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PERTURBATIONS OF UNIFORMLY MEAN ERGODIC SEMIGROUPS

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Abstract. We consider uniformly mean ergodic semigroups and establish conditions on bounded operators such that the semigroup perturbed by this bounded operator is again uniformly mean ergodic. Within this discussion, the so-called (topological) generalized inverse comes up.

Keywords: evolution equations; operator semigroups; mean ergodicity; perturbations; closed range operators; generalized inverse.

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INTRODUCTION

The interest of the operator theoretical approach to ergodic theory lies foremost in the convergence of the time means of linear bounded operators on Banach spaces, i.e., for a given Banach space X and a bounded linear operator $T \in \mathscr{L}(X)$ one considers

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n-1} T^m.$$

The basic idea here is that the time mean equals the space mean within a dynamical system. This so-called ergodic hypothesis goes back to Boltzmann [1] which has been mathematically

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accentuated later on by Birkhoff [2] and von Neumann [3]. The most common technique is the linearization of dynamical systems by means of Koopman operators which actually gives the chance to pass from dynamics to linear operators, and back. The fundamental concepts of this operator theoretical approach can for example be found in [4]. The interest in such operators has its origins appearing in the context of statistical mechanics and also probability theory highly motivate these kind of operators. In contrary to discrete dynamical systems, one has the time continuous processes which give rise to strongly continuous semigroups of linear operators (or C_0-semigroups for short). By definition, a family of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X is called a C_0-semigroups if it satisfies the semigroup property and is strongly continuous with respect to the Banach space norm, i.e., $T(t+s) = T(t)T(s)$ for all $t,s \geq 0$, $T(0) = \mathrm{I}$ and $t \mapsto T(t)x$ is continuous for each $x \in X$. Each semigroup yields an unbounded operator $(A, \mathrm{D}(A))$, the so-called generator, defined by

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad \mathrm{D}(A) := \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

The Hille–Yosida generation theorem gives a characterization of unbounded operators which are generators of C_0-semigroups. An extensive discussion on strongly continuous operator semigroups, their various properties and applications can for example be found in the monographs by Engel and Nagel [5], Pazy [6] or Goldstein [7], just to mention a few. The time means of C_0-semigroups, which are important to study the long-term behaviour of dynamical process, corresponding to a C_0-semigroup are then of the form

$$C(r) := \frac{1}{r} \int_0^r T(s) \, \mathrm{d}s, \quad r > 0,$$

and one is interested in the limit of $C(r)$ for $r \to \infty$. We will restrict ourself to the special case when this limit exists with respect to the operator norm. Semigroups satisfying this condition are called *uniformly mean ergodic*. The main goal of this paper is to elaborate the question under which conditions uniform mean ergodicity of C_0-semigroups is stable under bounded perturbations, i.e., if $(A, \mathrm{D}(A))$ is the generator of a uniformly mean ergodic semigroup on a Banach space X and $B \in \mathscr{L}(X)$, under which conditions on B does the sum $(A+B, \mathrm{D}(A))$ generates again a uniformly mean ergodic semigroup. We will see that for bounded C_0-semigroups the uniformly mean ergodicity is equivalent to the closedness of the range of its generator $(A, \mathrm{D}(A))$.

Therefore, the questions of perturbations of uniformly mean ergodic semigroups consists both on the stability of boundedness and the closedness of the range under perturbations. The first part is for example treated in a more general context by Casarino and Piazzera [8, Thm. 2.1] whereas the closedness of the range of operators and perturbations get attention from many authors, cf. [9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

This article presents results on perturbations of uniformly mean ergodic semigroups on Banach spaces by combining results on perturbations of closed range operators as well as perturbations of operator semigroups. It is structured as follow: In the first section, we recall some basic definitions and results of uniformly mean ergodic semigroups. Moreover, we discuss bounded perturbations of C_0-semigroups and the stability of boundedness of semigroups under those perturbations. The last part of the first section consists of closed range operators and the minimum reduced modulus. Secondly, we discuss the main topic of this paper, the perturbations of uniformly mean ergodic semigroups. In particular, we discuss several types of perturbations. Last but not least we give an example for our results.

1. Preliminaries

1.1. Uniformly mean ergodic semigroups. In this section we recall the most important definitions and results about uniformly mean ergodic semigroups. We start with the definition of the so-called *Cesàro means* $(C(r))_{r>0}$ which we already mentioned in the introduction.

Definition 1.1. Let $(T(t))_{t\geq 0}$ be C_0-semigroup on a Banach space X. For each $r>0$ the operators

$$C(r) := \frac{1}{r}\int_0^r T(s)\,\mathrm{d}s, \tag{1.1}$$

defined pointwise by

$$C(r)x := \frac{1}{r}\int_0^r T(s)x\,\mathrm{d}s, \tag{1.2}$$

will be called the *Cesàro means* of the semigroup $(T(t))_{t\geq 0}$.

In what follows we define the main objects of this paper, the uniformly mean ergodic semigroups.

Definition 1.2. A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called *uniformly mean ergodic* if $\lim_{r\to\infty} C(r)$ exists with respect to the operator norm.

In what follows, we will always assume that the semigroups we are working with are bounded, i.e., there exists $M \geq 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. In this case, the Cesàro means, if they converge, actually converge to a projection P on X which is the so-called *mean ergodic projection*, i.e., $\|C(r) - P\| \to 0$ for $r \to \infty$. The following result summarizes some basic properties of the mean ergodic projection, cf. [5, Chapter V, Lemma 4.4] .

Lemma 1. *Let $(T(t))_{t\geq 0}$ be a bounded semigroup on a Banach space X with generator $(A, \mathrm{D}(A))$. If $(T(t))_{t\geq 0}$ is uniformly mean ergodic with mean ergodic projection P, then the following assertions hold true:*

(i) $P = T(t)P = PT(t) = P^2$ *for all* $t \geq 0$.
(ii) $X = \mathrm{Ran}(P) \oplus \mathrm{Ker}(P)$
(iii) $\mathrm{Ran}(P) = \mathrm{Ker}(A)$
(iv) $\mathrm{Ker}(P) = \overline{\mathrm{Ran}(A)}$

Since we assumed that the semigroup is bounded, one has the following charaterization of uniformly mean ergodicity of semigroups, cf. [19], [20, Thm. 2.30] or [5, Chapter V, Thm. 4.10].

Theorem 1.3. *For a bounded C_0-semigroup $(T(t))_{t\geq 0}$ with generator $(A, \mathrm{D}(A))$ on a Banach space X, the following assertions are equivalent.*

(a) $(T(t))_{t\geq 0}$ *is uniformly mean ergodic.*
(b) $\lim_{\lambda\searrow 0} \lambda R(\lambda, A)$ *exists in the operator norm.*
(c) $\mathrm{Ran(A)}$ *is closed.*
(d) $0 \in \rho(A)$ *or* 0 *is a first-order pole of the resolvent.*

We intentionally throw the spotlight on Banach spaces [5, Chapter V, Exam. 4.7] since every bounded C_0-semigroup on reflexiv spaces is mean ergodic, hence the situation in Hilbert spaces are not of interest here.

1.2. Bounded perturbations of C_0-semigroups. The general challenge of perturbation theory for C_0-semigroups is the following: Let $(A, \mathrm{D}(A))$ be a generator of a C_0-semigroup $(T(t))_{t\geq 0}$ and $(B, \mathrm{D}(B))$ another operator. Find conditions on $(B, \mathrm{D}(B))$ such that $A+B$, on a certain domain is the generator of a C_0-semigroup. For the case, that B is a bounded linear operator, i.e., $B \in \mathscr{L}(X)$, the following theorem holds, cf. [5, Chapter III, Thm. 1.3].

Theorem 1.4. *Let $(A, \mathrm{D}(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. If $B \in \mathscr{L}(X)$, then $(A+B, \mathrm{D}(A))$ generates a C_0-semigroup.*

In other words, Theorem 1.4 says that $(A+B, \mathrm{D}(A))$ always generates a strongly continuous semigroup as long as $(A, \mathrm{D}(A))$ does and B is bounded. There are also other types of semigroup perturbations, e.g., Miyadera–Voigt type perturbations [21, 22], Desch–Schappacher perturbations [23] and Staffans–Weiss perturbations [24, 25, 26], just to mention a few. However, these kind of perturbations are beyond the scope of this paper and will not be treated in more detail.

With regard to Theorem 1.3, the first question that natural arises in the context of perturbations of uniformly mean ergodic semigroups if whether a the boundedness of semigroups is stable under bounded perturbations, i.e., if $(A, \mathrm{D}(A))$ generates a bounded semigroup $(T(t))_{t\geq 0}$ and $B \in \mathscr{L}(X)$ is the semigroup generated by $(A+B, \mathrm{D}(A))$ still bounded. The affirmative answer is given by an adaption of the work of Casarino and Piazzera [8, Thm. 2.1] under an additional assumption. For that, call to mind that for $t_0 > 0$ the operator-valued function space $\mathscr{X}_{t_0} := \mathrm{C}([0,t_0], \mathscr{L}_{\mathrm{s}}(X))$ of all continuous functions from $[0,t_0]$ to $\mathscr{L}_{\mathrm{s}}(X)$, i.e., $F \in \mathscr{X}_{t_0}$ if and only if $F(t) \in \mathscr{L}(X)$ for each $t \in [0,t_0]$ and $t \mapsto F(t)x$ is continuous for all $x \in X$, is a Banach space for the norm

$$\|F\| := \sup_{t\in[0,t_0]} \|F(t)\|, \quad F \in \mathscr{X}_{t_0}.$$

For a C_0-semigroup $(T(t))_{t\geq 0}$ and $B \in \mathscr{L}(X)$ the associated *Volterra operator* defined by

$$V_B F(t)x := \int_0^t T(t-s)BF(s)x \,\mathrm{d}s, \quad t \in [0,t_0], \; F \in \mathscr{X}_{t_0}, \; x \in X,$$

is a bounded operator on $\mathscr{X}_{t_0}$, cf. [5, Chapter III, Lemma 1.9]. With help of the Volterra operator we can prove the following theorem.

Theorem 1.5. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup with generator $(A, \mathrm{D}(A))$, i.e., there exists $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. Let $B \in \mathscr{L}(X)$ such that that there exists $q \in (0,1)$ such that*

$$\int_0^t \|T(s)Bx\| \,\mathrm{d}s \leq q\|x\|, \tag{1.3}$$

for all $t \geq 0$ and $x \in \mathrm{D}(A)$. Then $(A+B, \mathrm{D}(A))$ generates a bounded C_0-semigroup.

Proof. Recall from [5, Chapter III, Sect. 1] that the semigroup $(S(t))_{t\geq 0}$ generated by $(A+B, \mathrm{D}(A))$ can be expressed by

$$S(t) = \sum_{n=1}^{\infty} V^n T(t), \quad t \geq 0.$$

Observe that by (1.3) one obtains that

$$\|V_B F(t)x\| \leq \int_0^t \|T(t-s)BF(s)\| \,\mathrm{d}s \leq \|F\| \cdot \int_0^t \|T(s)B\| \,\mathrm{d}s < \|F\| \cdot q\|x\|,$$

which shows that $\|V_B\| \leq q < 1$. Therefore, we obtain that

$$\|S(t)\| \leq \sum_{n=0}^{\infty} \|V^n T(t)\| \leq M \sum_{n=0}^{\infty} q^n = \frac{M}{1-q} < \infty.$$

This shows, that $(S(t))_{t\geq 0}$ is indeed a bounded C_0-semigroup. □

Remark 1.6.
(i) It was remarked by Casariona and Piazzera, see [8, Rem. 2.3] that if $BT(t) = T(t)B$ for all $t \geq 0$ and B is surjective, then $(T(t))_{t\geq 0}$ is even exponentially stable, i.e., there exists $M \geq 1$ and $\omega > 0$ such that $\|T(t)\| \leq M\mathrm{e}^{-\omega t}$ for each $t \geq 0$.
(ii) Asymptotic properties which are preserved under bounded perturbations are extensive studied beginning with Phillips [27] and later on by Nagel and Piazzera [28]. Stability of asymptotic properties of operator semigroups as well as evolution families under unbounded perturbations is topic of several other papers, cf. [8, 29, 30, 31, 32].

1.3. Closed range operators and the reduced minimum modulus. By Theorem 1.3 the question of uniformly mean ergodicity of (bounded) semigroups is equivalent to the question of the closed range. Closed range operators has for example been studied by Friedman [12], Feldman [13] and Kato [9]. In particular, we have to answer the question under which conditions the range of $(A+B, \mathrm{D}(A))$ is closed if its of $(A, \mathrm{D}(A))$ is. Of course, one could think about using the closed range theorem, however, this theorem can not be applied easy in this context. For that reason, we discuss the notion of the so-called *reduced minimum modulus* $\gamma(A)$ of an operator $(A, \mathrm{D}(A))$ which definition goes back to Kato, cf. [9, 33].

Definition 1.7. Let $(A, \mathrm{D}(A))$ be a linear operator on a Banach space. The *reduced minimum modulus* $\gamma(A)$ is defined by

$$(1.4)\quad \gamma(A) := \inf_{x \notin \mathrm{Ker}(A)} \frac{\|Ax\|}{\mathrm{dist}(x, \mathrm{Ker}(A))} = \sup\{\alpha \geq 0 : \ \|Ax\| \geq \alpha \cdot \mathrm{dist}(x, \mathrm{Ker}(A)),\ x \in \mathrm{D}(A)\}.$$

The reduced minimum modulus attracted interest both in Hilbert spaces, see for example [34, 35], as well as Banach space, e.g., [36, Sect. 6] or [33, Chapter IV, Sect. 5]. It is important for the study of closed range operators, since closed range operators correspond to these operators having strict positive reduced minimum modulus, cf. [33, Thm. 5.2]

Theorem 1.8. *Let $(A, \mathrm{D}(A))$ be a closed operator on a Banach space X. Then $(A, \mathrm{D}(A))$ has closed range if and only if $\gamma(A) > 0$.*

Therefore, the question whether $(A + B, \mathrm{D}(A))$ has closed range translates into the showing that $\gamma(A + B) > 0$. The following quantities in the context of closed range operators also come up in the work of Kato [9].

Definition 1.9. Let $(A, \mathrm{D}(A))$ be a linear operator on a Banach space X. Then the numbers

$$\alpha(A) := \dim(\mathrm{Ker}(A)) \quad \text{and} \quad \beta(A) := \mathrm{codim}(\mathrm{Ran}(A)) = \dim(X/\mathrm{Ran}(A)),$$

are called *nullity* and *deficiency* of $(A, \mathrm{D}(A))$, respectively.

It can, for example, been shown that a closed operator $(A, \mathrm{D}(A))$ with $\beta(A) < \infty$ has immediately closed range, cf. [9, Lemma 332].

2. Bounded perturbations of uniformly mean ergodic semigroups

2.1. Perturbations by means of compact resolvents and quasi-compactness. First of all, we deal with perturbation results which follows by additional assumptions on the operator $(A, \mathrm{D}(A))$ or $B \in \mathscr{L}(X)$. Within this section we do not need the assumption that the range of $(A, \mathrm{D}(A))$ is closed. We recall the following definitions.

Definition 2.1. Let X be a Banach space and $\mathscr{K}(X)$ the set of compact linear operators on X. A C_0-semigroup $(T(t))_{t\geq 0}$ on X is called *quasi-compact* if

$$\lim_{n\to\infty}\inf_{K\in\mathscr{K}(X)}\|T(t)-K\|=0.$$

Definition 2.2. A linear operator $(A,\mathrm{D}(A))$ with $\rho(A)\neq\varnothing$ has *compact resolvent* if its resolvent $R(\lambda,A)$ is compact for one (and hence all) $\lambda\in\rho(A)$.

Equivalent conditions for quasi-compactness or compact resolvent are for example mentioned here, cf. [5, Chapter V, Prop. 3.5] and [5, Chapter II, Prop. 4.25]. As a matter of fact, both conditions introduced in Definition 2.1 and Definition 2.2 implies uniformly mean ergodicity, cf. [5, Chapter V, Cor. 4.11], which is our next result.

Lemma 2. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup with generator $(A,\mathrm{D}(A))$. If either $(T(t))_{t\geq 0}$ is quasi-compact or $(A,\mathrm{D}(A))$ has compact resolvent, then $(T(t))_{t\geq 0}$ is uniformly mean ergodic.*

By a combination of [5, Chapter III, Prop. 1.12] and [5, Chapter V, Prop. 3.6] one obtains the following result.

Proposition 2.3. *Let $(T(t))_{t\geq 0}$ be a bounded semigroup with generator $(A,\mathrm{D}(A))$. The following assertions hold true:*

(a) *If $(T(t))_{t\geq 0}$ is quasi-compact (and hence uniformly mean ergodic) and $B\in\mathscr{L}(X)$ is compact and satisfies* (1.3)*, then $(A+B,\mathrm{D}(A))$ generates a uniformly mean ergodic semigroup.*
(b) *If $(A,\mathrm{D}(A))$ has compact resolvent (and hence generates a uniformly mean ergodic semigroup) and $B\in\mathscr{L}(X)$ satisfies* (1.3)*, then $(A+B,\mathrm{D}(A))$ generates a uniformly mean ergodic semigroup.*

2.2. Perturbations by "small" operators.

Let us turn to perturbations of closed range operators which have for example been studied by Kato [9] or Gol'dman and Kračkovskiĭ [10]. Having Theorem 1.8 in mind, we can state our first perturbation theorem.

Proposition 2.4. *Let $(A,\mathrm{D}(A))$ be the generator of a bounded uniformly mean ergodic semigroup $(T(t))_{t\geq 0}$. Let $B\in\mathscr{L}(X)$ such that the following statements are true:*

(a) *B satisfies* (1.3),

(b) $\|B\| < \gamma(A)$,

(c) $\operatorname{Ker}(A) \subseteq \operatorname{Ker}(A+B)$.

Then $(A+B, \mathrm{D}(A))$ *generates a bounded uniformly mean ergodic semigroup.*

Proof. By condition (a) it is ensured that $(A+B, \mathrm{D}(A))$ generates again a bounded C_0-semigroup. Since $(A, \mathrm{D}(A))$ has closed range by Theorem 1.3, we notice that $\|Ax\| \geq \gamma(A) \cdot \operatorname{dist}(x, \operatorname{Ker}(A))$ for all $x \in \mathrm{D}(A)$. We then obtain for all $x \in \mathrm{D}(A+B) = \mathrm{D}(A)$ that

$$\begin{aligned} \|Ax + Bx\| &\geq |\|Ax\| - \|Bx\|| \\ &\geq |\gamma(A) \cdot \operatorname{dist}(x, \operatorname{Ker}(A)) - \|B\| \cdot \|x\|| \\ &\geq (\gamma(A) - \|B\|) \cdot \operatorname{dist}(x, \operatorname{Ker}(A+B)). \end{aligned}$$

By assumption (b) we obtain that $\gamma(A) - \|B\| > 0$ and hence by Theorem 1.8 that $\operatorname{Ran}(A+B)$ is closed. □

The following result also assumes that the norm of B is bounded by $\gamma(A)$, however, we do not longer assume that $\operatorname{Ker}(A) \subseteq \operatorname{Ker}(A+B)$. Neverthelesse, we need some assumptions on the mean ergodic projection coming from [9, Thm. 1], which actually stays in connection with Definition 1.9.

Proposition 2.5. *Let* $(A, \mathrm{D}(A))$ *be the generator of a bounded uniformly mean ergodic semigroup semigroup* $(T(t))_{t\geq 0}$ *with mean ergodic projection P such that either* $\dim(\operatorname{Ran}(P))$ *or* $\operatorname{codim}(\operatorname{Ker}(A))$ *is finite. If* $B \in \mathscr{L}(X)$ *satisfies* (1.3) *and* $\|B\| < \gamma(A)$, *then* $(A+B, \mathrm{D}(A))$ *generates a bounded uniformly mean ergodic semigroup on X.*

Proof. Firstly, $(A+B, \mathrm{D}(A))$ generates a bounded C_0-semigroup since (1.3) is satisfied. Furthermore, $(A, \mathrm{D}(A))$ has closed range and by [9, Thm. 1] also $(A+B, \mathrm{D}(A))$ has closed range. □

2.3. Perturbations by A-bounded operators.

Let us first consider operators $B \in \mathscr{L}(X)$ which norm can be estimated by the norm of A and $A+B$.

Proposition 2.6. *Let* $(A, \mathrm{D}(A))$ *be a generator of a bounded uniformly mean ergodic semigroup* $(T(t))_{t\geq 0}$ *on a Banach space X and let* $B \in \mathscr{L}(X)$ *such that condition* (1.3) *is satisfied and there* *exists $\lambda \in [0,1)$ and $\mu \in \mathbb{R}$ with*

$$\|Bx\| \le \lambda \|Ax\| + \mu \|(A+B)x\|, \quad x \in \mathrm{D}(A).$$

Then $\mu > 1$ and $(A+B, \mathrm{D}(A))$ generates a bounded uniformly mean ergodic semigroup.

Proof. Since B satisfies (1.3), the operator $(A+B, \mathrm{D}(A))$ generates a bounded C_0-semigroups. In view of Theorem 1.3 the goal is to prove that $\mathrm{Ran}(A+B)$ is closed. To do so, we first observe that

$$\|(A+B)x\| \ge \frac{1-\lambda}{1+\mu} \|Ax\|, \quad x \in \mathrm{D}(A),$$

by making use of the triangle inequality. From this one directly gets that

$$\gamma(A+B) \le \frac{1-\lambda}{1+\mu} \gamma(A).$$

By assumption $\mathrm{Ran}(A)$ is closed, hence $\gamma(A) > 0$ implying that $\gamma(A+B) > 0$, showing that $\mathrm{Ran}(A+B)$ is closed by an application of Theorem 1.8. □

The next result uses A-boundedness of $B \in \mathscr{L}(X)$, cf. [9, Thm. 1(a)].

Proposition 2.7. *Let $(A, \mathrm{D}(A))$ be the generator of a bounded uniformly mean ergodic semigroup semigroup $(T(t))_{t\ge 0}$ with mean ergodic projection P such that either* $\dim(\mathrm{Ran}(P))$ *or* $\mathrm{codim}(\mathrm{Ker}(A))$ *is finite. If $B \in \mathscr{L}(X)$ satisfies* (1.3) *and there exist $\mu, \nu > 0$ such that $\mu + \nu\gamma(A) < \gamma(A)$ and $\|Bx\| \le \mu \|x\| + \nu \|Ax\|$ for each $x \in \mathrm{D}(A)$, then $(A+B, \mathrm{D}(A))$ generates a bounded uniformly mean ergodic semigroup on X.*

2.4. Perturbations by strictly singular operators. Here, we discuss perturbations of uniformly mean ergodic semigroup by means of strictly singular operators. These operators are charaterized by the property that there exists no infinite-dimensional subspace $Y \subseteq X$ such that $Y \to B(Y)$ is an isomorphism, or equivalently, that whenever the restriction of B to a subspace $Y \subseteq X$ has a continuous inverse, Y is finite dimensional. More about these kind of operators can for example be found here [37, 38, 39, 40].

Definition 2.8. A linear bounded operator $B \in \mathscr{L}(X)$ on a Banach space X is called *strictly singular* if and only if the existence of $c > 0$ such that $\|Bx\| \ge c\|x\|$ for all $x \in Y \subseteq X$ implies $\dim(Y) < \infty$.

With this notion in combination with [9, Thm. 2] we obtain the following result.

Proposition 2.9. *Let* $(A,\mathrm{D}(A))$ *be the generator of a bounded uniformly mean ergodic semigroup semigroup* $(T(t))_{t\geq 0}$ *with mean ergodic projection* P *such that either* $\dim(\mathrm{Ran}(P))$ *is finite. If* $B\in\mathscr{L}(X)$ *is strictly singular and satisfies* (1.3)*, then* $(A+B,\mathrm{D}(A))$ *generates a bounded uniformly mean ergodic semigroup on* X.

Proof. By (1.3) the semigroup generated by $(A+B,\mathrm{D}(A))$ stays bounded. Since $\dim(\mathrm{Ran}(P))<\infty$ we conclude that $\alpha(A)<\infty$. By the strict singularity of $B\in\mathscr{L}(X)$ we conclude by [9, Thm. 2] the desired result. □

3. Examples

Within the discussion of the reduced minimum modulus, see Section 1.3, the concept of the generalized inverse occurs. In Hilbert spaces, these inverses are known to be the Moore–Penrose inverses. For a given operator $(A,\mathrm{D}(A))$ on a Banach space, the situation slightly changes since in general one needs to assume the existence of topological complements of $\mathrm{Ker}(A)$ and $\overline{\mathrm{Ran}(A)}$, cf. [41, Thm. 5.6].

Theorem 3.1. *Let* X *be a Banach space and* $(A,\mathrm{D}(A))$ *a densely defined linear operator. Assume that there exist topological complements of* $\mathrm{Ker}(A)$ *and* $\overline{\mathrm{Ran}(A)}$*, i.e., there exist subspaces* Y *and* Z *such that* $X=\mathrm{Ker}(A)\oplus V=\mathrm{Ran}(A)\oplus W$. *Let* P *be the projection onto* $\mathrm{Ker}(A)$ *and* Q *the projection onto* $\mathrm{Ran}(A)$. *Then* $(A,\mathrm{D}(A))$ *has an unique generalized inverse* $A^\dagger=A^\dagger_{P,Q}$ *which satisfies the following set of conditions*

$$\begin{cases} X=\mathrm{Ran}(A)\oplus\mathrm{Ker}(Q)=\mathrm{Ran}(A)\oplus W,\\ AA^\dagger A=A \text{ on } \mathrm{D}(A),\\ A^\dagger AA^\dagger=A^\dagger \text{ on } \mathrm{D}(A^\dagger),\\ A^\dagger A=(\mathrm{I}-P)_{|\mathrm{D}(A)},\\ AA^\dagger=Q_{|\mathrm{D}(A^\dagger)}.\end{cases}\tag{3.1}$$

Moreover, one has that $A^\dagger\in\mathscr{L}(X)$ *if and only if* $\mathrm{Ran}(A)$ *is closed.*

Due to Theorem 3.1 the (unique) existence of a generalized inverse is in general linked to the existence of topological complements. However, by Lemma 1 this is not a problem for generators of uniformly mean ergodic semigroups. In addition, the generalized inverse of uniformly mean ergodic semigroup generators are always bounded by a combination of Theorem 1.3 and Theorem 3.1. The connection to the reduced minimum modulus is that if $(A, \mathrm{D}(A))$ has closed range, i.e., $\gamma(A) > 0$, then

$$\left\|A^{\dagger}\right\| = \frac{1}{\gamma(A)}.$$

Example 1. Consider an exponentially stable semigroup $(T(t))_{t\geq 0}$ on a Banach space X with generator $(A, \mathrm{D}(A))$. Let $M \geq 1$ and $\omega > 0$ such that $\|T(t)\| \leq M\mathrm{e}^{-\omega t}$ for all $t \geq 0$. It is not hard to see that in this case $\|C(r)\| \to 0$ for $r \to \infty$ meaning that $(T(t))_{t\geq 0}$ is actually uniformly mean ergodic with mean ergodic projection $P = 0$. By Lemma 1 we obtain that therefore $\mathrm{Ker}(A) = \{0\}$ and $\mathrm{Ran}(A) = X$. By the Hille–Yosida theorem one has $0 \in \rho(A)$ which implies that $(A, \mathrm{D}(A))$ is invertible and in that case $A^{-1} = A^{\dagger}$. Following [8, Exa. 2.4] every bounded operator satisfying $\|B\| < \min\left\{\frac{\omega}{M}, \frac{1}{\|A^{-1}\|}\right\} = \min\left\{\frac{\omega}{M}, \gamma(A)\right\}$ automatically satisfies the conditions of Proposition 2.4 or Proposition 2.5 which means that $(A + B, \mathrm{D}(A))$ generates a bounded uniformly mean ergodic semigroup.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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