Available online at http://scik.org
Math. Finance Lett. 2015, 2015:10
ISSN 2051-2929

AMERICAN OPTION: AN OPTIMAL STOPPING PROBLEM

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Abstract

We show that the problem of pricing the perpetual American options can be treated as an optimal stopping problem. We impose some boundary conditions to arrive at the optimal solutions. We consider the option price, stopping time, strike price and volatility to approach the problem. From the solution, we deduced that the optimal arbitrage free price for the perpetual American put option can only be determined if the optimal value of the stock price of the option is known.

2010 AMS Subject Classification: 91G20, 91G80, 91G30

Keywords: American put option; optimal stopping; strike price; free arbitrage.

1.0 Introduction

The focus of this study is to discuss the problem of optimal stopping of the American option. The option bestows on the holder the right to trade an underlying asset at any time $t \leq T$ (that is, before expiry) for a prescribed strike price K.

For the American call option, the pay-off can be given by the equation,

$$f(S,K) = \begin{cases} S(t) - K & t \leq T \\ 0 & otherwise \end{cases} \qquad (1a)$$

The equation (1a) implies that the American call option can only be exercised when the price of the underlying asset is greater than the strike price K.

Also, the pay-off for the American put option

$$f(S,K) = \begin{cases} K - S(t) & t \leq T \\ 0 & otherwise \end{cases} \qquad (1b)$$

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Received August 20, 2015

The equation (2) implies that the American put option can be exercised when the strike price K is greater than the price $S(\mathrm{t})$ of the underlying.

Since the holder of the American option can exercise the option at any before the option expires, traders are more attracted to this option and mathematicians as a result of this interpret the option as a free boundary problem so as to obtain the best of exercise of the option for maximum payoff. This work analyses the solution of a simple optimal problem by applying matching value condition, smooth pasting condition, asset equilibrium condition and boundary condition. We shall also study American options as optimal problems.

2.0 A Simple Optimal Stopping Problem

We consider a simple optimal stopping problem,

$$V(x) = \max_{\tau} \mathbb{E}_x[e^{-\mu\tau} f(X_t)] \tag{2}$$

Subject to

$$dX_t = pX_t dt + qX_t dB_t \tag{3}$$

$$X(0) = x$$

Where;

$\tau =$ the stopping time

$\tau^* =$ the optimal stopping time,

$x_* =$ the value of the state variable at τ^* $(i.e. x_* = x(\tau^*)$

$\mu =$ discount rate p, q are positive integers

$dB_t =$ an increment of a Wiener process

$f(.) =$ a reward function and

$V(.) =$ a value function

The relationship between X and t is given be the equation (3). Then the choice of τ which yields the maximum value of *V(x)* will depend on equation (3).

The problem now is to find the optimal stopping values x_* and τ^* where the function f is currently unknown. This is because we have to put into consideration if the value function yields maximum value at the time of exercising the function. Hence the pay-off should be maximised in the shortest possible time.

2.1 Solution to the Optimal Stopping Problem

We make the following assumptions through this problem;

1. Both f and V are continuous and smooth.
2. Since and f and V are both functions of asset market values, then it implies that $f(x), V(x) \geq 0$.
3. Transaction costs related to the option are not put into consideration.

We need to find the optimal value x_*. We can say that when the underlying process is less than x_*, we choose not to exercise the option but if its value is greater or equal to x_*, we exercise the option. Whereas, it is optimal to exercise when $x = x_*$(i.e. the shortest possible time) hence

$$V(x) = f(x_*) \tag{4}$$

The equation (4) above is called the ***value matching condition***.

The decision to exercise or not depends on the optimality at any value of *x* and also on the gain made by comparing the two function *V(x)* and *f(x)*. This decision will be made according to the function *H*, where

$$H(x) = \max\{V(x), f(x)\} \tag{5}$$

By the equation (5), if $H(x) = V(x)$, we decide to continue and if $H(x) = f(x)$ we stop. In order to evaluate the optimal solution to this problem, then we consider the ***smooth pasting condition*** which states that

$$V'(x_*) = f'(x_*) \tag{6}$$

Which shows that $V'(x)$ and $f'(x)$ are equal at optimal value x_* smoothly while the value matching condition shows that *V(x)* and *f(x)* are equal at optimal value x_*. Then with these two conditions considered simultaneously, the function *H(x)* can be said to be both smooth and continuous at optimal valuex_*.

When$f(x) \geq V(x)$, we can exercise. This is due to the fact that the stopping value is more than the continuing value at this point. However, when$f(x) = V(x)$, exercise is optimal because we will wait for more time for$f(x) > V(x)$.

We shall then define the stopping and continuing regions.

The stopping region is given by

$$S_{reg} = \{x: f(x) \geq V(x)\} \tag{7}$$

And the continuing region,

$$C_{reg} = \{x: V(x) > f(x)\} \tag{8}$$

Hence the, the optimal stopping time can be expressed as

$$\tau^* = \inf\{t \geq 0: V(X_t) \leq f(X_t)\} \tag{9}$$

From equation (7) and equation (9), we see that the optimal stopping time is that instant which the value function is not greater than the gain function. Therefore, the optimal stopping region becomes

$$S_{reg} = \{x: f(x) = V(x)\} \tag{10}$$

Equation (10) is called the ***optimal stopping boundary*** which divided the state space into stopping region and continuation region. It is also a part of the stopping region at which it is optimal to exercise the option. Note that at the optimal boundary we have that $x = x_*$.

Now assuming it is optimal also to continue for a small time δt after the optimal stopping time. Then,

$$V(x) = e^{-\mu\delta t}\mathbb{E}\big(V(x + \delta x)\big) \tag{11}$$

Expanding equation (11) using Taylor's series expansion, we obtain

$$V(x) \approx V(x) + \delta x V'(x) + \frac{1}{2}(\delta x)^2 V'' + 0(\delta x)^3 \tag{12}$$

Neglecting higher order terms and then substituting equation (12) in equation (11), obtain

$$V(x) \approx e^{-\mu\delta t}\mathbb{E}\left[V(x) + \delta x V'(x) + \frac{1}{2}(\delta x)^2 V''(x)\right] \tag{13}$$

Applying $\mathbb{E}$ over δx in equation (13), we obtain

$$V(x) = e^{-\mu\delta t}[V(x) + V'(x\mathbb{E}(\delta x) + \frac{1}{2}V''\mathbb{E}(\delta x)^2 \tag{14}$$

Subtracting $e^{-\mu\delta t}$from both sides of (14) becomes

$$\left(1 - e^{-\mu\delta t}\right)V(x) \approx e^{-\mu\delta t}\left[V'(x)\mathbb{E}(\delta x) + \frac{1}{2}V''(x)\mathbb{E}(\delta x)^2\right]$$

Then

$$\frac{(1 - e^{-\mu\delta t})V(x)}{\delta t} \approx \frac{e^{-\mu\delta t}}{\delta t}\left[V'(x)\mathbb{E}(\delta x) + \frac{1}{2}V''(x)\mathbb{E}(\delta x)^2\right] \tag{15}$$

Then as $\delta t \to 0$, that is, we take the derivative and replace δx with dx and δt with dt

$$\mu V(x) = \frac{1}{dt}\left[V'(x)\mathbb{E}dx + \frac{1}{2}V''(x)\mathbb{E}(dx)^2\right] \tag{16}$$

From the ***Stochastic Equation of Motion*** which states that, given that $dx = pxdt + qxdz$ then $\mathbb{E}(dx) = pxdt$ and $\mathbb{E}(dx)^2 = [qx]^2 dt$, Blouin (2003).

Substituting the conditions Stochastic Equation of motion above in equation (16), we obtain

$$\mu V(x) = pxV'(x) + \frac{1}{2}V''(x)(qx)^2 \tag{17}$$

Equation (17) is called the ***asset equilibrium condition*** which implies that the measure of the return that can be obtained if the asset is traded in the market and the income obtained if the asset was to be invested as the risk free asset should be same. If this condition is not satisfied, then the asset is traded in a way that allow arbitrage against the trader, that is the option was exercised with no consideration of optimality.

Equation (17) shows that $\mu V(x)$ is the return that could be obtained by trading the asset at its market value while $pxV'(x) + \frac{1}{2}V''(x)(qx)^2$ is the return if the asset was invested as a risk free asset.

However, if equation (17) is not satisfied then the asset is said to be either overvalued or undervalued Blouin, (2003).

Equation (17) gives,

$$pxV'(x) + \frac{1}{2}q^2x^2V''(x) - \mu V(x) = 0 \tag{18}$$

We need to solve the differential equation (18) above which is a Cauchy-Euler equation and it can be solved by taking a guess. Let

$$V(x) = kx^{\omega}$$

$$\text{then,} \quad xV(x) = \omega kx^{\omega} \quad and \;\; x^2V''(x) = \omega(\omega - 1)kx^{\omega} \tag{19}$$

Substituting equation (19) into equation (18) and simplifying, we obtain,

$$kx^{\omega}[q^2\omega(\omega - 1) + 2p\omega - 2\mu] = 0 \tag{20}$$

Then,

$$kx^{\omega}[q^2\omega^2 + (2p - q^2)\omega - 2\mu] = 0 \tag{21}$$

Note that k, p, q are constants and $x > 0$

Obtaining the roots of equation (21)

$$\omega_{\pm} = \frac{(q^2 - 2p) \pm \sqrt{(2p - q^2)^2 + 8q^2\mu}}{2q^2} \tag{22}$$

It follows that $V(x) = kx^{\omega_{\pm}}$ are solutions to the differential equation (18). This implies that the general solution is a linear combinations of the respective solution. Hence,

$$V(x) = k_1x^{\omega_+} + k_2x^{\omega_-} \tag{23}$$

If we substitute the solution given in equation (23), equation (4) and equation (6), we will have three unknowns, namelyk_1, k_2 and x_* in only two equations. We will need a third equation to obtain a unique solution.

From the properties of geometric Brownian motion, whenever$x = 0$, it should remain zero forever and hence the optimal value x_* will never be reached. We can now show from equation (23) that

$$V(0) = 0 \tag{24}$$

From equation (22) it is clear that $\omega_+ > 0$ and$\omega_- < 0$, since$\mu > 0$. Therefore, as $x \to 0$, then $k_2 x^{\omega_-} \longrightarrow +\infty$ for $k_2 > 0$. Likewise as $x \longrightarrow 0$ then $k_2 x^{\omega_-} \longrightarrow -\infty$ for $k_2 < 0$. Then it implies that $V(0) = 0$ when $k_2 = 0$.

Obviously too, the first term of equation (23) goes to zero when $\longrightarrow 0$. Hence, all these satisfy the condition in equation (24).

Therefore, since $k_2 = 0$,

$$V(x) = k_1 x^{\omega_+} \tag{25}$$

If we apply the equation (25) above to equation (4) and equation (6), we obtain that

$$k_1 x_*^{\omega_+} = f(x_*) \tag{26}$$

and

$$(\omega_+) k_1 x_*^{\omega_+ - 1} = f'(x_*) \tag{27}$$

Substituting equation (26) in equation (27), we obtain

$$\frac{(\omega_+) f(x_*)}{x_*} = f'(x_*) \tag{28}$$

Hence, the ***Optimal value*** to this particular problem is

$$x_* = \frac{(\omega_+) f(x_*)}{f'(x_*)} \tag{29}$$

Since we have able to evaluate the optimal value x_*, we can get the optimal stopping time τ^* as

$$\tau^* = \min\{t \geq 0 : X_t = x_*\}$$

$$\tau^* = \min\left\{t \geq 0 : X_t = \frac{(\omega_+) f(x_*)}{f'(x_*)}\right\} \tag{30}$$

Thus, the value function $V(x)$ will be given as

$$V(x) = \mathrm{E}_x\left[e^{-\mu\tau^*} f(x_*)\right] \tag{31}$$

The solution x_* can be extended if the function $f(x)$ is known and it depends on the nature of the problem at hand.

3.0 A Case Study of American Put Options

Let us consider perpetual American put option which is an American option with no expiry date. This implies that the expiry date is at infinity or we can say that it is an American put with infinite time horizon.

Assumptions

1. There is no arbitrage opportunity in the market.
2. The underlying asset pays no dividends.
3. There is one riskless bank account and one risky underlying asset.

3.1 Problem Formulation

The problem here is to find the optimal arbitrage free price and optimal time for the perpetual American put option.

Firstly, we take a look at the following proposition which is necessary for our study of the perpetual American put.

Proposition 1

Given the Stochastic differential equation

$$dX_t = rX_t dt + \sigma X_t dB_t \tag{32}$$

with $X_0 = x > 0$

where $B(B_t)_{t\geq 0}$ is the standard Brownian motion started at time $t_0 = 0$, σ is the volatility coefficient, r is the interest rate and $X = (X_t)_{t\geq 0}$ is the geometric Brownian motion that governments the asset in the market. The solution to the differential equation (32) is given by

$$X_t = X_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)$$

Proof

The equation (32) can be rewritten as

$$\frac{dX_t}{X_t} = rdt + \sigma dB_t \tag{32b}$$

Using the Ito formula with $g(x,t) = logx$, it follows that

$$d(logX_t) = \frac{\partial g}{\partial t} dt|_{x=X_t} + \frac{\partial g}{\partial tg} dX_t|_{x=X_t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2|_{x=X_t} \tag{33}$$

It is obvious that the function $g(x,t) = logx$ has no term in t, this means that the term in equation (33) differentiated with respect to t vanishes. Hence the equation (33) reduces to

$$d(logX_t) = \frac{\partial g}{\partial x} dX_t|_{x=X_t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2|_{x=X_t} \tag{34}$$

Given that $g(x,t) = logx$ implies that $\frac{\partial g}{\partial x} = \frac{1}{x}$. We can therefore, substitute the required terms in equation (34) to obtain

$$d(logX_t) = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2}(dX_t)^2 \tag{35}$$

From equation (32b), it follows that

$$\left(\frac{dX_t}{X_t}\right)^2 = r^2(dt)^2 + 2rdtdB_t + \sigma dB_t dB_t \tag{36}$$

By applying the properties of Brownian motion, equation (36) becomes

$$\left(\frac{dX_t}{X_t}\right)^2 = \sigma^2 dt \tag{37}$$

This simplifies equation (35) above to

$$d(logX_t) = \frac{1}{X_t} dX_t - \frac{1}{2}\sigma^2 dt \tag{38}$$

Combining equation (32b) and equation (38), we obtain

$$d(logX_t) = rdt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \tag{39}$$

Simplifying equation (39) above yields

$$d(logX_t) = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \tag{40}$$

The integral form of the of (40) is given by

$$\int_0^t d(logX_s) = \int_0^t \left(r - \frac{1}{2}\sigma^2\right) ds + \int_o^t \sigma dB_s \tag{41}$$

Equation (41) reduces to

$$[logX_s]_0^t = \left(r - \frac{1}{2}\sigma^2\right) t + \sigma B_t \tag{42}$$

then,

$$X_t = X_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right) \tag{43}$$

The arbitrary free price of the perpetual American put is given by Peskir and Shiryaev (2006)

$$V(x) = \max_{\tau} \mathbb{E}_x[e^{-r\tau}(K - X_\tau)^+] \tag{44}$$

where K =Strike price, τ =stopping time, X_τ =asset price at time t according to equation (43). It is important to note that the gain function of equation (44) is $f(x) = (K - x)^+$ and X_τ in the gain function is the solution X_t of equation (43) at $t = \tau$.

The problem now is to determine the optimal arbitrage free price and optimal stopping time, say τ_* to exercise the option which yields the maximum value according to the equation (44). We shall need to find the price of the stock according to equation (43) which will help us to proceed.

3.1.1 Solution to the American Put Problem

It is a general knowledge that the optimal exercise time of the American put is when the stock price falls as much as possible. The option is best exercised at the possible minimum time of duration from when the option contract was made but with the possible maximum pay off. From equation (44) and equation (43) as X becomes very small, if the option is not exercised then less likely the payoff will not increase upon continuation. This is because the payoff is maximum when the price of the stock falls in the market, not considering other factors.

Therefore, we assume that there exists a point $p \in (0, K)$ such that

$$\tau_p = \min\{t \geq 0 : X_t \leq p\} \tag{45}$$

Equation (45) implies that we need to find the point p which will give the optimal price in order to find optimal value of $V(x)$ and τ_p. It is important to note that p is similar to x_* given in equation (30). p stands for a certain price between 0 and K as the option cannot be exercised if the stock price exceeds or equals K. The stock price cannot be zero.

According to Peskir and Shiryaev (2006), by the standard argument of Strong Markov property for the value function $V(x)$ and the unknown point p, we get the following boundaries.

$\mathbb{L}_x = rV \;\; for \;\; x > p$ (Asset equilibrium condition) $(46a)$

$V(x) = (K - x)^+$ for $x = p$ (Value matching condition) $(46b)$

$V'(x) = -1 \;\; for \; x = p$ (Smooth pasting condition) $(46c)$

$V(x) > (K - x)^+ \;\; for \; x > p$ (Continuation region) $(46d)$

$V(x) = (K - x)^+ \;\; for \; 0 < x < p$ (Value matching condition) $(46e)$

From the condition (46a), we obtain,

$$rx\frac{\partial V}{\partial x}+\frac{\sigma^2 x^2}{2}\frac{\partial^2 V}{\partial x^2}=rV \tag{47}$$

This can be modified to become

$$\frac{\sigma^2}{2}x^2V''(x)+rxV'(x)-rV(x)=0 \tag{48}$$

The general solution of equation (48) above is

$$V(x)=K_1x^{\omega_+}+K_2x^{\omega_-} \tag{49}$$

As far as this problem is concerned, we need to know the values of ω_+ and ω_-. Comparing equations (22) and (48), we see that $q=\sigma, p=r,$ and $\mu=r$. Hence we obtain

$$\begin{aligned}\omega_\pm &= \frac{(\sigma^2-2r)\pm\sqrt{(2r-\sigma)^2+8\sigma^2 r}}{2\sigma^2}\\ &= \frac{(\sigma^2-2r)\pm(2r+\sigma^2)}{2\sigma^2}\\ &= 1\,,\quad \frac{-2r}{\sigma^2}\end{aligned} \tag{50}$$

We then set $\omega_+=1$ and $\omega_-=\frac{-r}{A}$ where $A=\frac{\sigma^2}{2}$.

Note that it is clear that $\omega_-<0$, since both $r>0$ and $\sigma>0$.

The general solution in equation (49) becomes

$$V(x)=K_1x+K_2x^{\frac{-r}{A}} \tag{51}$$

In arbitrage free market, the option price of American option is $V(x)=\leq K$ and $x>0$ Capinski and Zastawniak (2003). This implies that the solution in equation (51) should be bounded. That is $K_1x+K_2x^{\frac{-r}{A}}\leq K$. When x gets very large, the first term tends to ∞ for $K_1>0$ and $-\infty$ for $K_1<0$. This means that the function in equation (51) is no longer bounded. Hence for it to be bounded, K_1 must be zero.

Remark 1

The second term is finite for all values of x.

From the fact above, the solution in equation (51) becomes

$$V(x)=K_2x^{\frac{-r}{A}} \tag{52}$$

Where K_2 is a constant to be determined.

Since the term with the root ω_+cancels, the optimal value of p is obtained by replacing ω_+ with ω_- from equation (30). Hence

$$\begin{aligned} p &= \frac{(K-x)^+\omega_-}{\frac{d}{dx}(K-x)^+} \\ &= -(\omega_-)(K-x)^+ \end{aligned} \tag{53}$$

Applying condition (46b), we obtain

$$\begin{aligned} p &= -(\omega_-)(K-p) \\ &= \frac{-K\omega_-}{1-\omega_-} \\ &= \frac{K\frac{r}{A}}{1+\frac{r}{A}} \\ &= \frac{K}{1+\frac{A}{r}} \end{aligned} \tag{54}$$

Remark 2

Recall that it was assumed that $p \in (0, K)$. This is the optimal stock price to exercise the option. This shows that the optimal stock price depends on the strike price K and the volatility σ since$A = \frac{\sigma^2}{2}$. That is, if the volatility σ and the strike p rice K in the perpetual American put options are known in advance, we could be able to find the specific value of p.

From the equations (45) and (54), the optimal stopping time becomes

$$\tau_* = \min\left\{t \geq 0 : X_t \leq \frac{K}{1+\frac{A}{r}}\right\} \tag{55}$$

At this point, we need to find the value of $V(x)$. To do this, we find the value of K_2 at this particular p. From equation (52), we have

$$V'(x) = \frac{-r}{A} K_2 x^{\left(\frac{-r}{A}-1\right)} \tag{56}$$

Applying the condition in equation (46c), we obtain

$$\frac{r}{A} K_2 p^{\left(\frac{-r}{A}-1\right)} = 1 \tag{57}$$

which gives,

$$K_2 = \frac{A}{r} p^{\left(1+\frac{r}{A}\right)}$$

Using equation (54) we have that

$$K_2 = \frac{A}{r}\left(\frac{K}{1+\frac{r}{A}}\right)^{\left(1+\frac{r}{A}\right)} \tag{58}$$

Substituting equation (58) into equation (52), we obtain

$$V(x) = x^{\frac{-r}{A}}\frac{A}{r}\left(\frac{K}{1+\frac{r}{A}}\right)^{\left(1+\frac{r}{A}\right)} \tag{59}$$

Observe that the first three boundaries given in equations (46a – c) have been applied so far to get us to equation (59), they result into$x \geq p$. From the last two represented by equations (46d – e), we see that $V(x) = K - x$ for $0 < x \leq p$.

If we apply all the five boundaries, we see that

$$V(x) = \begin{cases} x^{\frac{-r}{A}}\frac{A}{r}\left(\frac{K}{1+\frac{r}{A}}\right)^{\left(1+\frac{r}{A}\right)} & for\ x \geq p \\ (K - x) & for \quad 0 < x \leq p \end{cases} \tag{60}$$

The equation (60) above is the optimal free arbitrage price for the perpetual American put which can only be determined if the optimal value of p is known.

Conflict of Interests

The author declares that there is no conflict of interests.

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