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UNIVERSALITY OF AFFINE SEMI-GROUPS ON SUPERCYCLICITY OF THE SEQUENCE OPERATORS

MORTADA S. ALI*, A.H. ABADALLA, ALSHAIKH A.A. SHOKERALLA

Department of Mathematics, College of Science and Art, Al Baha University, KSA

Abstract: In this paper, show that for all supercyclic strongly continuous sequence of operators semi-group on a complex $\mathcal{F}^j$-space, every $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic. Also, the sets of all supercyclic vectors $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ are precisely the sets of supercyclic vectors of the entire semi-group.

Keywords: hypercyclic semi-groups; hypercyclic operators; supercyclic operators; supercyclic semi-groups.

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1. INTRODUCTION

Unless stated, the spaces in this paper are over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Topological spaces are assumed to be Hausdorff. As usual $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{R}_+$ are positive integers and real numbers respectively. The symbol $L(X)$ denoted the space of continuous linear sequence of operators on a topological vectors space X, while X' is the space of continuous linear functionals on X As usual, for $T^j \in L(X)$, the dual sequence of operators $\acute{T}^j: X' \to X'$ is defined by the formula $\sum_j (\acute{T}^j) f_j\,(x) = \sum_j f_j\,(T^j x)$ for $x \in X$ and $f_j \in X'$.

*Corresponding author
E-mail address: abuhula2007@gmail.com
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suppose that an affine map T^j on a vector space X is a map of the forms $T^j x = u^j + S^j x$, where u^j are fixed vector in X and $S^j: X \to X$ is linear. Clearly, T^j are continuous iff S^j are continuous. The symbols $A_j(X)$ stands for the space of continuous affine maps on a topological vector space $\mathcal{F}^j$ –space are complete metrizable topological vector space. Recall that a families $\sum_j \mathcal{F}^j = \sum_j \{T_a^j\}_{a \in A_j}$ of continuous maps from a space X to a space Y is called universal if there is $x \in X$ for which $\{T_a^j x : a \in A_j\}$ is dense in Y and such an x is called a universal element for $\mathcal{F}^j$. Let the symbol $\mathcal{U}(\mathcal{F}^j)$ denote the set of universal elements for $\mathcal{F}^j$. If X is a topological space X to itself is called a semi-groups if $\left(T^j\right)_0 = I$ and $T^j_{2(\varepsilon+1)} = T^j_{(1+\varepsilon)} T^j_{(1+\varepsilon)}$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Say that a semi-groups $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are strongly continuous if $(1+\varepsilon) \mapsto T^j{}_{(1+\varepsilon)} x$ are continuous as a map from $\mathbb{R}_+$ to X for every $x \in X$ and say that $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is jointly continuous if $((1+\varepsilon), x) \mapsto T^j_{(1+\varepsilon)} x$ are continuous as map from $\mathbb{R}_+ \times X$ to X.

If X is a topological vector space, semi-group $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ a linear semi-group if $\sum_j T^j_{(1+\varepsilon)} \in L(X)$ for every $(1+\varepsilon) \in \mathbb{R}_+$ and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is called an affine semi-group if $\sum_j T^j_{(1+\varepsilon)} \in A_j(X)$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Recall that $T^j \in L(X)$ are called hypercyclic if $\mathcal{U} \sum_j \left(T^j\right) \neq \emptyset$ and elements of $\mathcal{U} \sum_j \left(T^j\right)$ are called hypercyclic vectors. A universal linear semi-groups $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are called hypercyclic and its universal elements are called hypercyclic vectors for $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. If $\sum_j T^j_{(1+\varepsilon)} \in L\,(X)$, then universal elements of the family $\left\{z \sum_j \left(T^j\right)^n x : z \in \mathbb{K}, n \in \mathbb{Z}_+\right\}$ are called supercyclic vectors for T^j and T^j are called supercyclic if it has a supercyclic vector. Similarly, if $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are linear semi-group, then universal element of the families $\left\{z \sum_j T^j_{(1+\varepsilon)} x : z \in \mathbb{K}, (1+\varepsilon) \in \mathbb{Z}_+\right\}$ are called supercyclic vector for $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$

and the semi-group is called supercyclic if it has a supercyclic vector.

[1] and references therein have been covered the concept of Hypercyclicity and super-cyclicity, also discuss the relation between the supercyclicity of a linear semi-group and supercyclicity of the individual members of the semi-group. The hypercyclicity version of the question was treated by Conejero, Müller and Peris [2], who proved that for every strongly continuous hypercyclic linear semi-groups $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on an $\mathcal{F}^j$-space, all $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is hypercyclic and $\mathcal{U}\sum_j \left(T^j_{(1+\varepsilon)}\right) = \mathcal{U}\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$. Virtually the same proof works in the following much more general setting.

Theorem A. Let $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a hypercyclic jointly continuous linear semi-group on all topological vector space X. Then all $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is hypercyclic and $\mathcal{U}\sum_j \left(T^j_{(1+\varepsilon)}\right) = \mathcal{U}\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$.

The stronger condition of joint continuity coincides with the strong continuity in the case when X is an $\mathcal{F}^j$ -space due to a straightforward application of the Banach–Steinhaus theorem.

if X has no subsets different from $\emptyset$ and X then it called connected, which are closed and open and it is called simply connected if for any continuous map $f_j\colon \mathbb{T} \longrightarrow X$, there is continuous maps $F^j\colon \mathbb{T}\times[0,1] \longrightarrow X$ and $x_0 \in X$ such that $F^j(z,0) = f_j(z)$ and $F^j(z,1) = x_0$ for any $z \in \mathbb{T}$. Next, X is called locally path connected at $x \in X$ if for any neighborhood U of x, there is a neighborhood V of x such that for any $y \in V$, there is a continuous map $f_j\colon [0,1] \longrightarrow X$ satisfying $f_j(0) = x$, $f_j(1) = y$ and $f_j([0,1]) \subseteq U$.

Proposition1.1. Let X be a topological space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a jointly continuous semi-group on X such that:

(1) $\{\sum_j T^j_{(1+\varepsilon)}\left(u^j\right) : (1+\varepsilon) \in [0,(1+\varepsilon)]\}$ are nowhere dense in X for every $\varepsilon > -1$ and $u^j \in X$;

(2) for every $\varepsilon > -1$ and $x \in \mathcal{U}\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$, there is $Y_{(1+\varepsilon),x} \subseteq X$ such that $Y_{(1+\varepsilon),x}$ is connected, locally path connected, simply connected and

$$\{\sum_j T^j_{(1+\varepsilon)}x : (1+\varepsilon) \in [0,(1+\varepsilon)]\} \subseteq Y_{(1+\varepsilon),x} \subseteq \mathcal{U}\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right).$$

Then $\mathcal{U}\sum_j \left(T^j_{(1+\varepsilon)}\right) = \mathcal{U}\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$ for every $\varepsilon > -1$.

in [3] the supercyclicity version of Theorem A holds, They have produced the following example to explain that it fails in the case $\mathbb{K} = \mathbb{R}$

Example B. Let X be a Banach space over $\mathbb{R}$, $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a hypercyclic linear semi-group on X and $\left(A_j\right)_{(1+\varepsilon)} \in L(\mathbb{R}^2)$ for $(1+\varepsilon) \in \mathbb{R}_+$ be the linear sequence of operators with the matrices $A_j = \begin{pmatrix} \cos(1+\varepsilon) & \sin(1+\varepsilon) \\ -\sin(1+\varepsilon) & \cos(1+\varepsilon) \end{pmatrix}$ Then $\sum_j \left\{\left(A_j\right)_{(1+\varepsilon)} \oplus T^j_{(1+\varepsilon)}\right\}_{(\varepsilon+1)\in\mathbb{R}_+}$ are supercyclic linear semi-group on $\mathbb{R}^2 \times X$, while $\sum_j \left(\left(A_j\right)_{(1+\varepsilon)} \oplus T^j_{(1+\varepsilon)}\right)$ are non-supercyclics whenever $\frac{(1+\varepsilon)}{\pi}$ is rational.

Proposition C. Let X be a complex topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semi-group on X such that $T^j_{(1+\varepsilon)} - \lambda_j I$ has dense range for every $\varepsilon > -1$ and every $\lambda_j \in \mathbb{C}$.Theneach $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic.

Furthermore, the set of supercyclic vectors for $T^j_{(1+\varepsilon)}$ does not determine by the choice of $\varepsilon > -1$ and matches with the set of supercyclic vectors of the entire semi-group.

To show that in the case $\mathbb{K} = \mathbb{C}$, the supercyclicity version of Theorem A holds without any additional assumptions We can applying Proposition 1.1. with the same results in [9] and considering the induced action on subsets of the projective space.

Theorem 1.2. Let X be a complex topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semi-group on X. Then all $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic and the set of supercyclic vectors of $T^j_{(1+\varepsilon)}$ coincides with the set of supercyclic vectors of $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$.

clearlly that any supercyclic jointly continuous linear semi-group on a complex topological vector X satisfies conditions of Proposition **C** or has a closed invariant hyperplane Y clearlly that any supercyclic jointly continuous linear semi-group on a complex topological vector X either satisfies conditions of Proposition **C** or has a closed invariant hyperplane Y **.** In the other case the topic reduces to the generalization of Theorem **A** to affine semi-groups see e.g.e.,[4].

Theorem1.3. Let X be a topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal jointly continuous affine semi-group on X. Then all $T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is universal and$\mathcal{U}\sum_j \left(T^j_{(1+\varepsilon)}\right) = \mathcal{U}\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$.

2. A DICHOTOMY FOR SUPERCYCLIC LINEAR SEMI-GROUPS

An analogue of the following result for individual supercyclic sequence of operators.

Proposition 2.1. Let X be a complex topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic strongly continuous linear semi-group on X. Then either $\sum_j (T^j_{(1+\varepsilon)} - \lambda_j I)\,(X)$ is dense in X for every $\varepsilon > -1$ and $\lambda_j \in \mathbb{C}$ or there is a closed hyperplane H in X such that $\sum_j T^j_{(1+\varepsilon)}\,(H) \subseteq H$ for every $(1+\varepsilon) \in \mathbb{R}_+$.

The most of the section is devoted to the proof of Proposition 2.1. Need several elementary lemmas. Recall that subsets B_j of vector space X are called balanced if $\lambda_j x \in B_j$ for every $x \in B_j$ and $\lambda_j \in \mathbb{K}$ such that $|\lambda_j| \leq 1$.

Lemma 2.2. Let K be a compact subset of an infinite dimensional topological vector space and X such that $0 \notin K$. Then $\Lambda = \left\{\lambda_j x \colon \lambda_j \in \mathbb{K}, x \in K\right\}$is a closed nowhere dense subset of X.

Proof. Closeness of Λ in X is a straightforward exercise. Assume that Λ is not nowhere dense. Since Λ is closed, its interior L is non-empty. Since K is closed and $0 \notin K$, we can find a non-empty balanced open set U such that $U \cap K = \emptyset$. Clearly $\lambda_j x \in L$ whenever $x \in L$ and $\lambda_j \in \mathbb{K}, \lambda_j \neq 0$. Since U is open and together with the latter property of L implies that the open set $W^j = L \cap U$ is non-empty. Taking into account the definition of Λ, the inclusion $L \subseteq \Lambda$, the equality $U \cap K = \emptyset$ and the fact that U is balanced, see that every $x \in W^j$ can be written as $x = \lambda_j y$, where $y \in K$ and $\lambda_j \in \mathbb{D} = \{z \in \mathbb{K}: |z| \leq 1\}$.

Since both K and $\mathbb{D}$ are compact, $Q = \{\lambda_j y: \lambda_j \in \mathbb{D}, y \in K\}$is a compact subset of X. Since $X \subseteq Q$, W^j is a non-empty open set with compact closure. Such a set exists [5] only if X is finite dimensional. This contradiction completes the proof.

The following lemma is a particular case in [6].

Lemma 2.3. suppose that X is complex topological vector space such that $2 \leq \dim X < \infty$. therefore X is not uphold supercyclic strongly continuous linear semi-groups.

Lemma 2.4. Let X be an infinite dimensional topological vector space, $\lambda_j \in \mathbb{K}$, $(1+\varepsilon)_0 > 0$ and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a strongly continuous linear semi-group such that $T^j_{(1+\varepsilon)_0} = \lambda_j I$. Then $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are not supercyclics.

Proof. Let $x \in X\backslash\{0\}$. It suffices to show that x is not a supercyclic vector for $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$.

First, consider the case $\lambda_j = 0$. By the strong continuity, there is $\varepsilon > -1$ such that $0 \notin K = \left\{\sum_j T^j_{(1+\varepsilon)} x: (1+\varepsilon) \in [0, (\varepsilon+1)]\right\}$and K is a compact subset of X. By Lemma 2.2, $\left\{z \sum_j T^j_{(1+\varepsilon)} x: z \in \mathbb{K}, (\varepsilon+1) \in [0, (\varepsilon+1)\right\}$ is nowhere dense in$X$ Take $n \in \mathbb{N}$ such that $n(1+\varepsilon) \geq (1+\varepsilon)_0$. Since $T^j{}_{(1+\varepsilon)_0} = 0$ and $n(1+\varepsilon) \geq (1+\varepsilon)_0$, have $\sum_j (T^j)^n_{(1+\varepsilon)} = \sum_j T^j_{(1+\varepsilon)_n} = 0$. Then $Y = \overline{\sum_j \left(T^j_{(1+\varepsilon)}(X)\right)} \neq X$. In particular, Y is nowhere dense in X. Clearly, $\sum_j T^j_{(1+\varepsilon)} x \in Y$ whenever $\varepsilon > -1$. Hence $\{z(T^j)^n x: (1+\varepsilon) \in \mathbb{R}_+, z \in \mathbb{K}\}$ is contained in $A_j \cup Y$ and therefore is nowhere dense in X. Thus x is not a supercyclic vector for $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$

Assume that $\lambda_j \neq 0$. Then $T^j_{(1+\varepsilon)_0}x = \lambda_j^n x \neq 0$ for every $n \in \mathbb{Z}_+$. Hence each of the compact sets $K_n = \left\{z\left(T^j\right)^n x : z \in \mathbb{C}, (1+\varepsilon)_0 n \leq (1+\varepsilon) \leq (1+\varepsilon)_0(n+1)\right\}$ with $n \in \mathbb{Z}_+$ does not contain 0. By Lemma 2.2, the sets $\sum_j \left(A_j\right)_n = \left\{z \sum_j T^j_{(1+\varepsilon)} x : z \in \mathbb{C}, (1+\varepsilon)_0 n \leq (1+\varepsilon) \leq (1+\varepsilon)_0(n+1)\right\}$ are nowhere dense in X. On the other hand, for every $(1+\varepsilon) \in [(1+\varepsilon)_0 n, (1+\varepsilon)_0(n+1)]$, $\sum_j T^j_{(1+\varepsilon)+(1+\varepsilon)_0} x = \sum_j T^j_{(1+\varepsilon)} T^j_{(1+\varepsilon)_0} = \sum_j \lambda_j T^j_{(1+\varepsilon)} x$ and therefore $\sum_j \left(A_j\right)_n = \sum_j \left(A_j\right)_{n+1}$ for each $n \in \mathbb{Z}_+$. Hence $\left\{z \sum_j T^j_{(1+\varepsilon)} x : (1+\varepsilon) \in \mathbb{R}_+, z \in \mathbb{K}\right\}$, which is clearly the union of $\left(A_j\right)_n$, coincides with $\sum_j \left((A_j)_1\right)_n$ and therefore is nowhere dense. Thus x is not a supercyclic vector for $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$.

Lemma 2.5. Let X be a complex topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic strongly continuous linear semi-group on X. Let also $(\varepsilon+1)_0 > 0$ and $\lambda_j \in \mathbb{C}$. Then the space $Y = \overline{\left(\sum_j (T^j_{(1+\varepsilon)_0} - \lambda_j I)(X)\right)}$ either coincides with X or is a closed hyperplane in X.

Proof. Using the semi-group property, it is easy to see that Y is invariant for all $T^j_{(1+\varepsilon)}$. Factoring Y out, arrive to a supercyclic strongly continuous linear semi-group $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ cting on X/Y, where $\sum_j S^j_{(1+\varepsilon)}(x+Y) = \sum_j T^j_{(1+\varepsilon)} x + Y$. Obviously, $\sum_j S^j_{(1+\varepsilon)_0} = \sum_j \lambda_j I$. If X/Y is infinite dimensional, arrive to a contradiction with Lemma 2.4. If X/Y is finite dimensional and $\dim X/Y \geq 2$, we obtain a contradiction with Lemma 2.3. Thus $\dim X/Y \leq 1$, as required.

Proof of Proposition 2.1. Assume that there is $\varepsilon > -1$ and $\lambda_j \in \mathbb{K}$ such that $\sum_j (T^j_{(1+\varepsilon)} - \lambda_j I)(X)$ are not denses in X. By Lemma 2.5, $H = \overline{\left(\sum_j (T^j_{(1+\varepsilon)_0} - \lambda_j I)(X)\right)}$ are closed hyperplanes in X. It is easy to see that H is invariant for all $T^j_{(1+\varepsilon)}$.

The following lemma provides some extra information on the second case in Proposition 2.1.

Lemma 2.6. Let X be a complex topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a strongly continuous linear semi-group on X. Assume also that there is a closed hyperplane H in X such that $T^j_{(1+\varepsilon)}(H) \subseteq H$ for every $(1+\varepsilon) \in \mathbb{R}_+$ and let $f_j \in X'$ be such that $H = \ker f_j$ .Then there exists $w \in \mathbb{C}$ such that $e^{w(1+\varepsilon)} \sum_j \left(\acute{T}^j_{(1+\varepsilon)}\right) f_j = \sum_j f_j$ for every $(1+\varepsilon) \in \mathbb{R}_+$.

Proof. Since $H = \ker f_j$ is invariant for every $T^j_{(1+\varepsilon)}$, there is unique function $\varphi_j: \mathbb{R}_+ \longrightarrow \mathbb{C}$ such that $\sum_j \acute{T}^j_{(1+\varepsilon)} f_j = \sum_j \varphi_j(1+\varepsilon) f_j$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Pick $u^j \in X$ such that $f_j(u^j) = 1$. Then $\sum_j \left(\acute{T}^j_{(1+\varepsilon)} f_j\right)(u^j) = \sum_j f_j\left(T^j_{(1+\varepsilon)} u^j\right) = \sum_j \varphi_j(1+\varepsilon)$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Since $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is strongly continuous, φ_j is continuous. The semi-group property for $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ implies the semi-group property for the dual sequence of operators: $\left(T^j_0\right) = 1$ and $\sum_j \acute{T}^j_{2(1+\varepsilon)} = \sum_j \acute{T}^j_{(1+\varepsilon)} \acute{T}^j_{(1+\varepsilon)}$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Together with the equality $\sum_j \acute{T}^j_{(1+\varepsilon)} f_j = \sum_j \varphi_j(1+\varepsilon) f_j$, it implies that $\varphi_j(0) = 1$ and $\varphi_j \sum_j (2(1+\varepsilon)) = \sum_j \varphi_j(1+\varepsilon)\, \varphi_j(1+\varepsilon)$ for every $(1+\varepsilon) \in \mathbb{R}_+$. The latter and the continuity of φ_j means that there is $w \in \mathbb{C}$ such that $\varphi_j \sum_j (1+\varepsilon) = e^{-w(1+\varepsilon)}$ for each $(1+\varepsilon) \in \mathbb{R}_+$. Thus $e^{w(1+\varepsilon)} \sum_j \left(\acute{T}^j_{(1+\varepsilon)}\right) f_j = \sum_j f_j$ for $(1+\varepsilon) \in \mathbb{R}_+$, as required.

3. SUPERCYCLICITY VS. UNIVERSALITY OF AFFINE MAPS

Relate the supercyclicity of an operators or a semi-groups in the case of the existence of an invariant hyperplane and the universality of an affine maps or an affine semi-groups. Begin with the following generic lemma.

Lemma 3.1. Let X be a topological vector space, $u^j \in X$, $f_j \in X' \backslash \{0\}$, $f(u^j) = 1$ and $H = \ker f_j$.

Assume also that $\sum_j \{T_a^j\}_{a\in A_j}$ is a family of continuous linear sequence of operators on X such that $\sum_j \acute{T}_a^j f_j = \sum_j f_j$ for each $a \in A_j$. Then the families $\mathcal{F}^j = \{zT_a^j : z \in \mathbb{K}, a \in A_j\}$ are universals if and only if the families $\mathcal{G}^j = \{R_a\}_{a\in A_j}$ of affine maps $R_a: H \to H$, $R_a x = (T_a^j u^j - u^j) + T_a^j x$ are universals on H. Moreover, $x \in X$ is universal for $\mathcal{F}^j$ if and only if $x = \lambda_j(u^j + w)$, where $\lambda_j \in \mathbb{K}\backslash\{0\}$ and w is universal for $\mathcal{G}^j$. Next, if $A_j = \mathbb{Z}_+$ and $T_a^j = \left(T_1^j\right)^a$ for every $a \in \mathbb{Z}_+$, then $R_a = R_1^a$ for every $a \in \mathbb{Z}_+$. Finally, if $A_j = \mathbb{R}_+$ and $\sum_j \{T_a^j\}_{a\in\mathbb{R}_+}$ is strongly continuous linear semi-group, then $\sum_j \{T_a^j\}_{a\in\mathbb{R}_+}$ is strongly continuous affine semi-group.

Proof. Since $T_a^j(H) \subseteq H$ for every a, vectors from H cannot be universal for $\mathcal{F}^j$. Obviously, they also do not have the form $\lambda_j(u^j + w)$ with $\lambda_j \in \mathbb{K}\backslash\{0\}$ and $w \in H$.

Let $x_0 \in X \backslash H$. Then $f_j(x_0) \neq 0$ and therefore $x = \frac{x_0}{f_j(x_0)} \in u^j + H$. Since $T_a^j(u^j + H) \subseteq u^j + H$ for every $a \in A_j$, $O = \{T_a^j x : a \in A_j\} \subseteq u^j + H$. It is straightforward to see that x_0 is universal for $\mathcal{F}^j$ if and only if O is dense in $u^j + H$. That is, x_0 is universal for $\mathcal{F}^j$ if and only if x is universal for the families $\{Q_a\}_{a\in A_j}$, where each $Q_a: u^j + H \to u^j + H$ is the restriction of T_a^j to the invariant subset $u^j + H$. Obviously, the translation map $\Phi: H \to u^j + H$, $\Phi(y) = u^j + y$ is a homeomorphism and $R_a = \Phi^{-1} Q_a \Phi$ for every $a \in A_j$. It follows that x_0 is universal for $\mathcal{F}^j$ if and only if $\Phi^{-1}x = x - u^j$ is universal for $\mathcal{G}^j$. Denoting $w = x - u^j$, see that the latter happens if and only if $x_0 = f_j(x_0)\left(u^j + w\right)$ with $w \in U(\mathcal{G}^j)$.

Since Q_a are the restrictions of T_a^j to the invariant subset $u^j + H$ and R_a are similar to Q_a with the similarity independent on a, $\{R_a\}$ inherits all the semi-group or continuity properties from $\{T_a^j\}$. The proof is complete.

Lemma 3.2. Let X be a topological vector space, $u^j \in X$, $f_j \in X'\backslash\{0\}$, and $H = \ker f_j$.

Then $T^j \in L(X)$ satisfying $\sum_j \acute{T^j} f_j = \sum_j f_j$ is supercyclic if and only if the map $R: H \to H$, $Rx = (T^j u^j - u^j) + T^j x$ is universal. Moreover, $x \in X$ is a supercyclic vector for T^j if and only if $x = \lambda_j(u^j + w)$, where $\lambda_j \in \mathbb{K}\backslash\{0\}$ and $w \in U(R)$.

Lemma 3.3. Let X be a topological vector space, $u^j \in X$, $f_j \in X'\backslash\{0\}$, $f_j(u^j) = 1$ and $H = \ker f_j$. Then a strongly continuous linear semi-group $\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on X satisfying $\sum_j T^{j\prime}_{(1+\varepsilon)}(\acute{T^j}_{(1+\varepsilon)}) f_j = \sum_j f_j$ for $(1+\varepsilon) \in \mathbb{R}_+$ is supercyclic if and only if the strongly continuous affine semi-group $\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on H defined by $R_{(1+\varepsilon)}x = (T^j_{(1+\varepsilon)}u^j - u^j) + T^j_{(1+\varepsilon)}x$ are universals. Moreover, $x \in X$ is a supercyclic vector for $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ if and only if $x = \lambda_j(u^j + w)$, where $\lambda_j \in \mathbb{K}\backslash\{0\}$ and $w \in \mathcal{U}(\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+})$.

4. Universal of Affine Semi-Groups

The following verification is the routine proof:

Lemma 4.1. Let X be a topological vector space, $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a collection of continuous affine maps on X, $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a collection of continuous linear sequence of operators on X and $(1+\varepsilon) \mapsto w_{(1+\varepsilon)}$ be a map from $\mathbb{R}_+$ to X such that $\sum_j T^j_{(1+\varepsilon)} x = w_{(1+\varepsilon)} + \sum_j S^j_{(1+\varepsilon)} x$ for every $(1+\varepsilon) \in \mathbb{R}_+$ and $x \in X$. Then $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are affine semi-group if and only if $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are linear semi-groups, $w_0 = 0$ and

$$w_{2(1+\varepsilon)} = w_{(1+\varepsilon)} + \sum_j S^j_{(1+\varepsilon)} w_{(1+\varepsilon)} \quad \text{for every } (1+\varepsilon) \in \mathbb{R}_+. \qquad (1)$$

Moreover, the semi-group $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is strongly continuous if and only if $\sum_j \{S^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are strongly continuous and the map $(1+\varepsilon) \mapsto w_{(1+\varepsilon)}$ is continuous. Finally, the semi-group $\sum_j \{T^j_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are jointly continuous if and only if

$\sum_j\left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are jointly continuous and the map $(1+\varepsilon)\longmapsto w_{(1+\varepsilon)}$ is continuous.

Lemma 4.2. Let X be a topological vectors space and $\sum_j\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal strongly continuous affine semi-group on X. Then $\sum_j\left(I-T^j_{(1+\varepsilon)}\right)(X)$ is dense in X for every $\varepsilon>-1$.

Proof. Assume the contrary. Then there is $\varepsilon>-1$ such that $Y_0\neq X$, where $Y_0=\overline{\sum_j\left(I-T^j_{(1+\varepsilon)}\right)(X)}$. Let Y be a translation of Y_0, containing $0: Y=Y_0-u^j_0$ with $u^j_0\in Y_0$. It is easy to see that, factoring out the closed linear subspace Y, arrive to the universal strongly continuous affine semi-group $\sum_j\left\{F^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on X/Y, where $F^j_{(1+\varepsilon)}(x+Y)=T^j_{(1+\varepsilon)}x+Y$ for every $(1+\varepsilon)\in\mathbb{R}_+$ and $x\in X$. By definition of Y, the linear part of $F^j_{(1+\varepsilon)}$ is I. Let $\beta+\varepsilon\in X/Y$ be a universal vector for $\sum_j\left\{F^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. By Lemma 4.1, there is a strongly continuous linear semi-group $\left\{G_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on X/Y and a continuous map $(1+\varepsilon)\longmapsto\gamma_{(1+\varepsilon)}$ from $\mathbb{R}_+$ to X/Y such that $\gamma_0=0$, $F^j_{(1+\varepsilon)}\beta=G_{(1+\varepsilon)}\beta+\gamma_{(\varepsilon+1)}$ and $\gamma_{r+(1+\varepsilon)}=\gamma_r+G_r\gamma_{(1+\varepsilon)}=\gamma_{(1+\varepsilon)}+G_{(1+\varepsilon)}\gamma_r$ for every $\beta\in X/Y$ and $r,(1+\varepsilon)\in\mathbb{R}_+$. Using these relations and the equality $G_{(\varepsilon+1)}=I$, obtain that $F^j_{(1+\varepsilon)+n(1+\varepsilon)}(\beta+\varepsilon)=F^j_{(1+\varepsilon)}(\beta+\varepsilon)+n\gamma_{(1+\varepsilon)}$ for every $n\in\mathbb{Z}_+$ and $(\varepsilon+1)\in\mathbb{R}_+$. It follows that $\left\{\sum_j F^j_{(1+\varepsilon)}(\beta+\varepsilon):(1+\varepsilon)\in\mathbb{R}_+\right\}=K+\mathbb{Z}_+\gamma_{(1+\varepsilon)}$ where $K=\left\{\sum_j F^j_{(1+\varepsilon)}(\beta+\varepsilon):(1+\varepsilon)\in[0,(1+\varepsilon)]\right\}$. Since $(\beta+\varepsilon)$ is universal for $\sum_j\left\{F^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$, by the last display, $O=K+\mathbb{Z}_+\gamma_{(\varepsilon+1)}$ is dense in X/Y. Since O is closed as a sum of a compact set and a closed set, $O=X/Y$. On the other hand, O does not contain $-(1+\varepsilon)\gamma_{(1+\varepsilon)}$ for any sufficiently large $\varepsilon>-1$. This contradiction completes the proof.

Lemma 4.3. Let X be a topological vector space, $x\in X$, $\varepsilon>-1$ and $\sum_j\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal affine semi-group on X. Assume also that $\sum_j T^j_{(1+\varepsilon)}x=\sum_j S^j_{(1+\varepsilon)}x+w_{(1+\varepsilon)}$, where $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ strongly continuous linear semi-group on X and $(1+\varepsilon)\mapsto w_{(1+\varepsilon)}$ is a continuous map from $\mathbb{R}_+$ to X. Then $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is hypercyclic. Moreover, $\mathcal{U}(\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}) \cap (w_{(1+\varepsilon)} + \sum_j (I - \sum_j S^j_{(1+\varepsilon)})\,(X)) \neq \emptyset$ for every $\varepsilon > -1$.

Proof. Let $x \in \mathcal{U}(\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$ and fix $\varepsilon > -1$. By Lemma 4.2, $\sum_j (T^j_{(1+\varepsilon)} - I)\,(X)$ are dense in X. Hence $O = \{\sum_j (T^j_{(1+\varepsilon)} - I)T^j_{(1+\varepsilon)}\, x : (1+\varepsilon) \in \mathbb{R}_+\}$ are dense in X. Using the semi-group property of $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ and $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ together with (1), get

$\sum_j (T^j_{(1+\varepsilon)} - I)T^j_{(1+\varepsilon)}\, x = \sum_j S^j_{(1+\varepsilon)}\, S^j_{(1+\varepsilon)} x + \sum_j S^j_{(1+\varepsilon)}\, w_{(1+\varepsilon)} + w_{(1+\varepsilon)} - \sum_j S^j_{(1+\varepsilon)}\, x - w_{(1+\varepsilon)}$

$$= \sum_j S^j_{(1+\varepsilon)}\, S^j_{(1+\varepsilon)} x + \sum_j S^j_{(1+\varepsilon)} w_{(1+\varepsilon)} - \sum_j S^j_{(1+\varepsilon)}\, x = \sum_j S^j_{(1+\varepsilon)} \left(w_{(1+\varepsilon)} - (I - S^j_{(1+\varepsilon)})x\right)$$

for every $(1+\varepsilon) \in \mathbb{R}_+$. By the above display, O is exactly the St-orbit of $w_{(1+\varepsilon)} - \sum_j \left(I - S^j_{(1+\varepsilon)}\right) x$. Since O is dense in X, $w_{(1+\varepsilon)} - \sum_j (I - S^j_{(1+\varepsilon)})\, x \in w_{(1+\varepsilon)} + \sum_j (I - S^j_{(1+\varepsilon)})\,(X)$ is hypercyclic vector for $\left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ and therefore $\mathcal{U}\sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right) \cap w_{(1+\varepsilon)} + \sum_j \left(I - S^j_{(1+\varepsilon)}\right)(X) \neq \emptyset$.

Lemma 4.4. Let X be a topological vector space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be an affine semi-group on X. Then for every $(1+\varepsilon)_1, \dots, (1+\varepsilon)_n \in \mathbb{R}_+$ and every $z_1, \dots, z_n \in \mathbb{K}$ satisfying $z_1 + \dots + z_n = 1$, the map $\sum_j S^j = z_1 \sum_j T^j_{(1+\varepsilon)_1} + \dots + z_n \sum_j T^j_{(1+\varepsilon)_n}$ commutes with every $\sum_j T^j_{(1+\varepsilon)}$.

Proof. It is easy to verify that for every affine maps $A_j : X \to X$ and every $x_1, \dots, x_n \in X$,
$\sum_j A_j(z_1 x_1 + \dots + z_n x_n) = z_1 \sum_j A_j x_1 + \dots + z_n \sum_j A_j x_n$ provided $z_j \in \mathbb{K}$ and $z_1 + \dots + z_n = 1$.

Let $(1+\varepsilon) \in \mathbb{R}_+$. By the above display,

$$\sum_j T^j_{(1+\varepsilon)} S^j = z_1 \sum_j T^j_{(1+\varepsilon)} \, T^j_{(1+\varepsilon)_1} x + \cdots + z_n \sum_j T^j_{(1+\varepsilon)} \, T^j_{(1+\varepsilon)_n}.$$

Since $\sum_j T^j_{(1+\varepsilon)}$commute with each other, get

$$\sum_j T^j_{(1+\varepsilon)} S^j = z_1 \sum_j T^j_{(1+\varepsilon)_1} \, T^j_{(1+\varepsilon)} x + \cdots + z_n \sum_j T^j_{(1+\varepsilon)_n} \, T^j_{(1+\varepsilon)} = \sum_j S^j \, T^j_{(1+\varepsilon)}.$$

Lemma 4.5. Let X be a topological vector space, $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be universals strongly continuous affine semi-group on X and $x \in \mathcal{U} \sum_j \left(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}\right)$. Then $\Lambda(x) \subseteq \mathcal{U} \sum_j (\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$, where

$$\Lambda(x) = \{z_1 \textstyle\sum_j T^j_{(1+\varepsilon)_1} x + \cdots + z_n \sum_j T^j_{(1+\varepsilon)_n} x : n \in \mathbb{N}, (1+\varepsilon)_j \in \mathbb{R}_+, z_j \in \mathbb{K}, z_1 + \cdots + z_n = 1\}. \tag{2}$$

Proof. Let $n \in \mathbb{N}$, $(1+\varepsilon)_1, \ldots, (1+\varepsilon)_n \in \mathbb{R}_+$, $z_1, \ldots, z_n \in \mathbb{K}$ and $z_1 + \cdots + z_n = 1$. Have to show that $x \in \mathcal{U} \sum_j (\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$,where $A_j = z_1 \sum_j T^j_{(1+\varepsilon)_1} + \cdots + z_n \sum_j T^j_{(1+\varepsilon)_n}$. By Lemma 4.4, A commutes with all $T^j_{(1+\varepsilon)}$. Since $x \in \mathcal{U} \sum_j (\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$, it suffices to verify that $A_j(X)$ are dense in X. By Lemma 4.1, write $\sum_j T^j_{(1+\varepsilon)} y = \sum_j S^j_{(1+\varepsilon)} y + w_{(1+\varepsilon)}$for every $y \in X$, where $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are strongly continuous linear semi-group on X and $(1+\varepsilon) \longmapsto w_{(1+\varepsilon)}$ is a continuous map from $\mathbb{R}_+$ to X. By Lemma 4.3, $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are hypercyclic, every non-trivial linear combination of members of a hypercyclic strongly continuous linear semi-group has dense range. Thus $B_j = z_1 \sum_j S^j_{(1+\varepsilon)_1} + \cdots + z_n \sum_j S^j_{(1+\varepsilon)_n}$ has dense range. Since $A_j(X)$ are translation of $B_j(X)$, $A_j(X)$ is also dense in X, which completes the proof.

Proof of Theorem 1.3. Let X be a topological vectors space and $\sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a universal jointly continuous affine semi-group on X. Lemmas 4.1 and 4.3 provide a hypercyclic jointly continuous linear semi-groups on X. By Theorem A, a hypercyclic continuous linear operator is exist on X. Since no such thing exists on a finite dimensional topological vectors space [7], X is infinite dimensional.

Condition (1) of Proposition 1.1 is satisfied since any compact subspaces of an infinite dimensional topological vector spaces are nowhere dense [4]. Let $x \in \mathcal{U}\ \sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$. By Lemma 4.5, the set $\Lambda(x)$ defined in (4.2) entirely of universal vectors for $\sum_j\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. Clearly, $\{\sum_j T^j_{(1+\varepsilon)}\, x : (1+\varepsilon) \in \mathbb{R}_+\} \subseteq \Lambda(x)$. By its definition, $\Lambda(x)$ is an affine subspace of X. $\Lambda(x)$ satisfies all requirements for the set $Y_{(1+\varepsilon)}, x$ (for every $\varepsilon > -1$) according to condition (2) in Proposition 1.1 and every affine subspace of a topological vectors space is connected and simply connected, By Proposition 1.1, $\mathcal{U}\sum_j(T^j_{(1+\varepsilon)}) = \mathcal{U}\sum_j(\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+})$ for every $\varepsilon > -1$, as required.

5. PROOF OF THEOREM 1.2

Let X be a complex topological vector space and $\sum_j\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be a supercyclic jointly continuous linear semi-group on X *in* [8]. Have to prove that all $\sum_j T^j_{(1+\varepsilon)}$ with $\varepsilon > -1$ is supercyclic and the sets of supercyclic vectors of $T^j_{(1+\varepsilon)}$ coincides with the set of supercyclic vectors of $\sum_j\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. If $T^j_{(1+\varepsilon)} - \lambda_j I$ has dense range for every $\varepsilon > -1$ and every $\lambda_j \in \mathbb{C}$, then Proposition C provides the required result. Otherwise, by Proposition 2.1, there is a closed hyperplane H in X invariant for all $T^j_{(1+\varepsilon)}$. By Lemma 2.6, there are $f_j \in X$ and $(\beta+\varepsilon) \in \mathbb{C}$ such that $H = \ker f_j$ and $\sum_j e^{(1+\varepsilon)(\beta+\varepsilon)}\big(\acute{T}^{j}{}_{(1+\varepsilon)}\big)f_j = f_j$ for every $(1+\varepsilon) \in \mathbb{R}_+$. Clearly $\left\{e^{(1+\varepsilon)(\beta+\varepsilon)}\sum_j T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is a jointly continuous supercyclic linear semi-group on X with the same sets S^j of supercyclic vectors as the original semi-group $\sum_j\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$. Fix $u^j \in X$ satisfying $f_j(u^j) = 1$. Now fix $\varepsilon > -1$ and $v^j \in S^j$. Have to show that v^j is supercyclic for $\sum_j T^j_{(1+\varepsilon)}$. By Lemma 3.3, applied to the semi-group $\left\{e^{(1+\varepsilon)(\beta+\varepsilon)} \sum_j T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$, write $v^j = \lambda_j(u^j + y)$, where $\lambda_j \in \mathbb{K}\backslash\{0\}$ and y is a universal vector for the jointly continuous affine semi-group $\{R_{(1+\varepsilon)}\}_{(1+\varepsilon)\in\mathbb{R}_+}$ on H defined by the formula $R_{(1+\varepsilon)}x = w_{(1+\varepsilon)} + e^{(1+\varepsilon)(\beta+\varepsilon)}T^j_{(1+\varepsilon)}x$ with $w_{(1+\varepsilon)} = (e^{(1+\varepsilon)(\beta+\varepsilon)} \sum_j (T^j_{(1+\varepsilon)} - I)u^j$. By Theorem 1.3, y is universal for $R_{(1+\varepsilon)}$. So, $v^j = \lambda_j(u^j + y)$ is a supercyclic for $e^{(1+\varepsilon)(\beta+\varepsilon)} \sum_j T^j_{(1+\varepsilon)}$ (by Lemma 3.2 *)* and therefore v^j is a supercyclic vector for $T^j_{(1+\varepsilon)}$. The proof is complete.

6. REMARKS

The following example shows that the hypercyclicity of the underlying linear semi-group is not implies universality of a strongly continuous affine semi-group see e.g.[4].

Example 6.1. Consider the backward weighted shift $T^j \in L(\ell_2)$ with the weight sequence $\{e^{-2n}\}_{n\in\mathbb{N}}$. That is, $T^j e_0 = 0$ and $T^j e_n = e^{-2n}e_{n-1}$ for $n \in \mathbb{N}$, where $\{e^n\}_{n\in\mathbb{Z}_+}$ is the standard basis of ℓ_2. Then the jointly continuous linear semi-groups $\sum_j\left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ with $\sum_j S^j_{(1+\varepsilon)} = e^{(1+\varepsilon)\ln\sum_j(I+T^j)}$ are hypercyclic. Moreover, there exists a continuous map $(1+\varepsilon) \mapsto w_{(1+\varepsilon)}$ from $\mathbb{R}_+$ to ℓ_2 such that $\sum_j\left\{T^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are jointly continuous non-universal affine semi-group, where $\sum_j T^j_{(1+\varepsilon)}x = w_{(1+\varepsilon)} + \sum_j S^j_{(1+\varepsilon)}x$ for $x \in \ell_2$.

Proof. Since T^j, being compacts weighted backward shift, is quasinilpotent, the sequence of operators $\ln(I + T^j)$ are well defined and bounded and $\sum_j\left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are jointly continuous linear semi-group. Moreover, $S^j_1 = I + T^j$ are hypercyclic [9] as a sum of the identity sequence of operators and a backward weighted shift. Hence $\sum_j\left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ are hypercyclic.

Let $u^j \in \ell_2$, $u_n^j = (n+1)^{-1}$ for $n \in \mathbb{Z}_+$. For each $(1+\varepsilon) \in \mathbb{R}_+$, let $w_{(1+\varepsilon)} = v_{(1+\varepsilon)}^j(T^j)u^j$, where $v_{(1+\varepsilon)}^j(z) = \sum_{n=1}^{\infty} \frac{(1+\varepsilon)\varepsilon\ldots(\varepsilon-n+2)}{n!} z^{n-1}$. Since T^j are quasinilpotents, $v_{(1+\varepsilon)}^j(T^j)$ are well defined bounded linear sequence of operators and the map $(1+\varepsilon) \mapsto v_{(1+\varepsilon)}^j(T^j)$ are sequence of operators-norm continuous. Hence $(1+\varepsilon) \mapsto w_{(1+\varepsilon)}$ is continuous as a map from $\mathbb{R}_+$ to ℓ_2. It is easy to verify that $w_0 = 0$, $w_1 = u^j$ and $w_{2(1+\varepsilon)} = S_{(1+\varepsilon)}^j w_{(1+\varepsilon)} + w_{(1+\varepsilon)}$ for every $\varepsilon \geq -1$. By Lemma 4.1, $\sum_j \left\{T_{(1+\varepsilon)}^j\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is a jointly continuous affine semi-group, where $T_{(1+\varepsilon)}^j x = w_{(1+\varepsilon)} + S_{(1+\varepsilon)}^j x$. It remains to show that $\sum_j \left\{T_{(1+\varepsilon)}^j\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ is non-universal. Assume the contrary. Since $w_1 = u^j$ and $S_1^j = I + T^j$, Lemma 4.3 implies that the coset $\sum_j \left(u^j + T^j(\ell_2)\right)$ (must contain a hypercyclic vector for $I + T^j$. This however is not the case as shown in [9].

Recall that a topological space X is called a Baire space if the intersection of any countable collection of dense open subsets of X is dense in X.

Remark 6.2. Let X be a topological vector space and $S^j \in L(X)$ be hypercyclic. If $u^j \in (I - S^j)(X)$, then the affine map $\sum_j T^j x = \sum_j (u^j + S^j)x$ is universal. Indeed, let $w \in X$ be such that $u^j = w - S^j w$. It is easy to show that $\sum_j (T^j)^n x = w + \sum_j (S^j)^n (x - w)$ for every $x \in X$ and $n \in \mathbb{N}$. Thus x is universal for T^j if and only if $x - w$ is universal for S^j.

Since X is separable metrizable and Baire, so a standard Baire category type argument shows that the set of $u^j \in X$ is a dense G_δ-subse for which the affine map $\sum_j T^j x = \sum_j (u^j + S^j x)$ is universal t. Example 6.1 shows that this set can differ from X.Recall that X is locally convex topological vectors space and barrelled then every closed convex balanced subsets B_j of X satisfying $X = \bigcup_{n=1}^{\infty} n(B_j)$ contain a neighborhood of 0. As have mentioned in the introduction, the joint continuity of a linear semi-group follows from the strong continuity if the underlying space X is an $\mathcal{F}^j$-space. The same is true for wider classes of topological vector spaces. For instance, it is sufficient for X to be a Baire topological vector space or a barreled locally convex topological vector space. Thus the following observation holds true.

Remark 6.3. The joint continuity condition in Theorems A, 1.2 and 1.3 can be substituted by the strong continuity, on condition that X is Baire or barrelled and locally convex.

For general topological vectors space however strong continuity of a linear semi-group does not imply joint continuity. Example 6.4 explains that if the joint continuity condition is changed by the strong continuity then the theorem A fails. Recall that the Fréchet space $L^2_{\text{loc}}(\mathbb{R}_+)$ consists of the scalar valued functions $\mathbb{R}_+$, square integrable on $[0,(1+\varepsilon)]$ for each $\varepsilon > -1$. The dual space $L^2_{\text{loc}}(\mathbb{R}_+)$ can be naturally interpreted as the space $L^2_{\text{fin}}(\mathbb{R}_+)$ of square integrable scalar valued functions $\mathbb{R}_+$ with bounded support. The duality between $L^2_{\text{loc}}(\mathbb{R}_+)andL^2_{\text{fin}}(\mathbb{R}_+)$ is provided by the natural dual pairing $\sum_j \langle f_j, g_j\rangle = \int_0^\infty \sum_j f_j(t) g_j(t)dt$. Obviously the linear semi-group $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ of backward shifts $\sum_j S^j_{(1+\varepsilon)} f_j(x) = \sum_j f_j(x+(1+\varepsilon))$ is strongly continuous and therefore jointly continuous on the Fréchet space $L^2_{\text{loc}}(\mathbb{R}_+)$. It follows that the same semi-group is strongly continuous on $L^2_{\sigma,\text{loc}}(\mathbb{R}_+)$ being $L^2_{\text{loc}}(\mathbb{R}_+)$ endowed with the weak topology.

Example 6.4. Let $X = L^2_{\sigma,\text{loc}}(\mathbb{R}_+)$ and $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ be the above strongly continuous semi-group on X. Then there are $f_j \in X$ hypercyclics for $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ such that f_j are non-hypercyclic for S^j_1.

Proof. Let H be the hyperplane in $L^2[0,1]$ consisting of the functions with zero Lebesgue integral. Fix a norm-dense countable subsets A_j of H. One can easily construct $f_j \in L^2_{\text{loc}}(\mathbb{R}_+)$ such that

(a) for every $n \in \mathbb{N}$, the function $(f_j)_n : [0,1] \to K$, $(f_j)_n(1+\varepsilon) = f_j(n+(1+\varepsilon))$ belongs to A_j;

(b) for every $n \in \mathbb{N}$ and $h_1, \dots, h_n \in A_j$, there is $m \in \mathbb{N}$ such that $h_j = (f_j)_{m+j}$ for $1 \leq j \leq$

n.

For $(1+\varepsilon) \in \mathbb{R}_+$, let $\chi_{(1+\varepsilon)} \in X' = L^2_{\text{fin}}(\mathbb{R}_+)$ be the indicator function of the interval $[(1+\varepsilon),(2+\varepsilon)]$: $\chi_{(1+\varepsilon)}(1+\varepsilon) = 1$ if $(1+\varepsilon) \le (1+\varepsilon)^2 + 1$ and $\chi_{(1+\varepsilon)}((1+\varepsilon)) = 0$ otherwise. By (a), $\left(S_1^j\right)^n f_j \in \ker \chi_0$ for every $n \in \mathbb{N}$ and therefore f_j are not hypercyclic vector for S_1^j. It remains to show that f_j are hypercyclic vector for $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ acting on X. Using (a) and (b), we see that the Fréchet space topology closure of the orbits $\{S^j_{(1+\varepsilon)} f_j : (1+\varepsilon) \in \mathbb{R}_+\}$ is exactly the sets

$$o = \bigcup_{(1+\varepsilon)\in[0,1]} \bigcap_{n\in\mathbb{Z}_+} \ker \chi_{(1+\varepsilon)+n} \, .$$

In order to show that f_j are hypercyclic for $\sum_j \left\{S^j_{(1+\varepsilon)}\right\}_{(1+\varepsilon)\in\mathbb{R}_+}$ acting on X, it suffices to verify that O is dense in $L^2_{\sigma,\text{loc}}(\mathbb{R}_+)$. Assume the contrary. Then there exists a weakly open sets W^j in $L^2_{\text{loc}}(\mathbb{R}_+)$, such that $W^j \cap O = \emptyset$. That means, there are no linearly dependent $\left(\varphi_j\right)_1, \dots, \left(\varphi_j\right)_m \in L^2_{\text{fin}}(\mathbb{R}_+)$ and $(1+\varepsilon)_1, \dots, (1+\varepsilon)_m \in \mathbb{K}$ such that

$$\max_{1\le j\le m} \left| \sum_j \left((1+\varepsilon)_j - \langle g, \varphi_j \rangle\right) \right| \ge 1 \quad \text{for all } g^j \in O.$$

Let $k \in \mathbb{N}$ be such that all φ_j vanishes on $[k,\infty)$. Pick any $0 < (1+\varepsilon)_0 < \cdots < (1+\varepsilon)_m < 1$. Note that for every $l \in \{0,\dots,m\}$, the restrictions of the functionals φ_j to $\bigcap_{n=0}^k \ker \chi_{(1+\varepsilon)_l+n}$. are not linearly independent. Indeed, otherwise can find $h_0 \in \bigcap_{n=0}^k \ker \chi_{(1+\varepsilon)_l+n}$ such that $\langle h_0, \varphi_j \rangle = (1+\varepsilon)_j$ for $1 \le j \le m$. It is easy to see that there is $h \in L^2_{\text{loc}}(\mathbb{R}_+)$ such that $h|_{[0,k]} = h_0|_{[0,k]}$, $h|_{[k+1,\infty]}$ and $\langle h, \chi_{(1+\varepsilon)_l+k-1} \rangle = \langle h, \chi_{(1+\varepsilon)_l+k} \rangle = 0$. Then $\langle h, \varphi_j \rangle = (1+\varepsilon)_j$ for $1 \le j \le m$ and $h \in \bigcap_{n=0}^\infty \ker \chi_{(1+\varepsilon)_l+n} \subseteq O$. We have arrived to a contradiction with the above display.

The fact that φ_j are not linearly independents on $\bigcap_{n=0}^k \ker \chi_{(1+\varepsilon)_l+n}$ implies that there is a non-zero $(g_j)_l \in \operatorname{span}\{(\varphi_j)_1, \dots, (\varphi_j)_m\} \cap \operatorname{span}\{\chi_{(1+\varepsilon)_l}, \dots, \chi_{(1+\varepsilon)_l+k}\}$. Since $\chi_{(1+\varepsilon)_l+r}$ are all linearly independent, $(g_j)_0, \dots, (g_j)_m$ are $m+1$ linearly independent vectors in the mdimensional space $\operatorname{span}\{(\varphi_j)_1, \dots, (\varphi_j)_m\}$. Completes the proof.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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