Available online at http://scik.org
Math. Finance Lett. 2015, 2015:9
ISSN: 2051-2929

SOLUTIONS TO SOME PORTFOLIO OPTIMIZATION PROBLEMS WITH STOCHASTIC INCOME AND CONSUMPTION

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Abstract. We solve an optimal portfolio choice problem for an investor with either power or log utility over terminal wealth in close form, facing imperfectly hedgeable stochastic income. The returns on the income and the stock are imperfectly correlated, therefore the market is incomplete. We describe how an investor accommodates or adjusts the Merton portfolio of the stock and risk-free asset through an interpolating hedging demand, in reaction to the stochastic income. The solutions to the investor thrilling problem of seeking the optimal portfolio are formulated and worked out using the stochastic control theory. The Bellman principle of dynamic optimality is utilized through the Hamilton-Jacobi-Bellman (HJB) partial differential equation. We apply the results to some unconstrained portfolio optimization problem with power and log utility functions which lead to four propositions as the main results. All the two models discussed shows that, there is an inverse relation between the risk and the value of Merton's investment strategy.

Keywords: Stochastic control; Dynamic programming; Utility functions.

2010 AMS Subject Classification: 93E20, 49L20, 91B16.

1. Introduction

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Received May 14, 2015

Portfolio Theory is a theory of investment which attempts to maximize portfolio expected return for a given amount of portfolio risk, or equivalently minimize risk for a given level of expected return, by carefully choosing the right proportions of various assets or investments. The theory was first discovered and developed by Hary Markowitz in 1950s. Portfolio optimization is an optimal investment strategy. That is, is a systematic plan to efficiently allocate investable assets among investment choices such as stocks, bonds, real estate and commodities for the purpose of acquiring optimal returns. An investor investing in bonds and stocks need maximum utility of her wealth. Investors focused on assessing the risks and returns of each individual securities in the construction for their portfolios need to identify those securities or assets that offered the best opportunities for gain with least risk and then construct portfolio from them. Modern Portfolio Theory (MPT) concentrates on risk at least as much as the returns [2]. In addition, MPT could be described as risk management rather than return management.

Harry Markowitz introduced the portfolio selection theory in 1952 with his *Mean-Variance* analysis, that aims to minimize the risk (modelled by the variance) under a constraint on the expected gain of the portfolio. The main problem is to find the best way to invest in a set of assets. Later, Robert Merton introduced stochastic control in 1969 and 1971, [7]. He discussed explicit solutions to the optimal portfolio problem in a 2-dimensional market, that is, with risky and risk-free as investment alternatives. The price of the risky asset (e.g. stock) follows a geometric Brownian motion. The investor wants to maximize her terminal wealth under specified utility function e.g. power, log-utility function, etc. Two main tools to be used are the dynamic programming principle, useful to solve problems from dynamic optimization and Hamilton-Jacobi-Bellman (HJB). The HJB equation was pioneered by Richard Bellman, [1] from the dynamic programming principle in continuous time and it generalizes the works of William Hamilton and Carl Gustav Jacobi in Classical Mechanics. Similar works can also be seen in [3, 5, 6].

In this paper we investigate Merton's classical portfolio optimization problem with consumption strategy and stochastic income. The work of this paper is much based on the works of [4, 8]. On the work of Henderson, he discussed the explicit solutions to the optimal portfolio choice problem for an investor with negative exponential utility over terminal wealth facing imperfectly hedgeable stochastic income without consumption strategy. The returns on income and the stock are imperfectly correlated, so the market is incomplete. [8] also explicitly solved the problem of pricing in an incomplete market, along with dynamic portfolio optimization problem which can be formulated using stochastic control theory. Both [4, 8] applied Hamilton-Jacobi-Bellman partial differential equation to establish their model results. In similar fashion, our approach to solve the stochastic optimization problem goes via the dynamic programming method and the associated Hamilton-Jacobi-Bellman (HJB) equation. We confine our results to complete markets.

2. Model description

We assume that we are in a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where Ω is a set of random events, $\mathscr{F}$ is a σ-algebra on Ω, $\mathbb{P}$ a probability measure on the measurable space $(\Omega, \mathscr{F})$ and $\mathbb{F}$ is the filtration. We then define the market as an $\mathscr{F}_t$-adapted 2-dimensional Itô process $M_t = (N_t, S_t)$ at time $t \in [0, T]$ where the dynamics of a riskless asset at time t is modeled by the equation:

$$dN_t = rN_t dt, \ (N_0 = 1) \tag{2.1}$$

where r is the interest rate and the dynamics of the price of the risky asset S_t at

any time t is modeled by the equation:

$$dS_t = S_t(\mu dt + \sigma dB_t), (S_0 = s_0), \tag{2.2}$$

where μ is the drift parameter or the appreciation rate, σ is the volatility, and B_t a standard Brownian motion. As in [4], we assume that the investor also receives income over time, the income rate at time t is $\alpha(X_t, t)$ where X_t is the state variable such that

$$dX_t = \phi(X_t, t)dt + \psi(X_t, t)dY_t. \tag{2.3}$$

The correlation between dB_t and dY_t is ρdt, where $\rho \in [-1, 1]$. Y_t can also be written in the form $Y_t = \rho B_t + \sqrt{1-\rho^2} Z_t$ where Z_t is a standard Brownian motion independent of B_t. We assume that $\phi(X_t)$ and $\psi(X_t)$ are continuous and satisfy Lipschitz and growth conditions in X to ensure a unique solution. In the case where $|\rho| < 1$, the presence of the second motion Z_t means that the income can not be perfectly be hedged via the stock S_t and the market the investor is faced with is incomplete. The wealth, W_t is generated by an investor allocating his current wealth accordingly (in the following fashion: By letting π_t to be the fraction of cash that is invested in the risky asset, and the remainder, $1-\pi_t$ to be the fraction of the current wealth invested in the risk-free asset (bond) at time t), and by the inflow of the stochastic income $\alpha(X_t,t)$. Let c_t be consumption process.

2.1. Problem statement

This paper investigates an investor's problem of seeking the optimal portfolio $u = (\pi_t, c_t)$ that would definitely maximize his utility of terminal wealth in the presence of the consumption $c_t \geq 0$, and the non-negative and continues stochastic income $\alpha(X_t,t)$ in a given market $(M_t)_{t\in[0,T]}$ in a time horizon T. These would also be based on a rational investor's preference (utility). We discuss the power, and log utilities which are incorporated with the stochastic income for maximizing the wealth of the trader, hence extending the work of [4] who dwelt on negative utility function.

2.2. Optimal portfolio

We consider the income rate $\alpha(X_t,t)$ as an Itô process. Apply Itô's formula to obtain the dynamics for income rate $\alpha(X_t,t)$ as follows

$$
\begin{aligned}
d\alpha(x,t) &= \frac{\partial\alpha(x,t)}{\partial t}dt + \frac{\partial\alpha(x,t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2\alpha(x,t)}{\partial x^2}(dX_t)^2 \\
&= \frac{\partial\alpha(x,t)}{\partial t}dt + \frac{\partial\alpha(x,t)}{\partial x}\phi(X_t,t)dt + \frac{\partial\alpha(x,t)}{\partial x}\psi(X_t,t)dY_t \\
&+ \frac{1}{2}\frac{\partial^2\alpha(x,t)}{\partial x^2}\psi^2(X_t,t)dt,
\end{aligned}
$$

where the following rules are taken into consideration;

$dY_t \cdot dt = dt \cdot dY_t = dt \cdot dt = 0$ and $dY_t \cdot dY_t = dt$,

$$d\alpha(x,t) = \left(\frac{\partial\alpha(x,t)}{\partial t} + \frac{\partial\alpha(x,t)}{\partial x}\phi(X_t,t) + \frac{1}{2}\frac{\partial^2\alpha(x,t)}{\partial x^2}\psi^2(X_t,t)\right)dt$$
$$+\frac{\partial\alpha(x,t)}{\partial x}\psi(X_t,t)dY_t,$$

or in short we have

$$d\alpha(X_t,t) = (\dot{\alpha} + \alpha_x\phi(X_t,t) + \frac{1}{2}\alpha_{xx}\psi^2(X_t,t))dt + \alpha_x\psi(X_t,t)dY.$$

The dynamics of wealth process comes out from the following computation: Consider 2-dimensional processes, market $(M_t)_{t\in[0,T]}$ and an adapted trading strategy $\theta = (\Theta_t^0, \Theta_t)$. The consumption $c_t \geq 0$ and the stochastic income rate $\alpha(X_t,t)$ is non-negative and continuous. We then define the corresponding continuous and adapted wealth process with respect to the self financing trading strategy θ as

$$dW_t = \Theta_t^0 dN_t + \Theta_t dS_t - c_t dt + \alpha(X_t,t)dt \tag{2.4}$$

or

$$W_t = \int_0^t \Theta_\varsigma^0 dN_\varsigma + \int_0^t \Theta_\varsigma dS_\varsigma + \int_0^t (\alpha(X_\varsigma,\varsigma) - c_\varsigma)d\varsigma.$$

The above equation (2.4), shows that the wealth process of an investor is found by the sum of the dynamics of the stock, riskless asset and income rate deducting the consumption at time t in a finite horizon.

Suppose π_t and $1-\pi_t$ are fractions of the current wealth that an investor has, and find it appropriate to invest in risky asset (stock) and riskless asset at time t respectively. Thus, from (2.4) we have

$$dW_t = r\Theta_t^0 N_t dt + \Theta_t S_t(\mu dt + \sigma dB_t) + (\alpha(X_t,t) - c_t)dt$$

of which implies that

$$dW_t = r(1-\pi_t)W_t dt + \pi_t W_t(\mu dt + \sigma dB_t) + (\alpha(X_t,t) - c_t)dt.$$

This is possible since both the fractions of the cash held at each time satisfies the following: $\pi_t = \dfrac{\Theta_t S_t}{W_t}$ invested in the risk asset and $1-\pi_t = \dfrac{\Theta_t^0 N_t}{W_t}$ traded in the risk-free asset. Finally, the wealth process evolves as

$$dW_t = ((r + \pi_t(\mu - r))W_t - c_t + \alpha(X_t,t))dt + \pi_t\sigma W_t dB_t,\ W_0 = w_0. \tag{2.5}$$

The use of the function $\alpha(x,t)$ allows flexibility in modeling but also introduces some indeterminacy as there can be many characterizations of the same model.The investor receives the value of the state variable X_t itself over time by taking $\alpha(x,t) = x$.

Consider the problem of an investor with utility over terminal wealth which can be maximized by the selection of the investment strategy (π_t, c_t).

2.3. Reward functions

Suppose, $\nu : [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $\eta : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|\nu(t,w,c)| + |\eta(t,w)| \leq C(1+|w|^2)$$

for all $(t,w,u) \in [0,T] \times \mathbb{R}^n \times \mathbb{U}$, and $C > 0$ being a constant.

Definition 2.1. (The reward function) The reward function is the function defined on $[0,T] \times \mathbb{R}^n \times \mathbb{U}$ by

$$J(t,w,u) = \mathbb{E}_{t,w}\left[\int_t^T \nu(s,W_s,c_s)ds + \eta(T,W_T)\right], \tag{2.6}$$

where $\int_t^T \nu(s,W_s,c_s)ds < \infty$ is the running costs and $\eta(T,W_T) < \infty$ is the terminal benefits. The reward function evolves as the expected sum of the running costs and/or the terminal benefits. Thus, the reward function gives the expected utility for a given initial wealth $x > 0$.

2.4. Admissible control

Definition 2.2. The admissible set $\mathscr{U}(t,w) = \mathscr{U}$ is the subset of all possible controls $(u_s)_{s\in[t,T]}$ for which the reward function is well defined. The SDE equation (2.5) has unique solution and such that the function

$$\varphi : t \to \nu(t,W_t,c_t),\ is\ in\ L^1_{\mathscr{F}}(0,T;\mathbb{R}),$$

where $L^1_{\mathscr{F}}(0,T;\mathbb{R})$ is a set of all $\mathscr{F}$-measurable functions that are integrable with respect to t. We permit an admissible control to be adapted process to model the fact that the investor has a fair knowledge on the market, and should be such that the total consumption over the trading time is finite. Therefore, the integrant of the total wealth process with respect to the Brownian motion is a martingale.

2.5. Value functions and HJB equations

Definition 2.3. The value function of the control problem is the greatest possible payoff function defined as

$$V(t,w) = \sup_{u \in \mathscr{U}} J(t,w,u). \tag{2.7}$$

The intention is to describe the value function and find the maximizer u^* such that

$$V(t,w) = J(t,w,u^*)$$

of which is the utility attained by the investor if the optimal policy is followed. Since (W_t, X_t) are jointly Markov for time t in $[0,T]$, the generator of the equation is given by

$$\begin{aligned}\mathscr{L} f(t,w) = \frac{\partial v}{\partial t} + (w[r+\pi(\mu-r)] - c + \alpha(x,t))\frac{\partial v}{\partial w} + \phi(x,t)\frac{\partial v}{\partial x} + \frac{(\pi w \sigma)^2}{2}\frac{\partial^2 v}{\partial w^2} + \\ \frac{\psi^2(x,t)}{2}\frac{\partial^2 v}{\partial x^2} + \psi(x,t)\rho\pi w\sigma\frac{\partial^2 v}{\partial w \partial x} + U(c).\end{aligned} \tag{2.8}$$

The derived utility function for the wealth function $v(t,w)$ satisfies the non-linear HJB equation;

$$\sup_{(\pi,c)\in\mathbb{R}\times R_+} [\mathscr{L} v(t,w)] = 0, \tag{2.9}$$

that is,

$$\begin{aligned}\sup_{(\pi,c)\in\mathbb{R}\times R_+} [\dot{v} + (w[r+\pi(\mu-r)] - c + \alpha(x,t))v_w + \phi(x,t)v_x + \frac{(\pi w\sigma)^2}{2}v_{ww} \\ + \frac{\psi^2(x,t)}{2}v_{xx} + \psi(x,t)\rho\pi w\sigma v_{wx} + U(c)] = 0,\end{aligned} \tag{2.10}$$

where $U(\cdot)$ is the utility function, $V_w = \partial V/\partial w$, $V_{ww} = \partial^2 V/\partial w^2$, $V_{wx} = V_{xw} = \partial^2 V/\partial w \partial x$, $V_x = \partial V/\partial x$ and $V_{xx} = \partial^2 V/\partial x^2$. Differentiating with respect to π and c gives the first order conditions

$$\frac{\partial}{\partial \pi} : (\mu - r)v_w + \pi w \sigma^2 v_{ww} + \psi(X_t,t)\rho\sigma v_{wx} = 0$$

which implies that

$$\pi^* = \frac{(r-\mu)v_w - \psi(X_t,t)\rho\sigma v_{wx}}{\sigma^2 w v_{ww}} \tag{2.11}$$

and

$$\frac{\partial}{\partial \pi} : -V_w + U'(c) = 0,$$

which yields

$$c^* = (U')^{-1}(V_w). \tag{2.12}$$

Hence the optimal optimal portfolio is given by $u^* = (\pi^*, c^*)$. This is the all-important answer we are looking for: the weights of the portfolio.

We now put (2.11) and (2.12) into JHB equation (2.10), and simplifying we get

$$\begin{aligned} &\dot{v} + (\alpha(x,t) + rw - (U')^{-1}(V_w))v_w + \phi(x,t)v_x + \frac{\psi^2(x,t)}{2}v_{xx} \\ &- \frac{(v_w(\mu - r) + \psi(x,t)\rho\sigma v_{wx})^2}{2\sigma^2 v_{ww}} + U((U')^{-1}(V_w)) = 0 \end{aligned} \tag{2.13}$$

with

$$v(T,w,x) = U(w,x).$$

It is also convenient to write $\theta_t = \frac{(r-\mu)v_w - \rho\sigma\psi(X_t,t)v_{wx}}{\sigma^2 v_{ww}}$ by allowing $\pi^* w = \theta$. We then present the following

$$\theta_M = \frac{(r-\mu)v_w}{\sigma^2 v_{ww}}$$

and

$$\theta_H = \frac{-\psi(X_t,t)\rho\sigma v_{wx}}{\sigma^2 v_{ww}}.$$

Similarly as discussed in [4], the optimal portfolio $u \in U$ or θ_t is comprised of two components; θ_M and θ_H. θ_M is called Merton Investment strategy as is of the form of the Merton (1969) strategy and θ_H is the hedging component, can also be interpreted as intertemporal hedging demand. It depends on the volatility and the indirect utility function V. The term V_{wx} measures the sensitivity of the marginal utility wealth to the stochastic income, or the attitude towards change in stochastic income.

If $\psi(x,t) = 0$ or $\rho = 0$, the hedging term θ_H no longer exist, thus the existence of non-traded income has no effect on the stock portfolio. The stock income cannot be hedged against in the case where the stock returns and income are uncorrelated. Even though, the expression θ denotes the optimal strategy for the portfolio choice problem, it is expressed in terms of derivatives of the value function solving HJB equation. Also, since the indirect utility function can depend on investor preferences and the investment horizon, the market can depend on the market parameters.

3. The explicit solutions of $V(t, W_t)$

In this section, we find the explicit solutions to the portfolio choice problem $v(t,W_t)$.

3.1. Model 1

Proposition 3.1. *The optimal investment in the stock for the portfolio choice (2.7) with power utility function* $U(w) = \dfrac{(w)^\kappa}{\kappa}$, $0 < \kappa < 1$, *with the candidate solution* $v(t,w) = w(h(t))^{\kappa-1}$ *to the HJB equation (2.9) is*

$$\begin{aligned}\theta^\star &= \frac{-(\mu-r)w-(\kappa-1)\rho\sigma\psi w_x}{(\kappa-1)\sigma^2} \\ &= \frac{(r-\mu)w}{(\kappa-1)\sigma^2} - \frac{\rho\psi w_x}{\sigma} \\ &= \theta^*_M + \theta^*_H\end{aligned}$$

and

$$c^\star = h(t)w,$$

where $h(t) = \frac{\kappa}{1-\kappa}\Big(\exp\big(\frac{\kappa}{1-\kappa}(T-t)\big)\Big)^{-1}$.

Proof. We find the explicit solution to the portfolio choice problem $v(t,W_t) = \mathbb{E}_{t,w}[U(W_T)]$. Taking $U_2(w) = \dfrac{(w)^\kappa}{\kappa}$, $0 < \kappa < 1$ as a utility function. The value function problem is given by

$$v(t,w,u) = \sup_{u\in U}\mathbb{E}_{t,w}\left[\frac{(W_T)^\kappa}{\kappa}\right],\ \kappa \in (0,1).$$

The associated HJB equation

$$\sup_{u\in U}[\mathscr{L}v(t,w,u)] = 0,$$

where $\mathscr{L}v(t,w,u)$ is the generator as in (2.8), with the terminal condition $v(T,w) = \dfrac{(w_T)^\kappa}{\kappa}$, we look for the candidate solution of the above HJB equation of the form $v(t,w) = \dfrac{(w)^\kappa}{\kappa}(h(t))^{\kappa-1}$. The optimal portfolio $(\pi^\star, c^\star)$ is evaluated directly using relevant derivatives of $v(t,w)$ inserted in (2.11) and (2.12). The HJB equation has the following form

$$(h(t)w)^{\kappa}\Big(\frac{(\kappa-1)\dot{h(t)}}{\kappa(h(t))^2}+A(h(t)w)^{-1}+\frac{1-\kappa}{\kappa}\Big)=0,$$

where

$$A=w(r+\pi^{\star}(\mu-r))+\alpha+\phi w_x+\frac{(\kappa-1)\sigma\pi^{\star}}{2}(\sigma\pi^{\star}w+2\rho\psi w_x)$$
$$+\frac{\psi^2}{2}((\kappa-1)w^-1{w_x}^2+w_{xx}).$$

Since $(h(t)w)^{\kappa}\neq 0$ then $\frac{(\kappa-1)\dot{h(t)}}{\kappa(h(t))^2}+A(h(t)w)^{-1}+\frac{1-\kappa}{\kappa}=0$ which simplifies to Bernoulli differential equation

$$\dot{h(t)}+\frac{\kappa A}{(\kappa-1)w}h(t)=(h(t))^2,$$

which has the solution

$$h(t)=\frac{\kappa}{1-\kappa}\Big(\exp\Big(\frac{\kappa}{1-\kappa}(T-t)\Big)-1\Big)^{-1}. \tag{3.1}$$

Finally, we have the value function as

$$v(t,w)=\beta w^{\kappa}\Big(\exp\Big(\frac{\kappa}{1-\kappa}(T-t)\Big)-1\Big)^{1-\kappa},\text{ with }\beta=\Big(\frac{\kappa}{\kappa-1}\Big)^{\kappa}(\kappa-1).$$

Hence the solution of the HJB equation is given by

$$v:(t,w)\to\frac{w^{\kappa}}{\kappa}(h(t))^{\kappa-1},\ 0<\kappa<1.$$

Proposition 3.2. *The optimal investment in the stock for the portfolio choice (2.7) with log utility functions $U_2(w)=\ln(w)$, with the candidate solution $v(t,w)=\ln(wh(t))$ is given by*

$$\theta=\frac{(\mu-r)w^2}{\sigma^2}-\frac{\rho\psi w_x w}{\sigma}$$
$$=\theta_M^*+\theta_H^*.$$

and

$$c^{\star}=w,$$

which is valid for $h(t)=h(T)+B(T-t)$, where

$$B=w^{-1}\big[w(r+\pi^{\star}(\mu-r))+\alpha+\phi w_x+\frac{\psi}{2}(\psi(w_{xx}-w_x w^{-1})-2\rho\sigma\pi^{\star}w_x)\big]$$
$$+ln(w)-\frac{2-(\sigma\pi^{\star})^2}{2}.$$

Proof. We consider the utility functions $U_2 = \ln(w)$ and the controlled wealth process

$$dW_t = ((r + \pi_t(\mu - r))W_t - c_t + \alpha(t, X_t))dt + \pi_t \sigma W_t dB_t,\ W_0 = w_0.$$

The problem a trader is faced with is to find the highest possible payoff $v(t,w) = \sup_{\pi,c} \mathbb{E}_{t,w}[\ln(W_T)]$.

We assume the solution of HJB equation (2.9) is of the form $v(t,w) = \ln(wh(t))$, which leads to the optimal portfolio (2.11) and (2.12) as

$$\pi^\star = \frac{(\mu - r)w - \rho\sigma\psi w_x}{\sigma^2} \text{ that is, } \theta = \frac{(\mu - r)w^2}{\sigma^2} - \frac{\rho\psi w_x w}{\sigma}$$

and the consumption becomes $c^\star = (U')^{-1}(v_w) = w$. Thus,

$$\frac{\dot{h(t)}}{h(t)} + B = 0. \tag{3.2}$$

Integrating (3.2) from t to T yields $h(t) = h(T) + B(T - t)$, where

$$B = w^{-1}\left[w(r+\pi^\star(\mu-r)) + \alpha + \phi w_x + \frac{\psi}{2}(\psi(w_{xx} - w_x w^{-1}) - 2\rho\sigma\pi^\star w_x)\right] + ln(w) - \frac{2-(\sigma\pi^\star)^2}{2}.$$

Hence the value function is

$$\begin{aligned} v(t,w) &= \ln(w) + \ln(h(t)) \\ &= \ln(w) + \ln(B(T-t)), \end{aligned}$$

taking $h(T) = 0$.

3.2. Model 2

The dynamics of wealth process W_t is generated by investor holding cash amount ϑ in the stock S_t, the remainder in the risk-free bond, the inflow of stochastic income rate $\alpha(X_t,t)$ and the consumption $c_t \geq 0$. Both the stochastic income $\alpha(X_t,t)$ and the consumption $c_t \geq 0$ are non-negative and continuous. We then define the corresponding continuous and adapted wealth process with respect to the self financing trading strategy ϑ as

$$\begin{aligned} dW_t &= (W_t - \vartheta_t)\frac{dN_t}{N_t} + \vartheta_t \frac{dS_t}{S_t} - c_t dt + \alpha(X_t,t)dt \\ &= r(W_t - \vartheta_t)dt + (\alpha(X_t,t) - c_t)dt + \vartheta_t \frac{dS_t}{S_t}, \end{aligned}$$

which gives

$$dW_t = rW_t dt + (\alpha(X_t,t) - (c_t + r\vartheta_t))dt + \vartheta_t \frac{dS_t}{S_t}. \tag{3.3}$$

The use of the function $\alpha(x,t)$ allows flexibility in modeling but also introduces some indeterminacy as there can be many characterizations of the same model. This is obtained by taking either $\alpha(x,t) = x$, $\phi(x,t) = \phi x$ and $\psi(x,t) = \psi x$ or $\alpha(x,t) = e^x$, $\phi(x,t) = \phi$ and $\psi(x,t) = \psi$ where ψ, ϕ are constants [4]. We will be focused on the lognormally distributed income obtain through $\alpha(x,t) = x$. More generally the investor receives the value of the state variable X_t itself over time by taking $\alpha(x,t) = x$.

Consider the problem of an investor with utility over terminal wealth which can be maximized by the selection of the portfolio ϑ. Define reward function (2.6)

$$J(t,w,\vartheta) = \mathbb{E}_{t,w}\left[\int_t^T U_1(s,W_s)ds + U_2(T,W_T)\right] = \mathbb{E}_{t,w}[U(W_T)],$$

where $U_1(t,W_t)$ and $U_2(t,W_t)$ are utility functions, and the indirect utility function (2.7)

$$v(t,W_t) = \sup_{\vartheta} J(t,w) = \mathbb{E}_{t,w}[U(W_T)],$$

of which is the utility attained by the investor if the optimal policy is followed. Since (W_t,X_t) are jointly Markov for time t in $[0,T]$, the generator of the equation (3.3) is given by

$$\begin{aligned}\mathscr{L}f(t,w) = \frac{\partial v}{\partial t} + (wr + \vartheta(\mu - r) - c + \alpha(x,t))\frac{\partial v}{\partial w} + \phi(x,t)\frac{\partial v}{\partial x} + \frac{(\vartheta\sigma)^2}{2}\frac{\partial^2 v}{\partial w^2} \\ + \frac{\psi^2(x,t)}{2}\frac{\partial^2 v}{\partial x^2} + \psi(x,t)\rho\vartheta\sigma\frac{\partial^2 v}{\partial w \partial x}.\end{aligned} \tag{3.4}$$

The derived utility function for the wealth function $v(t,W_t)$ satisfies the non-linear HJB equation

$$\sup_{(\vartheta_k, k\in[0,T])} [\mathscr{L}f(k,w)] = 0$$

$$\begin{aligned}\sup_{(\vartheta_k, k\in[0,T])} [\dot{v} + (wr + \vartheta(\mu - r) - c + \alpha(x,t))v_w + \phi(x,t)v_x + \frac{(\vartheta\sigma)^2}{2}v_{ww} \\ + \frac{\psi^2(x,t)}{2}v_{xx} + \psi(x,t)\rho\vartheta\sigma v_{wx} + U(c)] = 0,\end{aligned} \tag{3.5}$$

where $U(\cdot)$ is the utility function. Differentiating (3.5) with respect to ϑ and c gives the first order conditions

$$\frac{\partial}{\partial\vartheta} : (\mu - r)v_w + \vartheta\sigma^2 v_{ww} + \psi(X_t,t)\rho\sigma v_{wx} = 0,$$

which implies that

$$\vartheta^* = \frac{(r-\mu)v_w - \psi(x,t)\rho\sigma v_{wx}}{\sigma^2 w v_{ww}} \tag{3.6}$$

and

$$c^{*} = (U')^{-1}(v_w). \tag{3.7}$$

Hence the optimal optimal portfolio is given by $(\vartheta^{\star}, c^{\star})$. This is the all-important answer we are looking for: the weights of the portfolio. We now put (3.6) and (3.7) into JHB equation (3.5) and simplifying we get

$$\begin{aligned} \dot{v} + (\alpha(x,t) + rw)v_w + \phi(x,t)v_x + \frac{\psi^2(x,t)}{2}v_{xx} - \frac{(v_w(\mu - r) + \psi(x,t)\rho\sigma v_{wx})^2}{2\sigma^2 v_{ww}} \\ + U((U')^{-1}(v_w)) - ((U')^{-1}(v_w))v_w = 0 \end{aligned} \tag{3.8}$$

with

$$v(T,w) = U(w).$$

3.2.1. The optimal portfolio choice

We directly construct the optimal portfolio for the investor with power utility function, and the investor receives income $\alpha(x,t)$ over finite time. We consider the case where correlation between the stock and the income state variable is not perfect. For the perfect correlation see [4]. In exploiting this, we first determine the nature of W_T from (3.3).

The wealth process is given as

$$dW_t = rW_t dt + (\alpha(X_t,t) - (c_t + r\vartheta))dt + \vartheta\frac{dS_t}{S_t},$$

where $t \in [0,T]$. The integrating factor (IF) is obtain as $IF = e^{-\int_t^T rdk} = e^{-r(T-t)}$. Multiplying throughout by the IF yields

$$\begin{aligned} e^{-r(T-t)}dW = re^{-r(T-t)}dt + e^{-r(T-t)}\left[(\alpha(X_t,t) - (c_t + r\vartheta))dt + \vartheta\frac{dS_t}{S_t}\right] \\ e^{-r(T-t)}dW - re^{-r(T-t)}dt = e^{-r(T-t)}\left[(\alpha(X_t,t) - (c_t + r\vartheta))dt + \vartheta\frac{dS_t}{S_t}\right] \\ d(W_t e^{-r(T-t)}) = e^{-r(T-t)}\left[(\alpha(X_t,t) - (c_t + r\vartheta))dt + \vartheta\frac{dS_t}{S_t}\right]. \end{aligned}$$

Integrating from t to T

$$\int_t^T d(W_k e^{-r(k-t)}) = \int_t^T e^{-r(T-k)}\left[(\alpha(X_k,k) - (c_k + r\vartheta))dk + \vartheta\frac{dS_k}{S_k}\right],$$

which implies that

$$W_T e^{-r(T-t)} - W_t = \int_t^T e^{-r(k-t)}(\alpha(X_k,k) - (c_k + r\vartheta))dk + \int_t^T e^{-r(k-t)}\vartheta\frac{dS_k}{S_k}.$$

That is,

$$W_T = W_t e^{r(T-t)} + e^{r(T-t)}\left[\int_t^T e^{-r(k-t)}(\alpha(X_k,k) - (c_k + r\vartheta))dk + \int_t^T e^{-r(k-t)}\vartheta\frac{dS_k}{S_k}\right].$$

Finally,

$$W_T = W_t e^{r(T-t)} + \int_t^T e^{r(T-k)}(\alpha(X_k,k) - (c_k + r\vartheta))dk + \int_t^T e^{r(T-k)}\vartheta\frac{dS_k}{S_k}. \tag{3.9}$$

Proposition 3.3. *The optimal investment in the stock for the portfolio choice (2.7) with power utility function* $U(w) = \frac{w^\gamma}{\gamma}$, $0 < \gamma < 1$ *is*

$$\begin{aligned}\vartheta^* &= \frac{(r-\mu)(we^{r(T-t)} + g)}{(\gamma-1)\sigma^2 e^{r(T-t)}} - \frac{\psi(x,t)\rho g_x}{\sigma e^{r(T-t)}}\\ &= \frac{(\mu-r)(we^{r(T-t)} + g)}{(1-\gamma)\sigma^2 e^{r(T-t)}} - \frac{\psi(x,t)\rho g_x}{\sigma e^{r(T-t)}}\\ &= \vartheta_M^* + \vartheta_H^*\end{aligned}$$

and

$$\begin{aligned}c^* &= (v_w)^{\frac{1}{\gamma-1}}\\ &= (W_t e^{r(T-t)} + g)e^{\frac{r}{\gamma-1}(T-t)},\end{aligned}$$

with function $g(T-t,x)$ *as defined in equation (3.12) and solves the equation (3.14).*

Proof. Consider power utility function $U(w) = \frac{w^\gamma}{\gamma}$, $0 < \gamma < 1$. The value function can be written as

$$v(t,w,x) = \frac{1}{\gamma}\sup_{\vartheta_k, k\in[0,T]} \mathbb{E}_t[w_T^\gamma]. \tag{3.10}$$

Plugging equation (3.9) into equation (3.10), the value function become

$$\begin{aligned}v(t,w) = \frac{1}{\gamma}\sup_{\vartheta_k, k\in[0,T]} \mathbb{E}_t[(W_t e^{r(T-t)} &+ \int_t^T e^{r(T-k)}(\alpha(X_k,k) - (c_k + r\vartheta))dk\\ &+ \int_t^T e^{r(T-k)}\vartheta\frac{dS_k}{S_k})^\gamma].\end{aligned}$$

Thus

$$v(t,w) = \frac{1}{\gamma}(W_t e^{r(T-t)} + g(T-t))^\gamma,\ (\gamma \in (0,1)), \tag{3.11}$$

where

$$g(T-t)=\int_t^T e^{r(T-k)}(\alpha(X_k,k)-(c_k+r\vartheta))dk+\int_t^T e^{r(T-k)}\vartheta\frac{dS_k}{S_k} \tag{3.12}$$

with $g(0,y)=0$.

We consider the following important terms from the HJB equation (3.5)

- $\dot{v}=(W_te^{r(T-t)}+g)^{\gamma-1}(-rW_te^{r(T-t)}+\dot{g})$.
- $v_w=(W_te^{r(T-t)}+g)^{\gamma-1}e^{r(T-t)}$.
- $v_x=(W_te^{r(T-t)}+g)^{\gamma-1}g_x$.
- $v_{wx}=(\gamma-1)e^{r(T-t)}(W_te^{r(T-t)}+g)^{\gamma-2}g_x$.
- $v_{ww}=(\gamma-1)e^{2r(T-t)}(W_te^{r(T-t)}+g)^{\gamma-2}$.
- $v_{xx}=(\gamma-1)(W_te^{r(T-t)}+g)^{\gamma-2}{g_x}^2+(W_te^{r(T-t)}+g)^{\gamma-1}g_{xx}$.

We now put the above terms into the HJB equation (3.5)

$$\begin{aligned}&q^{\gamma-1}[-rwe^{r(T-t)}+\dot{g}+e^{r(T-t)}(wr+\vartheta(\mu-r)+\alpha(x,t))+g_x\phi(x,t)\\&+\frac{1}{2}(\vartheta\sigma)^2e^{2r(T-t)}(\gamma-1)q^{-1}+\frac{\psi^2(x,t)}{2}(g_{xx}+(\gamma-1)q^{-1}{g_x}^2)\\&+\psi(x,t)\rho\vartheta\sigma(\gamma-1)e^{r(T-t)}q^{-1}g_x]-cq^{\gamma-1}e^{r(T-t)}+U(c)=0,\end{aligned} \quad 3.13$$

where $q=W_te^{r(T-t)}+g$. Differentiating with respect to ϑ and c (refer to equation (3.6) and (3.7)) we have

$$\frac{\partial}{\partial\vartheta}:\vartheta^*=\frac{(r-\mu)(W_te^{r(T-t)}+g)}{(\gamma-1)\sigma^2e^{r(T-t)}}-\frac{\psi(x,t)\rho g_x}{\sigma e^{r(T-t)}},$$

where

$$\vartheta_M^*=\frac{(r-\mu)(W_te^{r(T-t)}+g)}{(\gamma-1)\sigma^2e^{r(T-t)}}$$

resemble Merton investment strategy and the hedging component by

$$\vartheta_H^*=-\frac{\psi(x,t)\rho g_x}{\sigma e^{r(T-t)}}.$$

Consumption strategy,

$$\frac{\partial}{\partial c}:-q^{\gamma-1}e^{r(T-t)}+U'(c)=0.$$

Therefore, one has

$$\begin{aligned} c^* &= (U')^{-1}(q^{\gamma-1}e^{r(T-t)}) \\ &= (q^{\gamma-1}e^{r(T-t)})^{\frac{1}{\gamma-1}} \\ &= (W_t e^{r(T-t)} + g)e^{\frac{r}{\gamma-1}(T-t)}. \end{aligned}$$

Taking back the ϑ and c into HJB equation, we find the function g that solves the PDE (refer to equation (3.8)).

$$\dot{g} + \alpha(x,t)e^{r(T-t)} + \phi(x,t)g_x + \frac{1}{2}\psi^2(x,t)[g_{xx} + (\gamma-1)(we^{r(T-t)}+g)^{-1}{g_x}^2] - \tilde{\rho} = 0 \quad (3.14)$$

where

$$\tilde{\rho} = (we^{r(T-t)}+g)\frac{[(\mu-r)+(\gamma-1)(we^{r(T-t)}+g)^{-1}g_x]^2}{2\sigma^2(\gamma-1)} + \frac{\gamma-1}{\gamma}(W_t e^{r(T-t)}+g)^{\gamma}e^{\frac{r\gamma}{\gamma-1}(T-t)}$$

Therefore the value function is given by

$$v(t,w) = \frac{1}{\gamma}(W_t e^{r(T-t)} + g(T-t))^{\gamma},\ (\gamma \in (0,1)) \quad (3.15)$$

With g being the function that satisfies equation (3.14) and $g(0,x) = 0$.

Proposition 3.4. *The optimal investment in the stock for the portfolio choice (2.7) with log utility function $U(x) = ln(x)$ is*

$$\begin{aligned} \vartheta^* &= \frac{(\mu-r)e^{r(T-t)} - \psi(x,t)\sigma\lambda(we^{r(T-t)}+h(T-t))^{-1}h_x e^{r(T-t)}}{\sigma^2(we^{r(T-t)}+h(T-t))^{-1}e^{2r(T-t)}} \\ &= \frac{(\mu-r)(we^{r(T-t)}+h(T-t))}{\sigma^2 e^{r(T-t)}} - \frac{\psi(x,t)\rho h_x}{\sigma e^{r(T-t)}} \\ &= \vartheta^*_M + \vartheta^*_H \end{aligned}$$

and

$$c^* = (we^{r(T-t)} + h(T-t))e^{-r(T-t)},$$

with function $h(T-t)$ as defined in equation (3.17).

Proof. We now determine the solution for the log utility function $U(w) = ln(w)$. Then the performance function is given by:

$$
\begin{aligned}
J(t,w) &= \mathbb{E}_{t,w}[U(W_T)] \\
&= \mathbb{E}_{t,w}[ln(W_t e^{r(T-t)} + \int_t^T e^{r(T-k)}(\alpha(X_k,k) - (c_k + r\vartheta))dk \\
&+ \int_t^T e^{r(T-k)}\vartheta \frac{dS_k}{S_k})].
\end{aligned}
$$

The value function is obtained as:

$$
v(t,W_t,X_t) = \sup_{\vartheta} J(t,w) = \mathbb{E}_{t,w}[ln(W_t e^{r(T-t)} + h(T-t))], \tag{3.16}
$$

where

$$
h(T-t) = \int_t^T e^{r(T-k)}(\alpha(X_k,k) - (c_k + r\vartheta))dk + \int_t^T e^{r(T-k)}\vartheta \frac{dS_k}{S_k} \tag{3.17}
$$

with $h(0,x) = 0$. The equation (3.16) can be written in the following form

$$
v(t,W_t) = ln(W_t e^{r(T-t)} + h(T-t)).
$$

By referring to the HJB equation (3.5) terms, and using the new version of v we have:

- $\dot{v} = (we^{r(T-t)} + h(T-t))^{-1}(\dot{h} - rwe^{r(T-t)})$.
- $v_w = (we^{r(T-t)} + h(T-t))^{-1}e^{r(T-t)}$.
- $v_x = (we^{r(T-t)} + h(T-t))^{-1}h_x$.
- $v_{ww} = -(we^{r(T-t)} + h(T-t))^{-2}e^{2r(T-t)}$.
- $v_{wx} = -(we^{r(T-t)} + h(T-t))^{-2}h_x e^{2r(T-t)}$.
- $v_{xx} = (we^{r(T-t)} + h(T-t))^{-1}[h_{xx} - (we^{r(T-t)} + h(T-t))^{-1}{h_x}^2]$.

Inserting the above terms into the HJB equation (3.5) yields

$$
\begin{aligned}
&(we^{r(T-t)} + h(T-t))^{-1}[-re^{r(T-t)} + \dot{h} + (wr + \vartheta(\mu - r) - c + \alpha(x,t))e^{r(T-t)} \\
&- \frac{(\vartheta\sigma)^2}{2}(we^{r(T-t)} + h(T-t))^{-1}e^{2r(T-t)} - \frac{\psi(x,t)^2}{2}(h_{xx} - (we^{r(T-t)} + h(T-t))^{-1}{h_x}^2) \\
&- \psi(x,t)\sigma\vartheta\rho(we^{r(T-t)} + h(T-t))^{-1}h_x e^{r(T-t)}] + U(c) = 0.
\end{aligned} \tag{3.18}
$$

Differentiating with respect to ϑ and c, we obtain

$$(\mu-r)e^{r(T-t)}-\vartheta\sigma^2(we^{r(T-t)}+h(T-t))^{-1}e^{2r(T-t)}$$
$$-\psi(x,t)\sigma\rho(we^{r(T-t)}+h(T-t))^{-1}h_x e^{r(T-t)}=0,$$

that is,

$$\vartheta=\frac{(\mu-r)e^{r(T-t)}-\psi(x,t)\sigma\rho(we^{r(T-t)}+h(T-t))^{-1}h_x e^{r(T-t)}}{\sigma^2(we^{r(T-t)}+h(T-t))^{-1}e^{2r(T-t)}} \tag{3.19}$$

and $-(we^{r(T-t)}+h(T-t))^{-1}e^{r(T-t)}+U'(c)=0$. Thus,

$$\begin{aligned} c^* &= (U')^{-1}((we^{r(T-t)}+h(T-t))^{-1}e^{r(T-t)}) \\ &= (we^{r(T-t)}+h(T-t))e^{-r(T-t)}. \end{aligned} \tag{3.20}$$

Plugging back the optimal portfolio (ϑ^*, c^*) in the HJB equation (3.18) gives the function $h(T-t)$.

$$\begin{aligned} p(T-t)[-re^{r(T-t)}+\dot{h}+(wr+\vartheta(\mu-r)+\alpha(x,t))e^{r(T-t)} \\ -\frac{(\vartheta\sigma)^2}{2}p(T-t)e^{2r(T-t)}-\frac{\psi(x,t)^2}{2}(h_{xx}-p(T-t)h_x{}^2) \\ -\psi(x,t)\sigma\vartheta\rho p(T-t)h_x e^{r(T-t)}]+r(T-t)-(1-\ln(p(T-t)))=0, \end{aligned} \tag{3.21}$$

where $p(T-t)=(we^{r(T-t)}+h(T-t))^{-1}$. Equation (3.21) is not easy to solve for the function $h(T-t)$. Therefore, the value function is

$$v(t,W_t)=ln(W_t e^{r(T-t)}+h(T-t)).$$

4. Discussion of results

In the first model, if the investor's utility function is power utility. Our admissible portfolio is given by proposition (3.1). We observe that, μ (the drift parameter) and r (the rate of interest) has a positive impact on the Merton investment strategy of the portfolio which subsequently lead to a notable impact on the whole portfolio. If $\mu<r$ with variance σ small enough, then θ_M^*, and θ will be maximal . Similarly, if $\mu>r$ and the parameter of volatility σ small enough, then Merton investment strategy θ_M^* will be minimal and so does the whole portfolio π^*. It is also fundamental to note that, if the drift parameter μ and the interest rate r have equal magnitude (i.e, $\mu=r$), then the Merton investment strategy varnishes, and only the hedging term remains. In general terms, if the variance σ^2 becomes larger, then the portfolio becomes smaller. Therefore, we have an inverse relation between the risk and the portfolio. That is, if there is too much risk in the market we invest little. The Hedging term θ_H^* depends on the sensitivity measure term v_{wx}. v_{wx} is partial derivative of the value function with respect to w and x jointly. If the variance σ^2 is small enough than the risk premium $\mu - r$ then, the portfolio θ is maximal.

The second model shows that if the investor has power utility, then proposition (3.3) becomes our admissible portfolio. It is observed that, as μ becomes larger and larger than r, Merton's investment strategy (ϑ_M^*) also becomes larger and so does the whole portfolio. This means that risk premium has a positive impact on the value of the portfolio. The difference between the bank rate and the drift of the risk stock has a notable contribution on the value of the portfolio. If the difference is too big, then the value of the portfolio is going to be big. When the volatility becomes bigger, then ϑ_M^* becomes smaller. Therefore there is an inverse relation between the risk and the value of ϑ_M^*. Hence, if there is too much risk in the market we invest little.

In analyzing the hedging component ϑ_H^* in detail. Just like in [4], ϑ_H^* depend on the model for stochastic income through g_x, while Merton's investment strategy (ϑ_M^*) depends on $we^{r(T-t)} + g$. If $g(T-t)$ is a zero function then the hedging term ϑ_H^* varnishes for $g_x = 0$ and Merton's investment strategy remains. The absence of the hedging term may also be due to the correlation between the stock and income $\rho = 0$. Thus, correlation has a positive impact on the value of the hedging term and negative impact (of the positive correlation) to the whole portfolio. If $T - t = 0$, then the portfolio ϑ^* lean much on the product of variance and the term $1 - \gamma$ for desirable results.

It is notable that, the risk premium $(\mu - r)$ has a positive impact on the Merton's investment strategy ϑ_M^* and the whole portfolio. As the risk premium grows then the Merton's investment strategy also grows and so does the whole portfolio. There is an inverse relation between the risk and the value of the Merton's investment strategy. That is, as the volatility becomes larger and larger, the ϑ_M^* becomes smaller and smaller. We invest little in the market with a lot of risk. As in proposition (3.3) and [4] the hedging term ϑ_H^* depend on the model for stochastic income through h_x. The correlation ρ also as positive impact on the hedging term. If $\rho = 0$, then the hedging term varnishes. In the case where $|\rho| < 1$, the income can not be perfectly be hedged via the stock S_t and the market the investor is faced with is incomplete. Hence for $|\rho| = 1$, then the market is complete and the investor can perform intertemporal hedging.

In both models, the investor is allowed to consume. Considering proposition (3.1), if the function $h(t)$, satisfy the inequality $0 \leq h(t) < 1$ for all $t \in [0,T]$ the we conclude that the investor is supposed to consume the fraction of the wealth or do not consume at all. If $h(t) = 1$, then the investor consumes all of the wealth, and the investment strategy collapses. Therefore, $c^* = w$ is not advisable, this is depicted by propositions (3.2) and (3.4) (in case $h(T-t) = 0$) which are all under log utility. Still under (3.1) if for all $t \in [0,T]$, $h(t) > 1$ then the market is incomplete and there is arbitrage. Thus, in general terms, investor is assumed to take precautions on how much to consume to overcome penury.

5. Conclusion

Any investor who invests in stock and bond would like to have maximum utility of her wealth. This invites the classical theory of portfolio optimization. We have discussed the Merton's classic setup and results considering the control problem which can be transformed into a partial differential equation (PDE), even though they are highly non-linear and in general very difficult to solve analytically. The verification theorem gives conditions for the solution of the HJB equation to be solution of the control problem and with the help of the viscosity solutions of the HJB equation we can describe the properties of some value functions. Stochastic control theory plays an important role in providing room for varied degree of the investor willingness for the risk together with different utility functions. For a restricted class of utility functions (such as power and log utilities), we were able to compute a close form of the value function, and that of the optimal policy. That is, under the first model the admissible portfolios to power and log utility functions are given in propositions (3.1) and (3.2) respectively as our findings. In the second model, our portfolios are given as propositions (3.3) and (3.4) of which optimizes the power and log utilities respectively. All the two models vividly depicts that, there is an inverse relation between the risk and the value of Merton's investment strategy ϑ_M^*. Hence, if there is too much risk in the market we invest little. One of our above mentioned findings is that the consumption value is is relatively proportional to the wealth process, this indicates that an investor is allowed to consume up to the size of her wealth, otherwise the investor would face a collapse in investment strategy. A collapse in portfolio may lead to unbearable ruin.

Conflict of Interests

The authors declare that there is no conflict of interests.

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