $a_2(\mathrm{B_j}, \omega^j) = \lim_{k\to\infty} (g^j)_k (\mathrm{B_j})$ in the sequence of operator norm, for all $\omega^j \in \Omega_j$. The desired measurability follows.

Suppose that Y is separable, that I is a Banach ideal in $\mathcal{L}(X, Y)$ and let $S \in \mathcal{L}(X, Y)$.

By (11),

$$\sum_j \left\|a_2(\mathrm{B_j}, \omega^j) S a_1(\mathrm{A_j}, \omega^j)\right\|_1 \leq \|S\|_I \left\| a_1 \sum_j (\cdot, \omega^j) \right\|_{\mathrm{B_j}(sp(\mathrm{A_j}))}$$

$$\leq 16\, \nu^j\, (\mathrm{A_j} \nu^j\, (\mathrm{B_j}) \left\|a_2(\cdot, \omega^j)\right\|_{\mathrm{B_j}(sp(\mathrm{B_j}))} \tag{19}$$

for $\omega^j \in \Omega_j$. Since I is continuously embedded in $\mathcal{L}(X,Y)$, by the Pettis Measurability Theorem, Lemma 4.1 and (19) define the double sequence of operator integral

$$\sum_j (\mathrm{T}^j)_{\varphi_j}^{A_j,B_j}(S)x := \sum_j \int_{\Omega_j} a_2\left(B_j,\omega^j\right) S a_1\left(A_j,\omega^j\right) x \, d\mu_j\left(\omega^j\right) Y \quad (x \in X) \tag{20}$$

Throughout, use $(T^j)_{\varphi_j}$ for $(T^j)_{\varphi_j}^{A_j,B_j}$ when the sequence of operators A_j and B_j are clear from the context.

Proposition 4.2. Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let $A_j \in \mathcal{L}_{1+\epsilon}(X), B_j \in \mathcal{L}_{1+\epsilon}(Y)$, and $\varphi_j \in \mathfrak{A}(sp)(A_j) \times sp)(B_j))$. Let I be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness prop-erty. Then (20) defines an operator $(T^j) \in \mathcal{L}(I)$ which are independent of the choice of representation of $\varphi_j$in (16), with

$$\sum_j \left\|(T^j)_{\varphi_j}^{A_j,B_j}\right\|_{\mathcal{L}(I)} \leq 16\, \nu^j\left(A_j\right)\nu^j\left(B_j\right) \left\|\sum_j \varphi_j\right\|_{\mathfrak{A}\left(sp\left(A_j\right)\times sp\left(B_j\right)\right)} \tag{21}$$

Proof. By (19) and the strong convex compactness property, $(T^j)_{\varphi_j}(S) \in \mathcal{L}(I)$ for all $S \in I$, with

$$\sum_j \left\|(T^j)_{\varphi_j}(S)\right\| \leq 16\nu^j(A_j)\nu^j(B_j) \left\|\sum_j S\right\|_1 \int_{\Omega_j} \left\|a_1(\cdot,\omega^j)\right\|_{B_j(sp(A_j))_I}$$
$$\left\|a_2(\cdot,\omega^j)\right\|_{B_j(sp(B_j))} d\mu_j(\omega^j).$$

Clearly $(T^j)_{\varphi_j}$ are linear, hence the result follows if establish that $(T^j)_{\varphi_j}$are independent of the representation of φ_j. For this it suffices to let $\varphi_j \equiv 0$., first consider$S = x *\otimes y$ for $x^* \in X^*$ and $y \in Y$, and let $x \in X$, $y^* \in Y^*$ and $\omega^j \in \Omega_j$. Recall that $E$and F are the sequence of spectral measures of A_j and B_j, respectively. Then

$$\sum_j \langle y^*, a^2\left(B_j,\omega^j\right)(S) a_1\left(A_j,\omega^j\right)x\rangle = \sum_j \int_{(sp)(B_j)} a_2\left(\eta,\omega^j\right) d\langle y^*, F(\eta)(1+\epsilon) a_1\left(A_j,\omega^j\right)x\rangle$$

$$= \sum_{j} \int_{(sp)(\mathrm{B_j})} a_2\left(\eta, \omega^j\right)\langle x^*, a_1\left(A_j, \omega^j\right)x\rangle\, d\langle y^*, F(\eta)y\rangle$$

$$\sum_{j} \int_{sp(\mathrm{B_j})} \int_{sp(\mathrm{B_j})} a_1\left(\lambda_j, \omega^j\right) a_2(\eta, \omega^j) d\langle x^*, E\left(\lambda_j\right)x\rangle d\langle y^*, F(\eta)y\rangle$$

By (12). Fubini's Theorem and the assumption on φ_j yield $(T^j)_{\varphi_j}(S) = 0$. By linearity, $(T^j)_{\varphi_j}(S) = 0$ for all $S \in X^* \otimes Y$. By Lemma 3.2, a general $S \in I$is theSOT-limit of a bounded (in$\mathcal{L}(X, Y)$) sequence $\{(S)_n\}_{n\in\mathbb{N}} \subseteq X^* \otimes Y$. The dominated convergence theorem shows that $(T^j)((S)x = \lim_{n\to\infty} (T^j)_{\varphi_j}((S)_n)x = 0$ for all $x \in X$, which implies that $(T^j)_{\varphi_j}$ are independent of the representation of $\varphi_j$and concludes the proof.

Note that, if A_j and $\mathrm{B_j}$ are normal sequence of operators on separable Hilbert spaces X and$Y$, then 21 improves to

$$\sum_{j} \left\|(T^j)_{\varphi_j}^{A_j,\mathrm{B_j}}\right\|_{\mathcal{L}(I)} \leq \left\|\sum_{j} \varphi_j\right\|_{\mathfrak{A}\left(sp(A_j)\times sp(\mathrm{B_j})\right)}, \tag{22}$$

by appealing to (14) instead of (11) in (19).

Remark 4.3. Let H be an infinite-dimensional separable Hilbert space and S_2 the ideal of Hilbert-Schmidt the sequence of operators on H. There is a natural definition of a double sequence of operator integral $(T^j)_{\varphi_j}^{A_j,\mathrm{B_j}} \in \mathcal{L}(S_2)$ for all $\varphi_j \in \mathcal{B}_j(\mathbb{C}^2)$ and normal sequence of operators$A_j, \mathrm{B_j} \in \mathcal{L}(H)$, such that $\sum_j \left(\mathcal{T}^j\right)_{\varphi_j}^{A_j,\mathrm{B_j}} = \sum_j (T^j)_{\varphi_j}^{A_j,\mathrm{B_j}}\, if \varphi_j \in A_j(sp(A_j) \times sp(\mathrm{B_j}))$.

One could wonder whether Proposition 4.2 can be extended to a larger class of functions than $sp(A_j) \times sp(\mathrm{B_j}))$ using an extension of the definition of $(T^j)_{\varphi_j}^{A_j,\mathrm{B_j}}$in (20) which coincides with $\left(\mathcal{T}^j\right)_{\varphi_j}^{A_j,\mathrm{B_j}}$on S_2. But it follows that $(T^j)_{\varphi_j}^{A_j,\mathrm{B_j}}$, extends to a bounded sequence of operator on $I = \mathcal{L}(H)$ if and only if $\varphi_j \in \mathfrak{A}(sp(A_j) \times sp(\mathrm{B_j}))$. Hence Proposition 4.2 cannot be extended to a larger function class than$\mathfrak{A}(sp(A_j) \times sp(\mathrm{B_j}))$ in general. However, for specific Banach ideals, e.g. ideals with the UMD property, results have been obtained for larger classes of functions [5, 14].

Remark 4.4. The assumption in Proposition 4.2 that X or Y has the bounded approximation property is only used, via Lemma 3.2, to ensure that each $S \in I$ is the SOT-limit of a bounded net of finite-rank sequence of operators. Clearly this is true for general Banach spaces X and Y if I is the closure in $\mathcal{L}(X, Y)$of $X^* \otimes Y$.

Consider an assumption on $X$and I, called condition $c^*_{\lambda_j}$, which guarantees that each $S \in I$ is the SOT-limit of a bounded net of finite-rank sequence of operators. It is shown in [15] that this condition is strictly weaker than the bounded approximation property, for certain non-trivial ideals. In the results throughout, where assume that X has the bounded approximation property, onemay assume instead that X has satisfies condition $c^*_{\lambda_j}$for I for some $\lambda_j \geq 1$.

Commutator and Lipschitz estimates. Let $(1+\epsilon)_1{:}, (1+\epsilon)_2{:}\, \mathbb{C}^2 \to \mathbb{C}$ be the coordinate projections $(1+\epsilon)_1((\lambda_j)_1, (\lambda_j)_2) := (\lambda_j)_1 (1+\epsilon)_2((\lambda_j)_1, (\lambda_j)_2) := (\lambda_j)_2$ for $((\lambda_j)_1, (\lambda_j)_2) \in \mathbb{C}^2$. Note that$f_j \circ (1+\epsilon)_1, f_j \circ (1+\epsilon)_2 \in \mathfrak{A}(\sigma_1 \times \sigma_2)$ for all $\sigma_1, \sigma_2 \subseteq \mathbb{C}$Borel and $f_j \in \mathrm{B_j}(\sigma_1 \cup \sigma_2)$. For A_j and $\mathrm{B_j}$ self adjoint sequence of operators on a Hilbert space and I a non-commutative $L_{(1+\epsilon)}$-space, the following lemma.

Lemma 4.5. Under the assumptions of Proposition 4.2, the following hold:

(1) The map $\varphi_j \longmapsto (\mathcal{T}^j)^{A_j, \mathrm{B_j}}_{\varphi_j}$ are morphism $\mathfrak{A}(sp(A_j) \times sp(\mathrm{B_j})) \to \mathcal{L}(I)$ of unital Banach algebras.

(2) Let $f_j \in \mathrm{B_j} sp(A_j) \cup sp(\mathrm{B_j}))$ and $S \in \mathcal{L}(X, Y)$. Then $(T^j)_{f_j \circ (1+\epsilon)_1}(S) = S\, f_j\, (A_j)$ and $(T^j)_{f_j \circ (1+\epsilon)_2}(S) = S f_j\, (\mathrm{B_j})$. In particular, $(T^j)_{(1+\epsilon)_1}(S) = S(A_j)$and $(T^j)_{(1+\epsilon)_2}(S) = \mathrm{B_j} S$.

Proof. The structure of the proof is the same as that of [12, Lemma 3]. Linearity in (1) is straight forward. Fix $(\varphi_j)_1, (\varphi_j)_2 \in \mathfrak{A}(sp(A_j) \times sp(\mathrm{B_j}))$ with corresponding measure spaces (Ω_j, μ_j) and bounded Borel functions $a_{1,j} \in \mathrm{B_j} sp(A_j) \times \Omega_j)$ and $a_{2,j} \in \mathrm{B_j} sp(\mathrm{B_j}) \times \Omega_j)$ for $j \in \{1, 2\}$. Then $\varphi_j := (\varphi_j)_1 (\varphi_j)_2$ also has a representation as in (3.8), with $\Omega_j = (\Omega_j)_1 \times (\Omega_j)_2$, $\mu_j = (\mu_j)_1 \times\ (\mu_j)_2$ the product measure and $a_1 = a_{1,1} a_{1,2}, a_2 = a_{2,1} a_{2,2}$. By multiplicativity of the functional calculus for A_j

$$\sum_j a_1\left(A_j, \left((\omega^j)_1, (\omega^j)_2\right)\right) = \sum_j \left(\cdot, (\omega^j)_1\right) a_{1,2} \left(\cdot, (\omega^j)_2\right) (A_j)$$

$= \sum_j a_{1,1}\left(A_j,\left(\omega^j\right)_1\right) a_{1,2}\left(A_j,\left(\omega^j\right)_2\right)$ for all $\left(\left(\omega^j\right)_1,\left(\omega^j\right)_2\right) \in \Omega_j$, and similarly for $a_2\left(\mathrm{B_j},\left(\left(\omega^j\right)_1,\left(\omega^j\right)_2\right)\right)$. Applying this to (20) yields

$$\sum_j (T^j)_{\varphi_j}(S)x = \sum_j \int_{\Omega_j} a_2(\mathrm{B_j},\omega^j)(S)a_1(A_j,\omega^j)\, x d\mu_j(\omega^j)$$

$$= \sum_j \int_{(\Omega_j)_1} a_{2,1}\left(\mathrm{B_j},\left(\omega^j\right)_1\right)(T^j)_{(\varphi_j)_2} S a_{1,1}(A_j,{\omega^j}_1)\, x d(\mu_j)_1\left(\left(\omega^j\right)_1\right) =$$

$$\sum_j (T^j)_{(\varphi_j)_1}\left((T^j)_{(\varphi_j)_2}(S)x\right)$$

for all $S \in I$ and $x \in X$, which proves (1). Part (2) follows from (20) and the factthat $(T^j)_{\varphi_j}$is independent of the representation of φ_j.

For $f_j\colon sp(A_j) \cup sp(\mathrm{B_j}) \longmapsto \mathbb{C}$define

$$\sum_j \varphi_{fj}\big((\lambda_j)_1,(\lambda_j)_2\big) := \sum_j \frac{f_j\big((\lambda_j)_2\big) - f_j\big((\lambda_j)_1\big)}{(\lambda_j)_2 - (\lambda_j)_1} \quad (23)$$

For $\left((\lambda_j)_1,(\lambda_j)_2\right) \in sp(Aj) \times sp\left(\mathrm{B_j}\right)$ with $(\lambda_j)_1 \neq (\lambda_j)_2$.

Theorem 4.6. Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let I be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $A_j \in \mathcal{L}_{1+\epsilon}(X)$ and $\mathrm{B_j} \in \mathcal{L}_{1+\epsilon}(Y)$, and let $f_j \in \mathrm{B_j} sp(A_j) \cup sp(\mathrm{B_j}))$be such that$\left(\varphi_j\right)_{fj}$ extends to an element of $\mathfrak{A}(sp(A_j) \times sp(\mathrm{B_j}))$.Then

$$\sum_j \left\|f_j(\mathrm{B_j})S - Sf_j(A_j)\right\|_1 \leq 16\, \nu^j(A_j)\nu^j(\mathrm{B_j}) \left\|\sum_j A_j \varphi_{fj)}\right\|_{\mathfrak{A}\left(sp(A_j)\times sp(\mathrm{B_j})\right)}$$

$$\left\|\mathrm{B_j}S - SA_j\right\|_I \quad (24)$$

for all $S \in \mathcal{L}(X,Y)$ such that $\mathrm{B_j}S - SA_j \in I$.

In particular, if $X = Y$ and$\mathrm{B_j} - A_j \in I$,

$$\sum_j \left\|f_j(\mathrm{B_j}) - f_j(A_j)\right\|_I \leq 16\, \nu^j(A_j)\nu^j(\mathrm{B_j}) \left\|\sum_k f_j(\mathrm{B_j})S - S\, f_j(A_j)\right\|_I$$

$$\leq 16\, \nu^j(A_j)\nu^j(\mathrm{B_j}) \left\| \sum_{j} (\varphi_j)_{f_j} \right\|_{\mathfrak{A}\left(sp(A_j)\times sp(\mathrm{B_j})\right)} \left\|\mathrm{B_j} - A_j\right\|_I$$

Proof. Note that $\sum_{j}((1+\epsilon)_2 - (1+\epsilon)_1)(\varphi_j)_{f_j} = \sum_{j} f_j - f_j \circ (1+\epsilon)_1$. By Lemma 4.5,

$$\sum_j f_j(\mathrm{B_j})S - Sf_j(A_j) = \sum_j (\,T^j)_{(f_j)\circ(1+\epsilon)_2}\,(S) - (\,T^j)_{(f_j))\circ(1+\epsilon)_1}((S)$$

$$= \sum_j (\,T^j)_{((1+\epsilon)_2-(1+\epsilon)_1)(\varphi_j)_{f_j}}\,(S)$$

$$= \sum_j (\,T^j)_{(1+\epsilon)_2(\varphi_j)_{f_j}}\,(S) - (\,T^j)_{(1+\epsilon)_1(\varphi_j)_{f_j}}(S)$$

$$= \sum_j (\,T^j)_{(\varphi_j)_{f_j}}\left((\,T^j)_{(1+\epsilon)_2}(S) - (\,T^j)_{(1+\epsilon)_1}(S)\right)$$

$$= \sum_j (\,T^j)_{(\varphi_j)_{f_j}}\,(\mathrm{B_j}S - SA_j)$$

for each S ∈ I. Proposition 4.2 concludes the proof.

Letting X and Y be Hilbert spaces and A_j and $\mathrm{B_j}$ normal sequence of operators, generalize results to all Banach ideals with the strong convex compactness property. this includes all separable ideals and the so-called maximal sequence of operator ideals, which in turn is a large class of ideals containing the absolutely $((1+\epsilon),(1+\epsilon)$-summing the sequence of operators, the integral sequence of operators. Note that, for normal operators, we can improve the estimate in (24) by appealing to (22) instead of 21.

Corollary 4.7. Let $A_j \in \mathcal{L}(X)$ and $\mathrm{B_j} \in \mathcal{L}(Y)$ be normal sequence of operators on separable Hilbert spaces $X$and Y. Let I be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compact-ness property, and let $f_j \in \mathrm{B_j}(sp(A_j) \cup sp(\mathrm{B_j}))$ be such that $\varphi_j f_j$ extends to an element of$\mathfrak{A}(sp(A) \times sp(\mathrm{B_j}))$. Then

$$\sum_{j} \left\|f_j(\mathrm{B_j})S - Sf_j(A_j)\right\|_I \leq \left\| \sum_{j} (\varphi_j)_{f_j} \right\|_{\mathfrak{A}sp(A_j)\times sp(B))} \left\|\mathrm{B_j}S - Sa\right\|_I$$

for all $S \in \mathcal{L}(X,Y)$ such that $\mathrm{B_j}S - SA_j \in I$. In particular, if $X = Y$ and $\mathrm{B_j} - A_j \in I$,

$$\sum_{j} \left\|f_j(\mathrm{B_j}) - f_j(A_j)\right\|_I \;\leq \left\| \sum_{j} (\varphi_j)_{f_j} \right\|_{\mathfrak{A}(sp(A_j)\times sp(\mathrm{B_j}))} \left\|\mathrm{B_j} - A_j\right\|_I .$$

Combining Theorem 4.6 with Lemma 3.7 yields the following.

Corollary 4.8. There exists a universal constant $C \geq 0$ such that the following holds.
Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let I be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $f_j \in (\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})$, and let $A_j(X)$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(Y)$ be such that
$sp(A_j) \cup sp(\mathrm{B_j}) \subseteq \mathbb{R}$. Then

$$\sum_j \left\|f_j(\mathrm{B_j})S - S(A_j)f_j\right\|_I \leq C\nu^j(A_j)\nu^j(\mathrm{B_j}) \left\|\sum_j f_j\right\|_{\dot{B}^1_{\infty,1}(\mathbb{R})} \left\|\mathrm{B_j}S - SA_j\right\|_I$$

for all $S \in \mathcal{L}(X,Y)$ such that $\mathrm{B_j}S - SA_j \in I$. In particular, if $X = Y$ and

$\mathrm{B_j} - \mathrm{A_j} \in I$,(25)

$$\sum_j \left\|f_j(\mathrm{B_j}) - f_j\,(A_j)\right\|_I \leq C\nu^j(Aj)\nu^j(\mathrm{B_j}) \left\|\sum_j f_j\right\|_{(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})} \left\|\mathrm{B_j} - A_j\right\|_I.$$

In the case where the Banach ideal I is the space $\mathcal{L}(X,Y)$ of all bounded sequence of operators from X to Y, obtain the following corollary.

Corollary 4.9. There exists a universal constant $C \geq 0$ such that the following holds. Let$X$ and Y be separable Banach spaces such that either X or Y has the bounded approximation property. Let $f_j \in (\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})$, and let $A_j, \mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(X)$ be such that $sp(A_j) \cup sp(\mathrm{B_j}) \subseteq \mathbb{R}$.
Then$\sum_j \left\|f_j(\mathrm{B_j})S - S\,f_j(A_j)\right\|_{\mathcal{L}(X,Y)} \leq C\nu^j(A_j\nu^j(\mathrm{B_j})\left\|\sum_j f_j\right\|_{\dot{B}^1_{\infty,1}(\mathbb{R})}\left\|\mathrm{B_j}S - SA_j\right\|_{\mathcal{L}(X,Y)}$ for all $S \in \mathcal{L}(X,Y)$. In particular, if $X = Y$ then$\sum_j f_j(\mathrm{B_j}) - f_j(A_j)_{\mathcal{L}(X)} \leq$

$$C\nu^j(A_j\nu^j(\mathrm{B_j}) \left\|\sum_j f_j\right\|_{(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})} \left\|\mathrm{B_j} - A_j\right\|_{\mathcal{L}(X)}.(26)$$

Remark 4.10. Corollaries 26 and 27 yield global estimates, in the sense that (25)and (26) hold for all scalar type operators A_j and $\mathrm{B_j}$ with real spectrum, and the constant in the estimate depends on A_j and $\mathrm{B_j}$ only through their sequence of spectral constants$\nu^j(A_j)$ and $\nu^j(\mathrm{B_j})$. Local estimates follow if $f_j \in \mathrm{B_j}(\mathbb{R})$ are contained in the Besov class locally. More precisely, given scalar type sequence of operators $A_j(X)$ and $\mathrm{B_j} \in \mathcal{L}_{1+\epsilon}(Y)$with real spectrum, suppose there exists $g^j \in (\dot{\mathrm{B}_j})^1_{\infty,1}(\mathbb{R})$ with $g^j S = f_j S$ for all $S \in sp(A_j) \cup sp(\mathrm{B_j})$. Then $kf_j \sum_j \|f_j(\mathrm{B_j})S - Sf_j(A_j)\|_I \leq C\nu^j(A_j)\nu^j(\mathrm{B_j})\|\sum_j g^j\|_{(\dot{\mathrm{B}_j})^1_{\infty,1}(\mathbb{R})}\|\mathrm{B_j}S - SA_j\|_I$(27)

for all $S \in \mathcal{L}(X,Y)$ such that $\mathrm{B_j}S - SA_j \in I$. This follows directly from Theorem 4.6.

5. Spaces with an Unconditional Basis

Prove some results for specific scalar type sequence of operators, namely sequence of operators which are diagonalizable with respect to an unconditional Schauder basis. These results will be used in later sections. Assume all spaces to be infinite-dimensional, but the results and proofs carry over directly to finite dimensional spaces.

Diagonalizable operators. Let X be a Banach space with an unconditional Schauder basis $\{e_j\}_{j=1}^{\infty} \subseteq X$. For $j \in \mathbb{N}, let\, \mathcal{P}_j \in \mathcal{L}(X)$ be the projection given by $\mathcal{P}_j(x) := x_j e_j\; for\; all\; x = \sum_{k=1}^{\infty} x_k e_k \in X$.

Assumption 5.1. For simplicity, assume in this section that $\|\sum_{j\in\mathbb{N}} \mathcal{P}_j\|_{\mathcal{L}(X)} = 1$for all non-empty $N \subseteq \mathbb{N}$. This condition is satisfied in the examples consider inlater and simplifies the presentation. For general bases one merely gets additional constants in the results.

Sequence of operators $A_j \in \mathcal{L}(X)$ is diagonalizable (with respect to $\{e_j\}_{j=1}^{\infty} = 1$) if there exists$U \in \mathcal{L}(X)$ invertible and a sequence $\{(\lambda_j)_j\}_{j=1}^{\infty} \in \ell_\infty$ of complex numbers such that

$\sum_j UA_jU^{-1}x = \sum_j \sum_{j=1}^{\infty} (\lambda_j)_j \mathcal{P}_j x \;\; (x \in X)$,(28)

where the series converges since $\{e_k\}_{k=1}^{\infty}$is unconditional. In this case $A_j$are scalar type sequence of operator, with point spectrum equal to $\{(\lambda_j)_j\}_{j=1}^{\infty}, sp(A_j) = \overline{\{(\lambda_j)_j\}_{j=1}^{\infty}}$ and spectral measure E given by

$$E(\sigma) = \sum_j \sum_{(\lambda_j)_j \in \sigma} U^{-1}\mathcal{P}_j U(29)$$

For $\sigma \subseteq \mathbb{C}$ Borel. The set of all diagonalizable sequence of operators on X is denoted by $\mathcal{L}_{(1+\epsilon)}(X)$.We do not explicitly mention the basis $\{e_j\}_{j=1}^{\infty}$with respect to which sequence of operator is diagonalizable, since this basis will be fixed throughout. Often write $A_j \in$

$\mathcal{L}_{(1+\epsilon)}\left(X,\left\{\left(\lambda_j\right)_j\right\}_{j=1}^{\infty},U\right)$ in order to identify the sequence of operator U and the sequence $\left\{\left(\lambda_j\right)_j\right\}_{j=1}^{\infty}$ from above. For $A_j \in \mathcal{L}_{(1+\epsilon)}\left(X,\left\{\left(\lambda_j\right)_j\right\}_{j=1}^{\infty},U\right)$ and $f_j \in \mathcal{B}_j(\mathbb{C})$, it follows from (12) that $\sum_j f_j(A_j) = U^{-1}\sum_j\left(\sum_{j=1}^{\infty} f_j\left(\left(\lambda_j\right)_j\right)\mathcal{P}_j\right)U$.

Since any Banach space with a Schauder basis is separable and has the bounded approximation property, apply the results from the previous to diagonalizable sequence of operators, and obtain for instance the following.

Corollary 5.2. There exists a universal constant $C \geq 0$ such that the following holds. Let X and Y be Banach spaces with unconditional Schauder bases, and let I be a Banachideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $f_j \in \left(\dot{B}_J\right)_{\infty,1}^{1}(\mathbb{R})$, and let $A_j \in \mathcal{L}_{(1+\epsilon)}(X)$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(Y)$ be such that $sp(\,A_j sp(\mathrm{B_j}) \subseteq \mathbb{R}$. Then

$$\left\|f_j(\mathrm{B_j})S\ - S\ f_j(A_j)\right\|_I \leq C\nu^j(A_j)\nu^j(\mathrm{B_j})\ \left\|f_j\right\|_{\dot{B}_{\infty,1}^{1}(\mathbb{R})}\left\|\mathrm{B_j} sp A_j\right\|_I$$

for all $\left(\lambda_j\right)_j \in \mathcal{L}(X,Y)$ such that $\left\|\mathrm{B_j}S - SA_j\right\| \in I$. In particular, if $X = Y$ and $\mathrm{B_j} - A_j \in\ I,\ \sum_j\left\|f_j(\mathrm{B_j}) - f_j(A_j)\right\|_I \leq\ C\nu^j(A_j)\nu^j(\mathrm{B_j})\ \left\|\sum_j f_j\right\|_{(\dot{\mathrm{B}}_\mathrm{j})_{\infty,1}^{1}(\mathbb{R})}\left\|\mathrm{B_j} - A_j\right\|_I$.

Since this result does not apply to the absolute value function, and because of the importance of the absolute value function, we study Lipschitz estimates for more general functions.

Let Y be a Banach space with an unconditional Schauder basis $\left\{(f_j\,)_k\right\}_{k=1}^{\infty} \subseteq Y$, andlet the projections $Q_k \in \mathcal{L}(Y)$ be given by $Q_k(y) := y_k(f_j)_k$for all $y = \sum_{l=1}^{\infty} y_l(f_j)_l \in Y$and $k \in \mathbb{N}$. Let $\lambda_j = \left\{\left(\lambda_j\right)_j\right\}_{j=1}^{\infty}$ and $\mu_j = \left\{(\mu_j)_k\right\}_{k=1}^{\infty}$be sequences of complex numbers,and let $\varphi_j\colon \mathbb{C}^2 \to \mathbb{C}$. For $n \in \mathbb{N}$, define $(\,T^j)_{\varphi_j,n}^{\lambda_j,\mu_j} \in \mathcal{L}(\mathcal{L}(X,Y))$ by

$$\sum_{\boldsymbol{j}}(\,T^j)_{\varphi_j,n}^{\lambda_j,\mu_j}(S) \coloneqq$$

$$\sum_{j,k=1}^{n}(\varphi_j)_{f_j}\left(\left(\lambda_j\right)_j,(\mu_j)_k\right)Q_k S\mathcal{P}_j(\left(S \in \mathcal{L}(X,Y)\right)(31)$$

Note that $(\,T^j)_{\varphi_j,n}^{\lambda_j,\mu_j} \in \mathcal{L}(I)$ for each Banach ideal Iin $\mathcal{L}(X,Y)$.

Let $f_j \in \mathcal{B}_j(\mathbb{C})$ and extend the divided difference from (23), given by

$$\sum_{j}(\varphi_j)_{f_j}\big((\lambda_j)_1,(\lambda_j)_2\big):=\sum_{j}\frac{f_j\big((\lambda_j)_2\big)-f_j\big((\lambda_j)_1\big)}{(\lambda_j)_1-(\lambda_j)_2}$$

for $\big((\lambda_j)_1,(\lambda_j)_2\big)\in\mathbb{C}^2$ with $(\lambda_j)_1\neq(\lambda_j)_2$, to functions $\varphi_{j_{(f_j)}}:\mathbb{C}^2\to\mathbb{C}$.

Lemma 5.3. Let X and Y be Banach spaces with unconditional Schauder bases, and let I be a Banach ideal in $\mathcal{L}(X,Y)$. Let $\lambda_j=\left\{(\lambda_j)_j\right\}_{j=1}^{\infty}$ and $\mu_j=\{(\mu_j)_k\}_{k=1}^{\infty}$ be sequences of complex numbers, and let $A_j\in\mathcal{L}_{(1+\epsilon)}(X,\lambda_j,U)$, $\mathrm{B_j}\in\mathcal{L}_{(1+\epsilon)}(Y,\mu_j,V)$, $f_j\in\mathrm{B_j}(\mathbb{C})$ *and* $n\in\mathbb{N}$. Then

$$\sum_{j}\left\|f_j(\mathrm{B_j})S_n-S_nf_j\,(A_j)\right\|_I\;\leq\;\|U\|_{\mathcal{L}(X)}\|V^{-1}\|_{L(Y)}\left\|(\,T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}(V(\mathrm{B_j}S-SA_j)U^{-1})\right\|_I$$

for all $S\in I$ such that $\mathrm{B_j}S-SA_j\in I$, where

$S_n:=\sum_{j,k=1}^{n}V^{-1}\,Q_kVSU^{-1}\mathcal{P}_jU.$

Proof. Let $S\in I$ be such that $\mathrm{B_j}S-SA_j\in I$. For the duration of the proof write

$\mathcal{P}_j:=U^{-1}\mathcal{P}_jU\in\mathcal{L}(X)$ and $Q_k:=V^{-1}Q_kV\in\mathcal{L}(Y)$ for $j,k\in\mathbb{N}$. By (30), and using that $\mathcal{P}_j\mathcal{P}_k=0$ and $Q_jQ_k=0$ for $j\neq k$,

$$\sum_{j}f_j\big(\mathrm{B_j}\big)S_n-S_nf_j\big(A_j\big)=\sum_{j}\sum_{k=1}^{\infty}f_j\big((\mu_j)_k\big)\,Q_k\left(\sum_{i,l=1}^{n}Q_iS\mathcal{P}_i\right)-\sum_{j=1}^{\infty}f_j\left((\lambda_j)_j\right)\left(\sum_{i,l=1}^{n}Q_iS\mathcal{P}_i\right)\mathcal{P}_j$$

$$=\sum_{j}\sum_{j,kl=1}^{n}f_j\big((\mu_j)_k\big)-f_j\left((\lambda_j)_j\right)Q_kS\mathcal{P}_j$$

$$=\sum_{j}\sum_{j,k=1}^{n}\quad\sum_{(\mu_j)_k\neq\lambda_j}^{n}\frac{f_j\big((\mu_j)_k\big)-f_j\left((\lambda_j)_j\right)}{(\mu_j)_k-(\lambda_j)_j}\big((\mu_j)_kQ_kS\mathcal{P}_j-\lambda_jQ_kS\mathcal{P}_j\big)$$

$$=\sum_{j}\sum_{j,k=1}^{n}(\varphi_j)_{f_j}\left((\lambda_j)_j(\mu_j)_k\right)Q_k\left(\left(\sum_{i=1}^{\infty}\mu_{j_I}Q_I\right)S-S\left(\sum_{j=1}^{\infty}(\lambda_j)_j\,\mathcal{P}_j\right)\right)$$

$$=\sum_{j}\sum_{j,k=1}^{n}\varphi_{j_{(f_j)}}\left((\lambda_j)_j,(\mu_j)_k\right)Q_k\big(\mathrm{B_j}S-SA_j\big)\mathcal{P}_j$$

$$=\;V^{-1}\textstyle\sum_j(\,T^j)^{A_j,\mathrm{B_j}}_{(\varphi_j)_{f_j}}(V(\mathrm{B_j}S\;-SA_j)U^{-1})U.$$

where have used that $\mathrm{B_j}S-SA_j\in M_n$. Use the ideal property of I toconclude the proof.

For a sequence λ_j of complex numbers and $A_j\in\mathcal{L}_{(1+\epsilon)}(X,\lambda_j,U)$, define

$\sum_j \ K_{A_j} := inf \sum_j \{ \|U\|_{\mathcal{L}(X)} \|U^{-1}\|_{\mathcal{L}(X)} \big| A_j \in \mathcal{L}_{(1+\epsilon)}(X, \lambda_j, U) \}$.(32)

Call$K_{(A_j)}$ the diagonalizability constant of A_j. Using the unconditionality of the Schauder basis of X and Assumption 5.1, one can show that $K_{(A_j)}$ does not dependon the specific ordering of the sequence λ_j. Since the sequence$\lambda_j$is, up to ordering, uniquely determined by A_j, $K_{(A_j)}$ only depends on A_j. Moreover, by Assumption 5.1 and (29), $\|E(\sigma)\|_{\mathcal{L}(X)} \leq \|U^{-1}\|_{\mathcal{L}(X)} \|U\|_{\mathcal{L}(X)}$ for all$\sigma \subseteq \mathbb{C}$Borel and $U \in \mathcal{L}(X)$ such that $A_j \in \mathcal{L}_{(1+\epsilon)}(X, \lambda_j, U)$, where E is the sequence spectral measure of A_j. Hence $\sum_j \ \nu^j(A_j) \leq \sum_j K_{(A_j)}$,(33)

where$\nu^j(A_j)$ is the spectral constant of A_j

Derive commutator estimates for A_j and $\mathrm{B_j}$ in the sequence of operator norm, under a boundedness assumption which will be verified for specific $X$and Y .

Proposition 5.4. Let X and Y be Banach spaces with unconditional Schauder bases, $A_j \in \mathcal{L}_{(1+\epsilon)}(X, \lambda_j, U)$, $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(Y, \mu_j, V)$ and $f_j \in \mathrm{B_j}(\mathbb{C})$. Suppose that

$$C: = \sup_{n\in\mathbb{N}} \sum_j \left\| (\ T^j)^{\lambda_j \mu_j}_{(\varphi_j)_{f_j}, n} \right\|_{\mathcal{L}(\mathcal{L}(X,Y))} < \infty. (34)$$

Then

$$\sum_j \left\| f_j(\mathrm{B_j}) S - S f_j(A_j) \right\|_{\mathcal{L}(X,Y)} \leq C K_{(A_j)} K_{(\mathrm{B_j})} \left\| \sum_j \mathrm{B_j}(S - SA_j \right\|_{K_{(A_j)}}$$

$for\ all\ S \in \mathcal{L}(X,Y).$

Proof. Let $S \in \mathcal{L}(X,Y)$ and let $S_n \in \mathcal{L}(X,Y)$ be as in Lemma 5.3 for $n \in \mathbb{N}$. It is straight forward to show that, for each $x \in X$, $S_n x \to Sx$ as $n \to \infty$. Hence $\sum_j f_j(\mathrm{B_j}) S_n x - S_n f_j(A_j) x \to \sum_j f_j(\mathrm{B_j}) S\, x - S f_j(A_j) x\ as\ n \to \infty$, for each $x \in X$. Lemma 5.3 and (34) yield

$$\sum_j \left\| f_j(\mathrm{B_j}) S - S f_j(A_j) \right\|_{\mathcal{L}(X,Y)} \leq \lim_{n\to\infty} sup \left\| \sum_j f_j(\mathrm{B_j}) S_n - S_n f_j(A_j) \right\|_{\mathcal{L}(X,Y)}$$

$$\leq C \|U\| \|V\|^{-1} \sum_j \left\| V(\mathrm{B_j} S - SA_j) U^{-1} \right\|_{\mathcal{L}(X,Y)}$$

$$\leq C \|U\| \|U\|^{-1} \|V\| \|V\|^{-1} \sum_j \left\| \mathrm{B_j} S - SA_j \right\|_{\mathcal{L}(X,Y)}.$$

Taking the infimum over U and V concludes the proof.

Remark 5.5. Proposition 5.4 also holds for more general Banach ideals in $\mathcal{L}(X,Y)$. Indeed, let I be a Banach ideal in $\mathcal{L}(X,Y)$with the property that, if $\{S_m\}_{m=1}^{\infty} \subseteq I$is an I-bounded sequence which SOT-converges to some $S \in \mathcal{L}(X,Y)$as $m \to \infty$,then $S \in I$with $\|S\|_I \leq \lim_{m\to\infty} sup\,\|S_m\|_I$. With notation as in Proposition 5.4,if

$$C := \sup_{n\in\mathbb{N}} \left\| \sum_j (\,T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n} \right\|_{\mathcal{L}(I)} < \infty$$

Then

$$\sum_j \left\| f_j(\mathrm{B_j})S - Sf_j(A_j) \right\|_I \leq CK_{(A_j))}K_{\mathrm{B_j}} \left\| \sum_j \mathrm{B_j}S - S(A_j) \right\|_I$$

for all $S \in \mathcal{L}(X,Y)$ such that $\mathrm{B_j}S - S(A_j) \in I$. This follows directly from the proof ofProposition 5.4.

Estimates for the absolute value function. It is known that Lipschitz estimates for the absolute value function are related to estimates for so-called triangular truncation sequence of operators. For example, in [16] and [7] it was shown that the boundedness of the standard triangular truncation on many sequence of operator spaces is equivalent to Lipschitz estimates forf_j the absolute value function. Prove that, in the setting, triangular truncation sequence of operators are also related to Lipschitz estimates for f_j the absolute value function. Do so by relating $A_j$the assumption in (34) to so-called triangular truncation sequence of operators associated to sequences. Then bound the norms of these sequence of operators in later for specific $X$and Y.Let $\lambda_j = \{(\lambda_j)_J\}_{j=1}^{\infty}$ and $\mu_j = \{(\mu_j)_k\}_{k=1}^{\infty}$ be sequences of real numbers, and let X and$Y$ be as before.

For $n \in \mathbb{N}$define $(\,T^j)^{\lambda_j,\mu_j}_{\Delta,n} \in (X,Y))$ by

$$\sum_j (\,T^j)^{\lambda_j,\mu_j}_{\Delta,n}(S) := \sum_j \sum_{j,k=1}^{n} \sum_{(\mu_j)_k \leq \lambda_j} Q_k\, SP_j\big(S \in \mathcal{L}(X,Y)\big). (35)$$

Call $(\,T^j)^{A_j,B_j}_D$the triangular truncation associated to $\lambda_j$and μ_j.

For $f_j(1+\epsilon) := |(1+\epsilon)|$ for $(1+\epsilon) \in \mathbb{R}$, define $\varphi_{j_{(fj)}}: \mathbb{C}^2 \to \mathbb{C}$by

$$\sum_j \varphi_{j_{(f_j))}}\left((\lambda_j)_1,(\lambda_j)_2\right):=\begin{cases}\sum_j \frac{|(\lambda_j)_1|-|(\lambda_j)_2|}{(\lambda_j)_1-(\lambda_j)_2} & if(\lambda_j)_1 \neq (\lambda_j)_2 \\ 1 & \text{otherwise}\end{cases} \quad (36).$$

The following result relates the norm of $(T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}$ to that of $(T^j)^{\lambda,\mu_j}_{\Delta,n}$.

Proposition 5.6. There exists a universal constant $C \geq 0$ such that the following holds.

Let X and Y be Banach spaces with unconditional Schauder bases and let I be a Banachideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $\lambda_j$and $\mu_j$be bounded sequences of real numbers. Let $f_j((1+\epsilon)):=|(1+\epsilon)|$for $(1+\epsilon)\in\mathbb{R}$. Then$\sum_j \left\|(T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}(S)\right\|_I \leq$

$$C\left(\|S\|_I+\left\|\sum_j (T^j)^{\lambda_j,\mu_j}_{\Delta,n}(S)\right\|_I\right)$$

for all $n\in\mathbb{N}$and $S\in I$. In particular, if

$sup_{n\in\mathbb{N}}\left\|\sum_j (T^j)^{\lambda_j,\mu_j}_{\Delta,n}(S)\right\|_{\mathcal{L}(X,Y)}<\infty$, then (34)

holds.

Proof. Let $n\in\mathbb{N}$and $S \in I$, and write $\lambda_j=\{(\lambda_j)_j\}_{j=1}^{\infty}$ and $\mu_j=\{(\mu_j)_k\}_{k=1}^{\infty}$. Throughoutthe proof only consider λ_j and $(\mu_j)_k$ for $1\leq j,k\leq n$, but to simplify thepresentation not mention this explicitly from here on. Decompose$\sum_j (T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}(S)$ as

$$\sum_j (T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}(S)=\sum_j\sum_{(\lambda_j)_{2_k},(\mu_j)_k\geq 0} Q_kSP_j-\sum_j\sum_{\mu_{k<0<\lambda_j}}\frac{(\mu_j)_k+(\lambda_j)_j}{(\mu_j)_k-(\lambda_j)_j}Q_kSP_j+$$

$$\sum_j\sum_{(\lambda_j)_j<0<\mu_k}\frac{(\mu_j)_k+(\lambda_j)_j}{(\mu_j)_k-(\lambda_j)_j}Q_kSP_j-\sum_j\sum_{(\lambda_j)_k,(\mu_j)_k\leq 0}Q_kSP_j+\sum_j\sum_{(\lambda_j)_k\ ,(\mu_j)_k=0}Q_kSP_j$$

Note that some of these terms may be zero. By the ideal property of I and Assumption28,

$$\sum_j\left\|\sum_{(\lambda_j)_j,(\mu_j)_k\geq 0}Q_kSP_j\right\|_I\leq\left\|\sum_j\sum_{(\mu_j)_k\geq 0}Q_k\right\|_{\mathcal{L}(Y)}\|S\|_I\left\|\sum_{(\lambda_j)_j\geq 0}P_j\right\|_{\mathcal{L}(X)}\leq\|S\|_I. \qquad (37)$$

Similarly,$\sum_j\left\|\sum_{(\lambda_j)_k,(\mu_j)_k\leq 0}Q_kSP_j\right\|_Iand\sum_j\left\|\sum_{(\lambda_j)_k,(\mu_j)_k=0}Q_kSP_j\right\|_I$are all bounded by $\|S\|_I$.

To bound the other terms it is sufficient to show that

$$\sum_j \left\| \sum_{(\lambda_j)_j (\mu_j)_k > 0} \frac{(\mu_j)_k - (\lambda_j)_j}{(\mu_j)_k + (\lambda_j)_j} Q_k S P_j \right\|_I \leq C' \left(\|S\|_I + \left\| \sum_j (T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S) \right\|_I \right)$$

for some universal constant $C' \geq 0$. Indeed, replacing λ_j by $-\lambda_j$ and μ_j by $-\mu_j$ thenyields the desired conclusion. Let

$$\sum_j \Phi(S) := \sum_j \sum_{(\lambda_j)_j, (\mu_j)_k > 0} \frac{(\mu_j)_k - (\lambda_j)_j}{(\mu_j)_k + (\lambda_j)_j} Q_k S P_j$$

and define $g^j \in (W^j)^{1,2}(\mathbb{R})$ by $g^j(1+\epsilon) := \frac{2}{e^{|(1+\epsilon)|}+1}$for $(1+\epsilon) \in \mathbb{R}$. Note that $\Phi(S)$ is equal to$\sum_j \sum_{0<(\mu_j)_k \leq (\lambda_j)_j} \left(g^j \left(\log \frac{(\lambda_j)_j}{(\mu_j)_k} \right) - 1 \right) Q_k S P_j + \sum_j \sum_{0<(\lambda_j)_j<(\mu_j)_k} \left(\left(g^j \log \frac{(\lambda_j)_j}{(\mu_j)_k} \right) \right) Q_k S P_j$.

Let $(\psi_j)_{g^j} : \mathbb{C}^2 \to \mathbb{C}$be as in (17), and let $A_j := \sum_{j=1}^{\infty} (\lambda_j)_j P_j \in \mathcal{L}(X)$ and $\mathrm{B_j} := \sum_j \sum_{k=1}^{\infty} (\mu_j)_k Q_k \in \mathcal{L}(Y)$. Let $(T^j)^{A_j, \mathrm{B_j}}_{(\psi_j)_{g^j}}$ be as in (20). One can check that

$$\sum_j \Phi(S) = \sum_j (T^j)^{A_j, \mathrm{B_j}}_{(\psi_j)_{g^j}} \left((T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S) \right) - \sum_j \sum_{(\lambda_j)_j, (\mu_j)_k > 0} Q_k \, (T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S) P_j +$$

$$\sum_j \sum_{(\lambda_j)_j, (\mu_j)_k > 0} Q_k \left(S - (T^j)^{\lambda_j, \mu_j}_{\Delta, n} S \right) P_j - \sum_j (T^j)^{A_j, \mathrm{B_j}}_{(\psi_j)_{g^j}} \left((S - (T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S) \right) .$$

Any Banach space with a Schauder basis are separableA_j and has the bounded approximation property, hence Lemma 3.6 and Proposition 4.2 yield

$$\sum_j \left\| (T^j)^{A_j, \mathrm{B_j}}_{(\psi_j)_{g^j}} \left((T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S) \right) \right\|_I \leq 16\sqrt{2}\, \nu^j(A_j) \nu^j(\mathrm{B_j}) \left\| \sum_j g^j \right\|_{(\omega^j)^{1,2}(\mathbb{R})} \left\| (T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S)) \right\|_I .$$

By (33), $\nu^j(A_j) = \nu^j(\mathrm{B_j}) = 1$. Similarly,

$$\sum_j \left\| (T^j)^{A_j, \mathrm{B_j}}_{(\psi_j)_{g^j}} \left(S - (T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S) \right) \right\|_I$$

$$\leq 16\sqrt{2}\, \nu^j(A_j) \nu^j(\mathrm{B_j}) \left\| \sum_j g^j \right\|_{(W^j)^{1,2}(\mathbb{R})} \left(\left\| \|S\|_I + (T^j)^{\lambda_j, \mu_j}_{\Delta, n}(S) \right\|_I \right).$$

By the same arguments as in (28),

$$\sum_j \left\| \sum_{\lambda_j,(\mu_j)_k>0} Q_k(\,T^j)^{\lambda_j,\mu_j}_{\Delta,n}\,(S)P_j \right\|_I + \sum_j \left\| \sum_{\lambda_j,\mu_k>0} Q_k\left(S-(\,T^j)^{\lambda_j,\mu_j}_{\Delta,n}(S)\right)P_j \right\|_I$$

$$\leq\; 2\left\| \sum_j \|S\|_I \;+(\,T^j)^{\lambda_j,\mu_j}_{\Delta,n}(S)) \right\|_I .$$

Combining all these estimates yields

$$\sum_{\boldsymbol{j}} \|\varphi_j\,(S)\|_I \leq \left(2+32\sqrt{2}\,\left\|\sum_{\boldsymbol{j}} g^j\right\|_{(W^j)^{1,2}(\mathbb{R})}\right) \big\|\,\|S\|_I+(\,T^j)^{\lambda_j,\mu_j}_{\Delta,n}(S))\big\|_I$$

as desired.

6. The Absolute Value Function on $\boldsymbol{\mathcal{L}\big(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\big)}$

Study the absolute value function on the space $\mathcal{L}\big(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\big)$. Show that the absolute value function is sequence of operator Lipschitz on $\mathcal{L}\big(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\big), on\ \mathcal{L}(\ell_1)$ and on $\mathcal{L}(c_0)$ Also obtain results for $\epsilon \geq 0$.

The key idea of the proof is entirely different from the techniques used, which are based on a special geometric property of the reflexive Schatten von Neumann ideals, a property which $\mathcal{L}\big(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\big)$ does not have. Instead, we prove our results by relating estimates for the sequence of operators from (35) to the standard triangular truncation sequence of operator, defined in (38) below.

For this use the theory of Schur multipliers on $\mathcal{L}\big(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\big)$developed.Then appeal to results about the boundedness of the standard triangular truncation on $\mathcal{L}\big(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\big)$

Schur multipliers. For $0 \leq \epsilon < \infty$, let $\{e_j\}_{j=1}^{\infty}$ be the standard Schauder basisof $\ell_{(1+\epsilon)}$, with the corresponding projections $P_j(x) := x_j e_j$ for $x = \sum_{k=1}^{\infty} x_k e_k$ and

$j \in \mathbb{N}$. Consider this basis and the corresponding projections on all $\ell_{(1+\epsilon)}$spaces simultaneously, for simplicity of notation. Note that Assumption 5.1 is satisfied for this basis. For $0 \leq \epsilon \leq \infty$, all sequence of operator $S \in \mathcal{L}\big(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\big)$ can be represented by aninfinite matrix $\widetilde{(S)} = \{(S)_{jk}\}_{j,k=1}^{\infty}$, where

$(S)_{jk} := (S(e_k), e_j)$ forj, $k \in \mathbb{N}$. For an infinite matrix $M = \{m_{jk}\}_{j,k=1}^{\infty}$ the product $M * \breve{(S)} :=$ $\{m_{jk}\breve{(S)}_{jk}\}$ is the Schur product of thematrices M and $\breve{(S)}$. The matrix M is a Schur multiplier if the mapping $\breve{(S)} \longmapsto M * \breve{(S)}$ is a bounded sequence of operator on $\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$. Throughout, identify Schur multipliers withtheir corresponding operators.

The notion of a Schur multiplier is a discrete version of a double sequence of operator integral in [17, 18]. Schur multipliers on the space $\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$ arealso called $((1+\epsilon),(1+\epsilon))$-multipliers. Denote by$\mathcal{M}((1+\epsilon),(1+\epsilon))$ the Banach space of $((1+\epsilon),(1+\epsilon))$ $-$multipliers with the norm

$$M_{((1+\epsilon),(1+\epsilon))} := sup\left\{\left\|M * \breve{(S)}\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})} \|(S)\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})} \leq 1\right\}.$$

Remark 6.1. Also consider $((1+\epsilon),(1+\epsilon))$-multipliers M for $\epsilon = \infty$ and $0 \leq \epsilon \leq \infty$.All sequence of operator$S \in \mathcal{L}(c_0, \ell_{(1+\epsilon)})$ corresponds to an infinite matrix $\breve{(S)} = \{(S)_{jk}\}_{j,k=1}^{\infty}$, and $M$is said to be a $(\infty,(1+\epsilon))$-multiplier if the mapping $\breve{(S)} \longmapsto M * \breve{(S)}$is a bounded operatoron $\mathcal{L}(c_0, \ell_{(1+\epsilon)})$, and define the Banach space$M(\infty,(1+\epsilon))$ in the obvious way. Below often not explicitly distinguish the case $\epsilon = \infty$ from $1 \leq \epsilon < \infty$, in orderto keep the presentation simple. The reader should keep in mind that for $\epsilon = \infty$ the space $\ell_{(1+\epsilon)}$ should be replaced by c_0.

Remark 6.2. It is straightforward to see that $\|M\|_{((1+\epsilon),(1+\epsilon))} \geq sup_{j,k\in\mathbb{N}}|m_{j,k}|$ for all $1 \leq \epsilon \leq \infty$,and $M \in \mathcal{M}((1+\epsilon),(1+\epsilon))$.

For $1 \leq \epsilon \leq \infty$,and $S \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$, define

$$\sum_j (\mathcal{T}^j)_\Delta (S) := \sum_{k\leq j} P_k S P_j \, , (38)$$

which is a well-defined element of $\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$ for suitable$1 \leq \epsilon \leq \infty$, by Proposition6.3 below. The sequence of operators $(\mathcal{T}^j)_\Delta$ arethe triangular truncation. This sequence of operator can be identified with the following Schur multiplier. Let$\left(\acute{\mathcal{T}}^{j}\right)_\Delta = \{\left(\frac{1+\epsilon}{\epsilon}\right)_{jk}\}_{j,k=1}^{\infty}$ be a matrix given by $\left(\frac{1+\epsilon}{\epsilon}\right)_{jk} = 1$ for $k \leq j$ and $\left(\frac{1+\epsilon}{\epsilon}\right)_{jk} = 0$ otherwise. It is clear that $(\mathcal{T}^j)_\Delta$ extends to abounded linear sequence of operator on $\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$if and only if $\left(\acute{\mathcal{T}}^{j}\right)_\Delta$are$((1+\epsilon),(1+\epsilon))-$ multiplier. For $n \in \mathbb{N}$ and $1 \leq \epsilon \leq \infty$ consider the operators $(\mathcal{T}^j)_{\Delta,n} \in$ $\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})\right), \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$, given by

$$\sum_j (\mathcal{T}^j)_{\Delta,n}(S) := \sum_{1\le k\le j\le n} \mathcal{P}_k S \mathcal{P}_j \; (S \in \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big).$$

The dependence of the $((1+\epsilon),(1+\epsilon))$ −normof $(\mathcal{T}^j)_\Delta$ on the indices $(1+\epsilon)$ and $(1+\epsilon)$ was determined in [19] as follows.

Proposition 6.3. Let $0 \le \epsilon \le \infty$. Then the following statements hold.

(i) . If $\epsilon = 0$ or $\epsilon = 0$ then $(\mathcal{T}^j)_\Delta \in \mathcal{M}((1+\epsilon),(1+\epsilon))$.

(ii) If $\epsilon \neq \infty$, then there is a constant $C > 0$ suchThat

$$\left\| \sum_j (\mathcal{T}^j)_{\Delta,n} \right\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)} \ge C \, ln\, n$$

for all $n \in \mathbb{N}$.

(iii) If $\epsilon \ge \infty$, then for each $\epsilon > 0$,

$$(\mathcal{T}^j)_\Delta : \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big) \to \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{1+\epsilon}\big) \; and \; (\mathcal{T}^j)_\Delta : \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big) \to \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big)$$

are bounded.

Also need the following result. Fora matrix $M = \{m_{jk}\}_{j,k=1}^{\infty}$, let $\overline{M} = \{\overline{m}_{jk}\}_{j,k=1}^{\infty}$ be obtained from M by repeating the first column, i.e., $\overline{m}_{j1} = m_{j1}$ and $\overline{m}_{jk} = m_{j(k-1)}$ for $j \in \mathbb{N}$and $k \ge 2$.

Proposition 6.4. Let $0 \le \epsilon \le \infty$, let $M = \{m_{jk}\}_{j,k=1}^{\infty}$ be such that$(\breve{S}) \longmapsto M * (\breve{S})$is a bounded mapping $\mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big) \to \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big)$. Then $(\breve{S}) \longmapsto M * (\breve{S})$ is also abounded mapping $\mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big) \to \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big)$, with

$$\|M\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)} = \|\widetilde{M}\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)}.$$

In particular, if $M \in \mathcal{M}((1+\epsilon),(1+\epsilon))$ then $\breve{M} \in \mathcal{M}((1+\epsilon),(1+\epsilon))$ with $\|M\|_{((1+\epsilon),(1+\epsilon))} = \|\widetilde{\mathcal{M}}\|_{((1+\epsilon),(1+\epsilon))}$.

Proof. The condition$\epsilon \ge 0$ is used to ensure that $|x_1|^{(1+\epsilon)} + |x_2|^{(1+\epsilon)} \le \big(|x_1|^{(1+\epsilon)} + |x_2|^{(1+\epsilon)}\big)^1$ for all $x_1, x_2 \in \mathbb{C}$.

Remark 6.5. By considering the transpose M′ of a matrix M , and using that $M' : \mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}\right) \to \mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)$ with

$$\|M'\|_{\mathcal{L}\left(\mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)},\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}\right),\mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)},\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)\right)} = \|M\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)}.$$

as is straight forward to check, one obtains from Proposition 6.4, that is, for $\epsilon \geq 0$, row repetitions do not alter the $\|\cdot\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)}$-normof a matrix. Moreover, since $\|(1+\epsilon)\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}$ is invariant under permutations of thecolumns and rows of $(1+\epsilon) \in \mathcal{L}\left(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\right)$, rearrangement of the rows and columns of $M \in \mathcal{L}\left(\mathcal{L}\left(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\right),\mathcal{L}\left(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\right)\right)$ does not change the norm $\|M\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)}$.

The following lemma is crucial to the main results.

Lemma 6.6. Let $0 \leq \epsilon \leq \infty$ with$\epsilon \geq 0$, let $\lambda_j = \{(\lambda_j)_j\}_{j=1}^{\infty}$ and $\mu_j = \{(\mu_j)_k\}_{k=1}^{\infty}$be sequences of real numbers. Then

$$\sum_{\boldsymbol{j}} \left\|(T^j)_{\Delta,n}^{\lambda_j,\mu_j}\right\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)} \leq \left\|\sum_{\boldsymbol{j}} (\mathcal{T}^j)_{\Delta,n}\right\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)}$$

for all $n \in \mathbb{N}$.

Proof. Note that $(T^j)_{\Delta,n}^{\lambda_j,\mu_j}(S) = M * S$ for all $S \in \mathcal{L}\left(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}\right)$, where $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is such that $(m_j)_k = 1\ if\ 1 \leq j,k \leq n$ and $(\mu_j)_k \leq (\lambda_j)_j$, and $(m_j)_k = 0$ otherwise. It suffices toprove that

$\sum_{\boldsymbol{j}}\|M\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)} \leq \left\|\sum_{\boldsymbol{j}}\mathcal{T}^j\right\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)}$.

Assume that M is nonzero, otherwise the statement is trivial. By Remark 6.5, rearrangement of the rows and columns of M does not change its norm. Hence, assume that $\sum_{\boldsymbol{j}}\lambda_j = \sum_{\boldsymbol{j}}\{(\lambda_j)_j\}_{j=1}^{n}$and $\mu_j = \{(\mu_j)_k\}_{k=1}^{n}$are decreasing. M has the property that, if $m_{jk} = 1$, then $m_{il} = 1$for all $i \leq j$ and $k \leq l \leq m_2$. By Proposition 6.4 and Remark 6.5, omit repeated rows and columns of M, and doing this repeatedly reduces M to $(\mathcal{T}^j)_{\Delta,n}$ forsome $1 \leq N \leq n$. Noting that $\sum_{\boldsymbol{j}}\left\|(\mathcal{T}^j)_{\Delta,N}\right\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},(1+\epsilon))\right)} \leq \left\|\sum_{\boldsymbol{j}}(\mathcal{T}^j)_{\Delta,n}\right\|_{\mathcal{L}\left(\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}),\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})\right)}$ concludes the proof.

6.2. **The cases** $\epsilon \geq 0, \epsilon = 0, \epsilon = \infty$. Combine the theory from the previous sections to deduce our main results. Throughout this section let

$f_j(1+\epsilon) := |(1+\epsilon)|\ for\ (1+\epsilon) \in \mathbb{R}$.

Theorem 6.7. Let $0 \leq \epsilon \leq \infty, \varepsilon = 0$or $\epsilon = \infty$. Then there existsa constant $C \geq 0$ such that the following holds. Let$A_j \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$ and $B_j \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$ have real spectrum. Then

$$\sum_j \|f_j(B_j)S - Sf_j(A_j)\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})} \leq$$

$$CK_{(A_j)}K_{(B_j)} \left\| \sum_j B_jS - SA_j \right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}$$

for all $S \in \mathcal{L}\left(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\right)$.

Proof. Simply combine Propositions 31 and 33 and Lemma 6.6 with Proposition 6.3 (i).

Single out the specific case in Theorem 6.7 where $\epsilon = 0$ or$\epsilon = \infty$and S is the identity sequence of operator.

Corollary 6.8. There exists a universal constant $C \geq 0$ such that

$$\sum_j \|f_j(B_j) - f_j(A_j)\|_{\mathcal{L}(\ell_1)} \leq C_{(A_j)}K_{B_j} \left\| \sum_j B_j - A_j \right\|_{\mathcal{L}(\ell_1)}$$

for all $A_j, B_j \in \mathcal{L}_{(1+\epsilon)}(\ell_1)$with real spectrum.

Corollary 6.9. There exists a universal constant $C \geq 0$ such that

$$\sum_j \|f_j(B_j) - f_j\ (A_j)\|_{\mathcal{L}(\ell_1)} \leq CK_{(A_j)}K_{B_j} \left\| \sum_j B_j - A_j \right\|_{\mathcal{L}(c_0)}$$

for all $A_j, B_j \in \mathcal{L}_{(1+\epsilon)}(c_0)$ with real spectrum.

In the case of Theorem 6.7 where $\epsilon = 1$, we can apply results tocompact self-adjoint sequence of operators. By the spectral theorem, any compact self-adjoint sequence of operator $A_j \in \mathcal{L}(\ell_2)$ has an orthonormal basis of eigenvectors, and therefore

$A_j \in \mathcal{L}_{(1+\epsilon)}(\ell_2, \lambda_j, U)$for some sequence $\lambda_j$of real numbers and an isometry $U \in \mathcal{L}(\ell_2)$.Thus Theorem 6.7 yields the following corollaries.

Corollary 6.10. Let $0 \leq \epsilon < 1$. Then there exists a constant $C \geq 0$ such that the following holds. Let $A_j \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$ and let $\mathrm{B_j} \in \mathcal{L}(\ell_2)$ be compact and self-adjoint. Then

$$\sum_j \|f_j(\mathrm{B_j})S - Sf_j(A_j)\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_2)} \leq CK_{(A_j)} \left\| \sum_j \mathrm{B_j}S - SA_j \right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_2)}$$

for all $S \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_2)$.

Corollary 6.11. Let $1 < \epsilon \le \infty$. Then there exists a constant $C \ge 0$ such that the following holds. Let $A_j \in \mathcal{L}(\ell_2)$ be compact andself-adjoint, and let $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$. Then

$$\sum_j \left\| f_j(\mathrm{B_j})S - Sf_j(A_j) \right\|_{\mathcal{L}(\ell_2, \ell_{(1+\epsilon)})} \le CK_{(A_j)} \left\| \sum_j \mathrm{B_j}S - SA_j \right\|_{\mathcal{L}(\ell_2, \ell_{(1+\epsilon)})}$$

for all $S \in \mathcal{L}(\ell_2, \ell_{(1+\epsilon)})$.

Remark 6.12. Corollaries 6.8 and 6.9 show that the absolute value function is sequence of operator Lipschitz on $\ell_1$and c_0, in the following sense. For fixed $\epsilon = 0$, there existsa constant $C \ge 0$ such that

$$\sum_j \left\| f_j(\mathrm{B_j}) - f_j(A_j) \right\| \le C \left\| \sum_j \mathrm{B_j} - A_j \right\|$$

for all diagonalizable sequence of operators A_j and $\mathrm{B_j}$ such that$K_{(A_j)}, K_{(\mathrm{B_j})} \le 1 + \epsilon$, and C is independent of $A_j$and $\mathrm{B_j}$.

For $\epsilon > 0$ a similar statement holds. Restricting $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$ and $f_j(\mathrm{B_j}) \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$ to sequence of operators from $\ell_{(1+\epsilon)}$ to $\ell_{(1+\epsilon)}$, and letting S be the inclusion mapping $\ell_{(1+\epsilon)} \hookrightarrow \ell_{(1+\epsilon)}$ inTheorem 6.7, one can suggestively write

$$\sum_j \left\| f_j(\mathrm{B_j}) - f_j(A_j) \right\|_{\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})} \le C \left\| \sum_j \mathrm{B_j} - A_j \right\|_{\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})},$$

for all $A_j \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)})$ with $K_{(A_j)}, K_{(\mathrm{B_j})} \le 1 + \epsilon$. This also applies to Corollaries6.10 and 6.11.

6.3. **The case** $\epsilon \ge 0$. Examine the absolute value functions f_j on $\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$ obtain the following result.

Proposition 6.13. Let $0 < \epsilon \le \infty$ then for each $\epsilon > 0$ there exists aconstant $C \ge 0$ such that the following holds .Let$A_j \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)}, \lambda_j, U)$and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)}, \mu_j, V)$ have real spectrum, and let $S \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$ besuch that

$V(B_jS - SA_j)U^{-1} \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$. Then

$$\sum_{j}\left\|f_j(\mathrm{B_j})S - Sf_j(A_j)\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})} \leq C\|U\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\|V^{-1}\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\left\|S(A_j)U^{-1}\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},(1+\epsilon))}.$$

In particular, if $\epsilon = 0$ and $V(\mathrm{B_j} - A_j)U^{-1} \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$, then

$$\left\|f_j(\mathrm{B_j}) - f_j(A_j)\right\|_{\mathcal{L}(\ell_{(1+\epsilon)})} \leq C\|U\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\|V^{-1}\|_{\mathcal{L}(\ell_{(1+\epsilon)})}V(\mathrm{B_j} - A_j)\|U^{-1}\|_{\mathcal{L}(\ell_{(1+\epsilon)},(1+\epsilon))}.$$

Proof. Let $R := V(\mathrm{B_j}(S) - S(A_j))U^{-1}$. With notation as in Lemma 5.3,

$$\sum_{j}\left\|f_j(S)_n(\mathrm{B_j}) \;\; - (S)_n f_j(A_j)\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}$$

$$\leq \|U\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\|V^{-1}\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\left\|\sum_{j}\,(T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}(R)\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}$$

for all $n \in \mathbb{N}$. Proposition 5.6, Lemma 6.6 and Proposition 6.3 (iii) yield a constant $C^{'} \geq 0$ such that

$$\sum_{j}\left\|(T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}(R)\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})} \leq C^{'}\left(\|R\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})} + \|R\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}\right).$$

Since $\mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}) \hookrightarrow \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$ contractively,

$$\sum_{j}\|f_j(\mathrm{B_j})S_n - S_n f_j(A_j)\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)}) \leq (1+\epsilon)} \leq$$

$$C\|U\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\|V^{-1}\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\left\|V\sum_{j}(\mathrm{B_j}S - SA_j)U^{-1}\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{1+\epsilon})}$$

for all $n \in \mathbb{N}$. Finally, as in the proof of Proposition 5.4, one lets tend to infinity to conclude the proof.

In the same way, appealing to the second part of Proposition 6.3 (iii), one deduces the following result.

Proposition 6.14. Let $0 \leq \epsilon < \infty$with $\epsilon \geq 0$. Then for all $\epsilon > 0$, there exists aconstant $C \geq 0$ such that the following holds. Let$A_j \in \mathcal{L}_d(\ell_{(1+\epsilon)}, \lambda_j, U)$and

$B_j \in \mathcal{L}_{(1+\epsilon)}(\ell_{(1+\epsilon)}, \mu_j, V)$have real spectrum, and let $S \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$ besuch that

$V(B_j(S - SA_j)U^{-1} \in \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$. Then

$$\sum_j \|f_j(\mathrm{B_j})S - Sf_j(A_j)\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}$$

$$\leq C\|U\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\|V^{-1}\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\left\|\sum_j V\big(\mathrm{B_j}S - SA_j\big)U^{-1}\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}$$

In particular, if $V(\mathrm{B_j} - A_j)U^{-1} \in \mathcal{L}\big(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\big)$, then

$$\sum_j \|f_j(\mathrm{B_j}) - f_j(A_j)\|_{\mathcal{L}(\ell_{(1+\epsilon)})} \leq C\|U\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\|V^{-1}\|_{\mathcal{L}(\ell_{(1+\epsilon)})}\left\|V\sum_j(\mathrm{B_j} - A_j)U^{-1}\right\|_{\mathcal{L}(\ell_{(1+\epsilon)},\ell_{(1+\epsilon)})}.$$

Single out the case where $\epsilon = 1$. Here write $f_j(A_j) = |A_j|$ for a normal sequence of operators$A_j \in \mathcal{L}(\ell_2)$, since then $f_j(A_j)$ is equal to $|A_j| := \sqrt{(A_j{}^*)A_j}$. Note alsothat the following result applies in particular to compact self-adjoint sequence of operators.

For simplicity of the presentation only consider $0 < \epsilon \leq \infty$, it should be clear how the result extends to other $\epsilon > 0$.

Corollary 6.15. For each $0 < \epsilon \leq \infty$ there exists a constant $C \geq 0$ such that the following holds. Let $A_j \in \mathcal{L}_{(1+\epsilon)}(\ell_2, \lambda_j, U)$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(\ell_2, \mu_j, V)$ be self-adjoint, with U and Vunitaries, and let $S \in \mathcal{L}(\ell_2)$. If $V\sum_j(\mathrm{B_j}S - SA_j)U^{-1} \in \mathcal{L}(\ell_2, \ell_{2-\epsilon})$, then

$$\sum_j \||\mathrm{B_j}|S - S|A_j|\|_{\mathcal{L}(\ell_2)} \leq C\left\|V\sum_j(\mathrm{B_j}S - SA_j)U^{-1}\right\|_{\mathcal{L}(\ell_2,\ell_{2-\epsilon})}$$

and if $V(\mathrm{B_j}S - SA_j)U^{-1} \in \mathcal{L}(\ell_{2+\epsilon}, \ell_2)$then

$$\sum_j \||\mathrm{B_j}|S - S|A_j|\|_{\mathcal{L}(\ell_2)} \leq C\left\|\sum_j V(\mathrm{B_j}S - SA_j)U^{-1}\right\|_{\mathcal{L}(\ell_{2+\epsilon},\ell_2)}.$$

In particular, if $V(\mathrm{B_j} - A_j)U^{-1} \in \mathcal{L}(\ell_2, \ell_{2-\epsilon})$, then

$$\sum_j \||\mathrm{B_j}| - |A_j|\|_{\mathcal{L}(\ell_2)} \leq C\left\|\sum_j V(\mathrm{B_j} - A_j)U^{-1}\right\|_{\mathcal{L}(\ell_2,\ell_{2-\epsilon})}$$

and if $V(\mathrm{B_j} - A_j)U^{-1} \in \mathcal{L}(\ell_{2+\epsilon}, \ell_2)$, then

$$\sum_j \| |B_j S - S|A_j| \|_{\mathcal{L}(\ell_2)} \leq C \left\| \sum_j V(B_j S - SA_j)U^{-1} \right\|_{\mathcal{L}(\ell_{2+\epsilon},\ell_2)} .$$

7. Lipschitz Estimates on the Ideal Of P-Summing Operators

Let H be a separable infinite-dimensional Hilbert space. It was shown that a matrix $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is a Schur multiplier on the Hilbert-Schmidt class $S_2 \subset \mathcal{L}(H)$ if and only if $sup_{j,k} |m_{jk}| < \infty$., S_2 coincides with the Banachideal $\Pi_{(1+\epsilon)}(H)$ of all $(1+\epsilon)$ -summing dequence of operators for all $0 \leq \epsilon < \infty$. Hence a matrix $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is a Schur multiplier on $\Pi_{(1+\epsilon)}(H)$ if and only if$sup_{j,k} |m_{jk}| < \infty$.In Lemma 7.2 show that the same statement is true for the Banach ideal $\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$ $in\, \mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$, for all $1 < \epsilon < \infty$. As a corollary obtain operator Lipschitz estimates on $\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$ for each Lipschitz function f on $\mathbb{C}$.

Let X and Y be Banach spaces and $1 \leq \epsilon < \infty$. An sequence of operator $S: X \to Y$ is$(1+\epsilon)$-absolutely summing if there exists a constant C such that for each $n \in \mathbb{N}$ and each collection $\{x_j\}_{j=1}^{n} \subseteq X$,

$$\left(\sum_{j=1}^{n} \|S(x_j)\|_Y^{(1+\epsilon)}\right)^{\frac{1}{(1+\epsilon)}} \leq C \sup_{\|x^*\|_{X^*} \leq 1} \left(\sum_{j=1}^{n} |\langle x^*, x_j\rangle|^{(1+\epsilon)}\right)^{\frac{1}{(1+\epsilon)}} . \quad (39)$$

The smallest such constant is denoted by $\pi_{(1+\epsilon)}$, and $\Pi_{(1+\epsilon)}(X,Y)$ is the space of $(1+\epsilon)$ -absolutely summing operators from X to Y . Let $\Pi_{(1+\epsilon)}(X) := \Pi_{(1+\epsilon)}(X,X)$. By Propositions2.3, 2.4 and 2.6 in [20], $(\Pi_{(1+\epsilon)}(X,Y), \pi_{(1+\epsilon)}(\cdot))$ is a Banach ideal in $\mathcal{L}(X,Y)$.

Below consider p-absolutely summing sequence of operators from $\ell_{(1+\epsilon)^*}$ $(1+\epsilon) o$ $\ell_{(1+\epsilon)}$. First present the following result.

Lemma 7.1. Let $1 < \epsilon < \infty$and $S = \{(S)_{jk}\}_{j,k=1}^{\infty}$. Then $S \in \Pi_{(1+\epsilon)}(\ell_{(1+\epsilon)^*}, \ell_{(1+\epsilon)})$if and only if

$$c_{(1+\epsilon)} := \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |(S)x_{jk}|^{(1+\epsilon)}\right)^{\frac{1}{(1+\epsilon)}} < \infty.$$

In this case,$\pi_{(1+\epsilon)}(S) = c_{(1+\epsilon)}$.

Proof. It is shown that, if $c_{(1+\epsilon)} < \infty$, then $S \in \pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$ with$\pi_{(1+\epsilon)}(S) \leq c_{(1+\epsilon)}$.

For the converse, let $n \in \mathbb{N}$and let $x_j := e_j \in \ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}$ for $1 \leq j \leq n$. By(39) (with $X = \ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}$ and $Y = \ell_{(1+\epsilon)}$),

$$\left(k \sum_{k=1}^{n} \sum_{j=1}^{\infty} \left|(S)x_{jk}\right|^{(1+\epsilon)}\right)^{\frac{1}{(1+\epsilon)}} \leq \pi_{(1+\epsilon)}(S).$$

Letting n tend to infinity concludes the proof.

For the following corollary of Lemma 7.1, recall that a matrix M is said to be a Schur multiplier on a subspace $I \subseteq \mathcal{L}(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)})$ if $S \longmapsto M * S$ is a bounded map onI. Recall also the definition of the standard triangular funcations $(\mathcal{T}^j)_\Delta$ from (38).

Corollary 7.2. Let $0 < \epsilon < \infty$ and let $M = \{m_{jk}\}_{j,k=1}^{\infty}$ be a matrix. Then M is a Schur multiplier on $\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$ if and only if $sup_{j,k}|m_{jk}| < \infty$. In this case,

$\|M\|_{\mathcal{L}\left(\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)\right)} = sup_{j,k}|m_{jk}|$.

In particular, $(\mathcal{T}^j)_\Delta \in \mathcal{L}\left(\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)\right)$

with$\sum_j \left\|(\mathcal{T}^j)_\Delta\right\|_{\mathcal{L}\left(\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)\right)} = 1$.

Observe that $(\mathcal{T}^j)_\Delta \notin \mathcal{L}\left(\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)\right)$ for $1 \geq \epsilon^2$, by Proposition 6.3 (ii). Nevertheless,$(\mathcal{T}^j)_\Delta$ are bounded on the ideal

$\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right) \subset \mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$for all $0 < \epsilon < \infty$

For a Lipschitz function $f_j: \mathbb{C} \to \mathbb{C}$, write

$$\sum_j \left\|f_j\right\|_{Lip_{(1+\epsilon)}} := \sup_{\substack{z_1, z_2 \in \mathbb{C} \\ z_1 \neq z_2}} \sum_j \frac{\left|f_j(z_1) - f_j(z_2)\right|}{|z_1 - z_2|}.$$

Moreover, let $(\varphi_j)_{f_j}: \mathbb{C}^2 \to \mathbb{C}$be given by

$$\sum_{j} (\varphi_j)_{f_j}\big((\lambda_j)_1, (\lambda_j)_2\big) := \begin{cases} \sum_{j} \frac{|(\lambda_j)_1| - |(\lambda_j)_2|}{(\lambda_j)_1 - (\lambda_j)_2} & if\ (\lambda_j)_1 \neq (\lambda_j)_2 \\ 0 & oherwise \end{cases}.$$

Prove main result concerning commutator estimates on$\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$

Theorem 7.3. Let $0 < \epsilon < \infty, A_j \in \mathcal{L}_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)$ and $A_j \in \mathcal{L}_{(1+\epsilon)}\big(\ell_{(1+\epsilon)}\big)$ Let $f_j: \mathbb{C} \to \mathbb{C}$beLipschitz. Then

$$\pi_{(1+\epsilon)} \sum_{j} \Big(f_j(\mathrm{B_j})S - Sf_j(A_j)\Big) \le \sum_{j} K_{(A_j)} K_{\mathrm{B_j}}\, f_{j_{(lip_{(1+\epsilon)})}} \pi_{(1+\epsilon)}\big(\mathrm{B_j}S - SA_j\big) (40)$$

for all $S \in \mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$ such that $\mathrm{B_j}S - SA_j \in \Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$.

Proof. Let $\lambda_j = \{(\lambda_j)_j\}_{j=1}^{\infty}$ and $\mu_j = \{(\mu_j)_k\}_{k=1}^{\infty}$ be sequences such that $A_j \in \mathcal{L}_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, (\lambda_j)_j, U\right)$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}\big(\ell_{(1+\epsilon)}, \mu_j, V\big)$ for certain $U \in \mathcal{L}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)$ and $V \in \mathcal{L}\big(\ell_{(1+\epsilon)}\big)$.

It follows directly from (39) that, if $\{(S)_m\}_{m=1}^{\infty} \subseteq \Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$ is a$\pi_{(1+\epsilon)}$ −bounded sequence which SOT-converges to some $S \in \mathcal{L}(X, Y)$, then S∈ $\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$ with$\pi_{(1+\epsilon)}(S) \le \lim\sup_{m\to\infty} \pi_{(1+\epsilon)}\ ((S)_m)$. Hence, by Remark 5.5, it suffices to prove that

$\sup_{n\in\mathbb{N}} \sum_{j} \left\| (T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n} \right\|_{\mathcal{L}\left(\Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)\right)} \le \left\|\sum_{j} f_j\right\|_{Lip}$, where

$$\sum_{j} (T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}},n}(S) = \sum \sum_{j,k=1}^{\infty} \varphi_{j_{(f_j)}}\Big((\lambda_j)_j, (\mu_j)_k\Big) P_k S P_j \left(S \in \Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)\right)$$

for $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and note that $(T^j)^{\lambda_j,\mu_j}_{\varphi_{j_{(f_j)}}j,n}(S) = M * S$ for $S \in \Pi_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}, \ell_{(1+\epsilon)}\right)$, , where $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is the matrix given by $m_{jk} = \varphi_{j_{(f_j)}}((\lambda_j)_j, (\mu_j)_k)$ for $1 \le j, k \le n$, and$m_{jk} = 0$ otherwise. Then

$$\sup_{j,k} |m_{jk}| \le \sup_{j,k} |m_{jk}| \sum_{j} \varphi_{j_{(f_j)0}}\, ((\lambda_j)_j, (\mu_j)_k)| \le \left\| \sum_{j} f_j \right\|_{Lip}.$$

Corollary 7.2 now concludes the proof.

Problem7.4. Let $0 \leq \epsilon \leq \infty$ be such that $\epsilon \geq 1$. Then the Schatten class $(S)_{(1+\epsilon)}$ coincides with the Banach ideal $\Pi_{(1+\epsilon),(1+\epsilon)}(\ell_2)$ of $((1+\epsilon),(1+\epsilon))$ −summing sequence of operators on ℓ_2 (for the definition of. Hence (see e.g. [1]), the standard triangular truncation is bounded on$\Pi_{(1+\epsilon),(1+\epsilon)}(\ell_2)$ for these indices. For which$0 \leq \epsilon \leq \infty$ is the standard triangular truncation bounded on $\Pi_{(1+\epsilon),(1+\epsilon)}\left(\ell_{(1+\epsilon)_1}, \ell_{(1+\epsilon)_2}\right)$? Are thereother non-trivial ideals I in $\mathcal{L}\left(\ell_{(1+\epsilon)}, \ell_{(1+\epsilon)}\right)$such that the standard triangular truncation is bounded on I ? As shown in Theorem 7.3, rest of this answers to the sequestions are linked to commutator estimates for diagonalizable sequence of operators.

8. Matrix Estimates

Apply the theory developed to finite dimensional spaces, and in doing so we obtain Lipschitz estimates which are independent of the dimension of the underlying space. The results yield dimension-independent estimates on finite dimensional spaces.

41. **Finite-dimensional spaces.** Let $n \in \mathbb{N}$and let X be an n-dimensional Banach space with basis $\{e_1, \ldots, e_n\} \subset X$. For $1 \leq k \leq n$, let $P_k \in \mathcal{L}(X)$ be the projectiongiven by $P_k(x_1 e_1 + \ldots + x_n e_n) := x_k e_k$ for $(x_1, \ldots, x_n) \in \mathbb{C}^n$. Recall that the sequence ofoperators $A_j \in \mathcal{L}(X)$ is diagonalizable if there exists $U \in \mathcal{L}(X)$ invertible such that

$$\sum_j U A_j U^{-1} = \sum_{k=1}^{n} \sum_j (\lambda_j)_k P_k \quad (41)$$

for some $\left((\lambda_j)_1, \ldots, (\lambda_j)_n\right) \in \mathbb{C}^n$. In this case write $A_j \in \mathcal{L}_{(1+\epsilon)}\left(\ell_{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)(X, \{(\lambda_j)_j\}_{j=1}^n, U)$.

Recallalso the definition of spectral and scalar type sequence of operators

Lemma 8.1. All $A_j \in \mathcal{L}(X)$ is a spectral sequence of operator. Furthermore, A_j are scalar type sequence of operator if and only if A_j are diagonalizable. If $A_j \in \mathcal{L}_{(1+\epsilon)}\left(X, \{(\lambda_j)_j\}_{j=1}^n, U\right)$ then the spectral measure E of A_j are given by $E(U) = 0 \; if \; U \cap sp(A_j) = \emptyset_j$ and $E(\{\lambda_j\}) = \sum_{(\lambda_j)_j = \lambda_j} U^{-1} P_j Ufor\lambda_j \in sp(A_j)$.

Proof. It was remarked that any diagonalizable sequence of operator is a scalar type sequence of operator, with spectral measure as specified, sequence of operators $T^j \in \mathcal{L}(Y)$ on an arbitrary

Banach space Y is a spectral sequence of operator if and only if $T^j = S + N$ for a scalar type sequence of operator $S \in \mathcal{L}(Y)$ and a generalized nilpotent sequence of operator $N \in \mathcal{L}(Y)$ such that $NS = SN$, and this decomposition is unique. Combining this with the Jordan decomposition for matrices yields that all$A_j \in \mathcal{L}(X)$ are spectral sequence of operator. If A_j are scalar type sequence of operator, then the Jordan decomposition yields a commuting diagonalizable $(1+\epsilon)$ and a nilpotent N such that $A_j = S + N$. Bythe uniqueness of such a decomposition, $N = 0$ and $A_j = S$ are diagonalizable.

Let$A_j \in \mathcal{L}_{(1+\epsilon)}\left(X, \left\{(\lambda_j)_j\right\}_{j=1}^n, U\right)$. As in (30),

$$\sum_j f_j\,(A_j) = U^{-1}\sum_j\left(\sum_{k=1}^n f_j\,((\lambda_j)_k)P_k\right)U.$$

Let Y be a finite-dimensional Banach space. A norm $\|.\|$on $\mathcal{L}(X,Y)$ is symmetric if

• $\sum_j \|RST^j\| \le \|R\|_{\mathcal{L}(Y)}\|S\|\left\|\sum_j T^j\right\|_{\mathcal{L}(X)}$ for all $R \in \mathcal{L}(Y), S \in \mathcal{L}(X,Y)$ and $T^j \in \mathcal{L}(X)$;• $\|x^* \otimes y\| = \|x^*\|_{X^*}\|y\|_Y$ for all $x^* \in X^*$ and $y \in Y$.

Clearly $(\mathcal{L}(X,Y), \|\cdot\|)$ is a Banach ideal in the sense of Section 10 if and only if k·k is symmetric.

The following result extends inequalities which were known for self-adjoint sequence of operators on finite-dimensional Hilbert spaces and unitarily invariant norms, to diagonalizable sequence of operators on finite-dimensional Banach spaces and symmetric norms. Note that, for general finite-dimensional Banach spaces X and Y, a symmetric norm on $\mathcal{L}(X,Y)$ is not unitarily invariant. Let $\mathfrak{A} := \mathfrak{A}(\mathbb{C}\times\mathbb{C})$ be as in Section 11, and for $f_j \in \mathcal{B}_j(\mathbb{C})$ let

$\sum_j \varphi_{j_{(f_j)}}\left((\lambda_j)_1, (\lambda_j)_2\right) := \sum_j \frac{f_j((\lambda_j)_2) - f_j((\lambda_j)_1)}{(\lambda_j)_2 - (\lambda_j)_1}$for $((\lambda_j)_1, (\lambda_j)_2) \in \mathbb{C}^2$ with

$(\lambda_j)_1 \neq (\lambda_j)_2$, as in (23). Recall the definition of the spectral constants $\nu^j(A_J)$ and $\nu^j(B_j)$. The following is a direct corollary of Theorem 4.6, since $\mathcal{L}(X,Y)$ has the strong convex compactness property.

Theorem 8.2. Let $f_j \in \mathcal{B}_j(\mathbb{C})$ be such that $(\varphi_j)_{f_j}$ extends to an element of $\mathfrak{A}$. Let X and Y befinite-dimensional Banach spaces,$\|\cdot\|$a symmetric norm on $\mathcal{L}(X,Y)$, and
let $A_j \in \mathcal{L}_{(1+\epsilon)}(X)$and $B_j \in \mathcal{L}_{(1+\epsilon)}(Y)$. Then

$$\sum_{j}\left\|f_j(\mathrm{B_j})-f_j(A_j)\right\| \le 16\,\nu^j(A_j)\nu^j(\mathrm{B_j})\left\|\sum_{j}\varphi_{j_{(f\,j)}}\right\|_{\mathfrak{A}}\left\|\mathrm{B_j}-A_j\right\|$$

for all $S \in \mathcal{L}(X, Y)$. In particular, if $X = Y$

$$(41)\sum_{j}\left\|f_j(\mathrm{B_j})-f_j(A_j)\right\| \le 16\,\nu^j\left(A_j\right)\nu^j\left(\mathrm{B_j}\right)\left\|\sum_{j}\varphi_{j_{(f\,j)}}\right\|_{\mathfrak{A}}\left\|\mathrm{B_j}-A_j\right\|.$$

Corollary 8.3. There exists a universal constant $C \geq 0$ such that the following holds. Let$X$ and Y be finite-dimensional Banach spaces and $\|\cdot\|$ a symmetric norm on $\mathcal{L}(X, Y)$. Let$f_j \in (\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})$, and let $A_j \in \mathcal{L}_{(1+\epsilon)}(X)$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(Y)$ be such that $sp(A_j) \cup sp(\mathrm{B_j}) \subseteq \mathbb{R}$. Then

$$\sum_{j}\left\|f_j(\mathrm{B_j})-f_j(A_j)\right\| \le C\,\nu^j(A_j)\nu^j(\mathrm{B_j})\left\|\sum_{j}f_j\right\|_{(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})}\left\|\mathrm{B_j}-A_j\right\|$$

for all $S \in \mathcal{L}(X, Y)$. In particular, if $X = Y$,

$$\sum_{j}\left\|f_j(\mathrm{B_j})-f_j(A_j)\right\| \le C\nu^j\left(A_j\right)\nu^j\left(\mathrm{B_j}\right)\left\|\sum_{j}f_j\right\|_{(\dot{\mathrm{B}}_{\mathrm{j}})^1_{\infty,1}(\mathbb{R})}\left\|\mathrm{B_j}-A_j\right\|.$$

Remark 8.4. Let $\sigma_1, \sigma_2 \subset \mathbb{C}$be finite sets. Then all$\varphi_j: \sigma_1 \times \sigma_2 \to \mathbb{C}$ belongs to$U(\sigma_1 \times \sigma_2)$. Indeed, one can find a representation as in (16) by letting Ω be finite and solving a system of linear equations. Therefore Theorem4.6 yields an estimate series

$$\sum_{j}\left\|f_j(\mathrm{B_j})-f_j(A_j)\right\| \le 16\,\nu^j(A_j)\nu^j(\mathrm{B_j})\left\|\sum_{j}\varphi_{j_{(f\,j)}}\right\|_{\mathfrak{A}(sp(A_j)\times sp(\mathrm{B_j}))}\left\|\mathrm{B_j}-A_j\right\|$$

as in (24) for all $f_j \in \mathrm{B_j}(\mathbb{C})$. This might lead one to think that the restriction in Theorem 8.2 that $\varphi_{j_{(f\,j)}}$ extends to an element of $\mathfrak{A}$ is not really necessary. However, for general $f_j \in B_j(\mathbb{C})$ the series norms$\sum_j \left\|\varphi_{j_{(f\,j)}}\right\|_{\mathfrak{A}(sp)\mathrm{A_j})\times sp(\mathrm{B_j}))}$ may blow up as the number ofpoints in $sp(A_j)$ and $sp(\mathrm{B_j})$ grows to infinity. Indeed, as remarked before, letting $f_j \in \mathrm{B_j}(\mathbb{C})$ be the absolute value function and considering the sequence of operator norm, adimension-independent estimate as in (41) does not hold for all self-adjoint sequence of operators on all finite-dimensional Hilbert spaces. Hence

$(\varphi_j)_f$ does not extendto an element of A_j, and one cannot expect to obtain Theorem 8.2 or Corollary 8.3 for all bounded Borel functions on $\mathbb{C}$.

The absolute value function. Apply the results for the absolute value function from previous to finite-dimensional spaces. let$f_j((1+\epsilon)) := |(1+\epsilon)|$for $(1+\epsilon) \in \mathbb{R}$.

First note that Lemma 5.3 and Proposition 5.6 relate commutator estimates to estimates for triangular truncation sequence of operators for general symmetric norms [21] on matrix spaces.

For $n \in \mathbb{N}$and $0 \leq \epsilon < \infty$ let $\ell^n_{(1+\epsilon)}$denote $\mathbb{C}^n$ with the p-series norms

$$\|(x_1, \ldots, x_n)\|_{(1+\epsilon)} := \left(\sum_{j=1}^{n} |x_j|^{(1+\epsilon)}\right)^{1/(1+\epsilon)} \qquad (x_1, \ldots, x_n) \in \mathbb{C}^n,$$

And $let\ \ell^n_{(1+\epsilon)}$be $\mathbb{C}^n$th the norm

$$\|x_1, \ldots, x_n\|_\infty : = \max_{1 \leq j \leq n} |x_j| \; (x_1, \ldots, x_n) \in \mathbb{C}^n.$$

Theorem 6.7 immediately yields the following result. It shows that, although the Lipschitz series estimates

$$\sum_j \left\|f_j(\mathrm{B_j}) - f_j(A_j)\right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)} \leq const \left\|\sum_j \mathrm{B_j} - A_j\right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)}$$

does not hold with a constant independent of the dimension n for f_j the absolutevalue function, $\epsilon = 1$ and all self-adjoint sequence of operators on ℓ^n_2, one can nevertheless obtain such estimates for $\epsilon = 0$ or $\epsilon = \infty$ by considering diagonalizable sequence of operators A_j and B for which $K_{(A_j)}, K_{(B_j)} \leq M$, for some fixed $M \geq 1$.

For A_j a diagonalizable sequence of operator, recall the definition of $K_{(A_j)}$ from (32).

Theorem 8.5. Let $0 \leq \epsilon \leq \infty$ with $\epsilon = 0$ or $\epsilon = \infty$. Then there exists a constant $C \geq 0$ such that the following holds. Let $n \in \mathbb{N}$and let $A_j \in \mathcal{L}_{(1+\epsilon)}\left(\ell^n_{(1+\epsilon)}\right)$and $B_j \in \mathcal{L}_{(1+\epsilon)}\left(\ell^n_{(1+\epsilon)}\right)$ have real spectrum. Then

$$\sum_{\boldsymbol{j}} \left\|f_j(\mathrm{B_j})S - Sf_j(A_j)\right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)} \leq CK_{(A_j)}, K_{\mathrm{B_j}} \left\|\sum_{\boldsymbol{j}} \mathrm{B_j}S - SA_j\right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)}$$

for all $S \in \mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)$. In particular,

$$\sum_j \|f_j(\mathrm{B_j}) - f_j(A_j)\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)} \leq CK_{(A_j)}, K_{(\mathrm{B_j})} \left\| \sum_j \mathrm{B_j} - A_j \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)}.$$

Of course Corollaries 6.10 and 6.11 imply that for $\epsilon = 1$ and A_j or $\mathrm{B_j}$self-adjoint, the estimates in Theorem 8.5 simplify.

For $\epsilon \geq 0$, Propositions 6.13 and 6.14 yield dimension-independent estimates.

State the estimates which follow from Proposition 6.13, the analogous result that follows from Proposition 6.14 should be obvious.

Proposition 8.6. Let $0 < \epsilon \leq \infty$ with $\epsilon \geq 0$. , there exists a constant $C \geq 0$ such that the following holds. Let $n \in \mathbb{N}$, and let $A_j \in \mathcal{L}_{(1+\epsilon)}(\ell^n_{(1+\epsilon)}, \lambda_j, U)$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(\ell^n_{(1+\epsilon)}\mu_j, V)$ have real spectrum. Then

$$\sum_j \|f_j(\mathrm{B_j})S - Sf_j(A_j)\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)}$$

$$\leq C\|U\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \|V^{-1}\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \left\| \sum_j V(\mathrm{B_j}S - SA_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{1+\epsilon}\right)}$$

for all $S \in \mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)$. In particular,

$$\sum_j \|f_j(\mathrm{B_j}) - f_j\,(A_j)\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{(1+\epsilon)}\right)}$$

$$\leq C\|U\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \|V^{-1}\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \left\| \sum_j V(\mathrm{B_j} - A_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{1+\epsilon}\right)}.$$

In the case $\epsilon = 1$, Corollary 6.15 implies the following. Again only consider $0 < \epsilon \leq \infty$, for simplicity, but the result extends in an obvious manner toother $\epsilon > 0$.

Write $f_j(A_j) = |A_j| = \sqrt{(A_j{}^*)A_j}$ for a normal operator $A_j$on ℓ^n_2 .

Result: Deduce that:

$$\left\| \sum_j V(\mathrm{B_j} - A_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{1+\epsilon}\right)} \leq \frac{1}{M} \|U\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \|V^{-1}\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \left\| \sum_j V(\mathrm{B_j} - A_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}, \ell^n_{1+\epsilon}\right)}$$

When M is constant, $M \geq 0$

Proof.

Since

$$\sum_j \|f_j(\mathrm{B_j}) - f_j(A_j)\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{(1+\epsilon)}\right)} \leq CK_{(A_j)}, K_{(\mathrm{B_j})} \left\| \sum_j \mathrm{B_j} - A_j \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{(1+\epsilon)}\right)} . \text{(a)}$$

$$\sum_j \|f_j(\mathrm{B_j}) - f_j\ (A_j)\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{(1+\epsilon)}\right)}$$

$$\leq C\|U\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \|V^{-1}\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \left\| \sum_j V(\mathrm{B_j} - A_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{1+\epsilon}\right)} \quad (b)$$

Divide (b) by (a)

$$K_{(A_j)}, K_{(\mathrm{B_j})} \left\| \sum_j \mathrm{B_j} - A_j \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{(1+\epsilon)}\right)}$$

$$= \|U\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \|V^{-1}\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \left\| \sum_j V(\mathrm{B_j} - A_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{1+\epsilon}\right)}$$

$$\left\| \sum_j V(\mathrm{B_j} - A_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{1+\epsilon}\right)} \leq \frac{1}{M} \|U\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \|V^{-1}\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)}\right)} \left\| \sum_j V(\mathrm{B_j} - A_j)U \right\|_{\mathcal{L}\left(\ell^n_{(1+\epsilon)},\ell^n_{1+\epsilon}\right)}$$

Where $M = K_{(A_j)}, K_{(\mathrm{B_j})}$

Corollary 8.7. For each $0 < \epsilon \leq \infty$ there exists a constant $C \geq 0$ such that the following holds. Let $n \in \mathbb{N}$ and let $A_j \in \mathcal{L}_d(\ell^n_2, \lambda_j, U)$ and $\mathrm{B_j} \in \mathcal{L}_{(1+\epsilon)}(\ell^n_2, \mu_j, V)$ be self-adjoint operators, with U and V unitaries. Then

$$\sum_j \left\| |\mathrm{B_j}| - |A_j| \right\|_{\mathcal{L}(\ell^n_2)} \leq C min \left\| \sum_j \mathrm{B_j} - A_j \right\|_{\mathcal{L}(\ell^\mathrm{n}_2)}, \min \|\mathrm{U}^{-1}\|_{\mathcal{L}(\ell^n_2, \ell^n_{2-\epsilon})}, \|V\|_{\mathcal{L}(\ell^n_{2+\epsilon}, \ell^\mathrm{n}_2)}. \quad (42)$$

Note that (42) in turn yields, for instance, the following series estimates:

$$\sum_j \left\| |\mathrm{B_j}| - |A_j| \right\|_{\mathcal{L}(\ell^n_2)} \leq$$

$$Cmin \left\| V \sum_j (\mathrm{B_j} - A_j) U^{-1} \right\|_{\mathcal{L}(\ell_2^n, \ell_{2-\epsilon}^n)}, \sum_j \left\| V(\mathrm{B_j} - A_j) U^{-1} \right\|_{\mathcal{L}(\ell_2^n, \ell_{2-\epsilon}^n)}.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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