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MARKOVIAN QUEUEING INVENTORY SYSTEM WITH TWO STAGE WORKING VACATIONS

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Abstract. This paper studies the behaviour a of two stage working vacations of a single server on Markovian queueing inventory system. The server starts taking two successive working vacations whenever the stock level and the customer level in the queue are both zero. The service rate in the 1^{st} stage working vacation is considered to be lesser than the service rate in regular busy period but greater than the service rate in the 2^{nd} stage working vacation. The nature of the inventory is perishable and the replenishment policy of any order is (s,Q). The steady state joint distribution of the inventory level, the status of the single server and the number of customers in the queue are obtained. Various characteristics of system performance in the steady state are derived and the long-run expected total cost rate is estimated. Several events with numerical examples, which provides the optimal behaviour of the system are presented.

Keywords: perishable goods; two stage working vacations; Markovian inventory; (s,Q) policy.

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1. INTRODUCTION

In the current scenario, while marketing a product, providing good service is mandatory. Service in this case, indicates the demonstration of the pros and cons, utility of the product etc. It also deals by adding convenience with additional accessories to the product purchased by the consumers indeed for practical utility like comfortability, luxury, premium security and deluxe or executive quality. At the time of servicing, the customer may be unaware of the technological aspects of the product they are purchasing.The customers lack of technological awareness results in loss of time and insufficient space which results in losing a customer. So it is one of the critical point to be considered. By increasing the rate of service, the waiting time of consumers and the losing of a customer shall be gradually reduced. Hence the revenue and the brand name of the institution shall be improved linearly.

The time required for servicing shall be planned by calculating the demand of consumers, rate of flow as well as the inventory of the product. Due to the concepts of inappropriate servicing at any period of time, the server may get exhausted rapidly. Hence for the necessary maintenance of server efficiency, up gradation of servers performance and channelizing the service, there is a need of adequate time period for the server. During vacation period, the server serves the customer at a lower service rate rather then completely stopping the service. This slow rate of servicing time is one of the classical vacation policies which is called as a working vacation in the queueing-inventory theory.

Servi and Finn [1], has first introduced the concept of working vacation with queueing model. They considered independent exponential distributions of inter arrival time, vacation times and servicing times during regular or working vacation. Tian and Zhang [2] approached the matrix analytic technique for analyzing the working vacation queues. Considering the results obtained from the investigations of Servi and Finn [1], Yutaka Baba [3], Wu and Takagi [4], Tian and Zhang [5], Sreenivasan et al [6] the basic principles of working vacations should be clearly understood. Banik et.al. [7] tested multiple working vacations with finite capacity $G1/M/1$. Banik derived some important system performance measures such as expected waiting time of a customer and the probability of balking etc. In the extension of Servi and Finn [1], Tian van do [8] discussed the working vacation queuing system along with retrial demand of service.

Further, Kathiresan et al [9] improved and investigated the working vacation of the system along with inventory and retrial demand. The system has analysed its characteristics such as expected inventory level, expected re-order rate and successful retrial rate based on the matrix geometric approach under steady state conditions. Jeganathan [11] studied queueing-inventory system with priority customers and multiple working vacations. Vijaya Laxmi and Soujanya [12] analyzed the retrial inventory model with MAP arrival and multiple working vacations. They considered independent exponential distributions of life time of an item and the lead time of reorder. Customers whose arrival during the stock out period or server busy period either enter the finite orbit or lost.

Till date, in a single server multiple working vacations, the service rate in busy period and different service rate in multiple working vacations have been theoretically explained. However, in real life situations, there is a chance with multiple rate of services in multiple working vacations. This is considered to be a server with multi stage working vacations. To explain the model, a simple real-life application is given. For example, chief dietician of a health care unit shall do the diet therapy in an efficient manner without any time delay during the regular period inspite of the increase in patients. If the arrival of patients is less, then the chief dietician role is partial and as per the guidelines, the duty dietician will take care of the patients, hence the rate of service is lesser obviously in case of the first stage working vacation. Further, in the same case of patients, as per the consultation of chief dietician and guidelines of duty dietician, the dietician inturn will do the task hence the service rate shall be greatly reduced in the second stage of working vacation. For any stage of working vacations, the chief dietician may also be involved in other duties, like, reviewing the patients health, involving in research, updating the facilities of clinic, review the clinic management etc. Hence, some researches have been conducted by various researchers based on multiple stages working vacations in queuing theory. This vacation policy is also known as multiple adaptive working vacations.

Tian and Zhang [5] analysed the advantages of multiple adaptive working vacations in $M/G/1$ queues with various cost structure of the set up cost and the cost of speeding up service rate. Wei Sun et al [10] studied the behaviour of double adaptive working vacations with balking strategies which is discussed in detail in single server Markovian queues. In this model, at the end of first stage working vacation, second stage working vacation is considered with much slower service rates than that of the first stage working vacation when the customers level and the inventory level are zero. Otherwise he starts with normal service period. After completion of second stage working vacation, the server either stays idle or goes to a regular period.

The main aim of this paper is to analyze the two stage working vacations of single server queueing system along with perishable inventory. The rest of the paper is structured as follows. An elaborate mathematical description of this model followed by mathematical notations is given in the next section. Under the steady state conditions, equations and solutions of the system are discussed in section 3. In section 4 measures of various system performances are derived. Finally, the numerical evidences are obtained for showing the advantages of the system performance especially the optimality of the system has been reached.

2. Model Description

We consider a Markovian queuing-inventory model consisting of a single server with a two stage working vacations with the maximum stock capacity of S units. Assume that the mode of arrival of customers belongs to Poisson process with rate λ and the demand of service in regular period follows an exponential distribution with service rate μ_1. The waiting hall for customers with finite capacity is assumed as $M \geq 1$. The arriving customer is considered as lost when the waiting hall is full. Assumptions has been made that the products are of perishable nature and follows exponential distribution with perishable rate γ. According to the ordering policy of (s,Q), $Q(=S-s)$ products are placed for any order and the re order level is s units $(Q>s)$. The replenishment time of an order is assumed to be exponentially distributed with rate β.

(1) After completion of any service in regular period,
   - The server is idle in regular period if any one of them (customers in the waiting hall and inventory items) are empty.
   - The server directly goes to the 1^{st} stage working vacation if both of them (customers in the waiting hall and inventory items) are empty.

(2) Activities of Server in the First Stage Working Vacation:

- Suppose the server is in the 1^{st} stage working vacation, then he is busy with the slower rate of service μ_2 $(i.e., \mu_2 < \mu_1)$ if atleast one customer and one item are available in the system.
- After completion of any service in the 1^{st} stage working vacation, the server goes to regular period if at least one of them is available. This is called as working vacation interruption in queueing-inventory theory. Otherwise(both are empty), he continues idle state in the 1^{st} stage working vacation.
- After completion of the 1^{st} stage working vacation, the server goes to regular period if atleast one of them is available. Otherwise, he goes to 2^{nd} stage working vacation. Activities of system are illustrated in Fig 1 and the transition diagram of regular period to the 1^{st} stage working vacation and vice versa can be illustrated in Fig 2.

(3) Activities of Server in the Second Stage Working Vacation:

- Suppose the server is in the 2^{nd} stage working vacation, the server is busy with service at the rate μ_3 which is lower than μ_2 as well as μ_1 if both customers and items are available. After completion of any service in the 2^{nd} stage working vacation, the server goes to regular period if at least one of them is available. Otherwise, the server continues idle state in the 2^{nd} stage working vacation.
- After completion of the 2^{nd} stage working vacation, the server goes to regular period if at least one of them is available. Otherwise, the server takes another working vacation. The rate of completion of any stage working vacation is θ. The transition of the 1^{st} stage working vacation to the 2^{nd} stage working vacation and the 2^{nd} stage working vacation to regular period is illustrated in Fig 3.

  Figures 2 and 3 represents the state-transition-rate diagram of this model. Here, we assume that $S = 4, s = 1$ and $M = 2$.

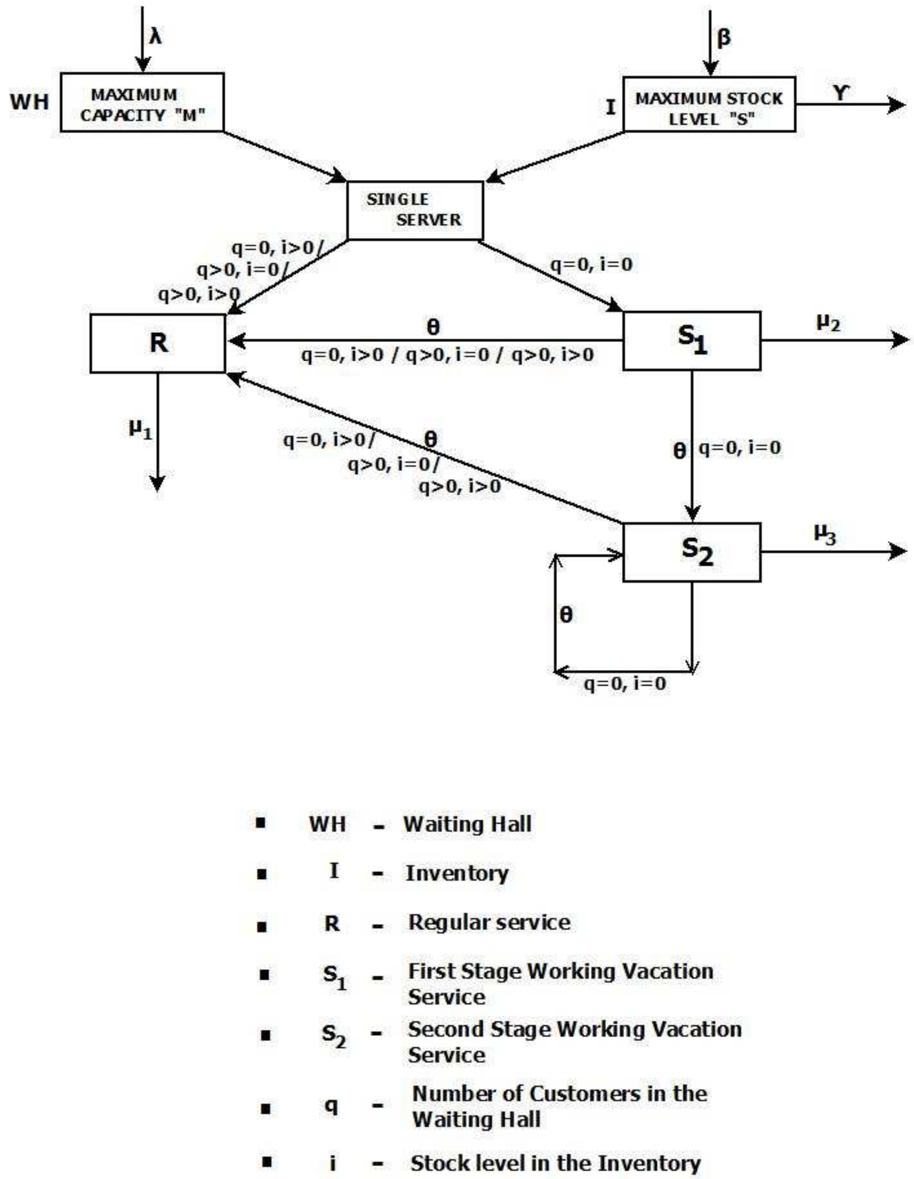

FIGURE 1. Activities of the System

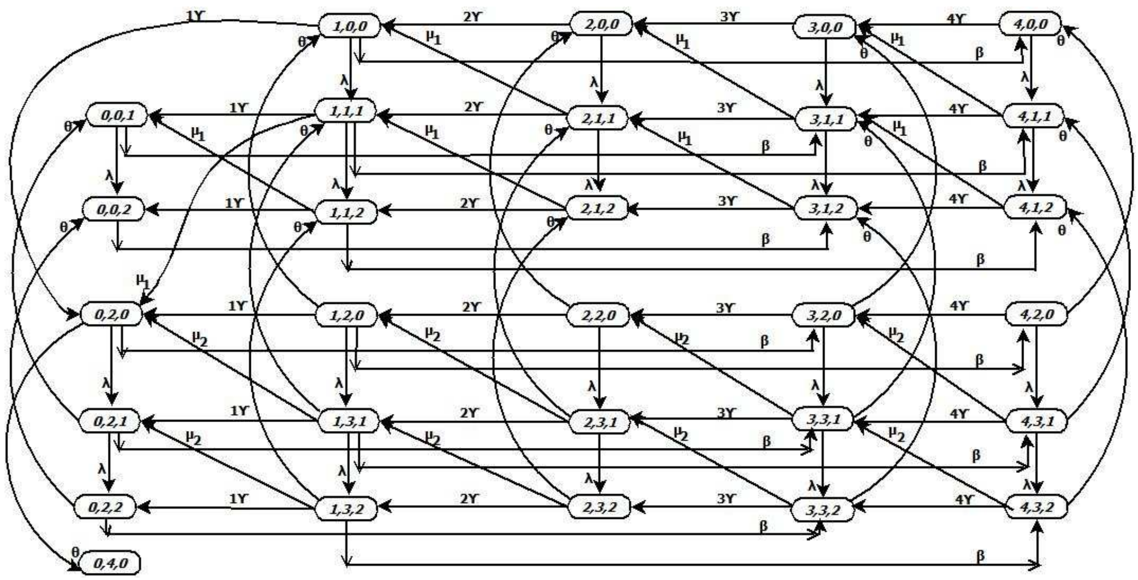

FIGURE 2. Transition diagram of Regular Period to 1^{st} Stage Working Vacation and Vice Versa

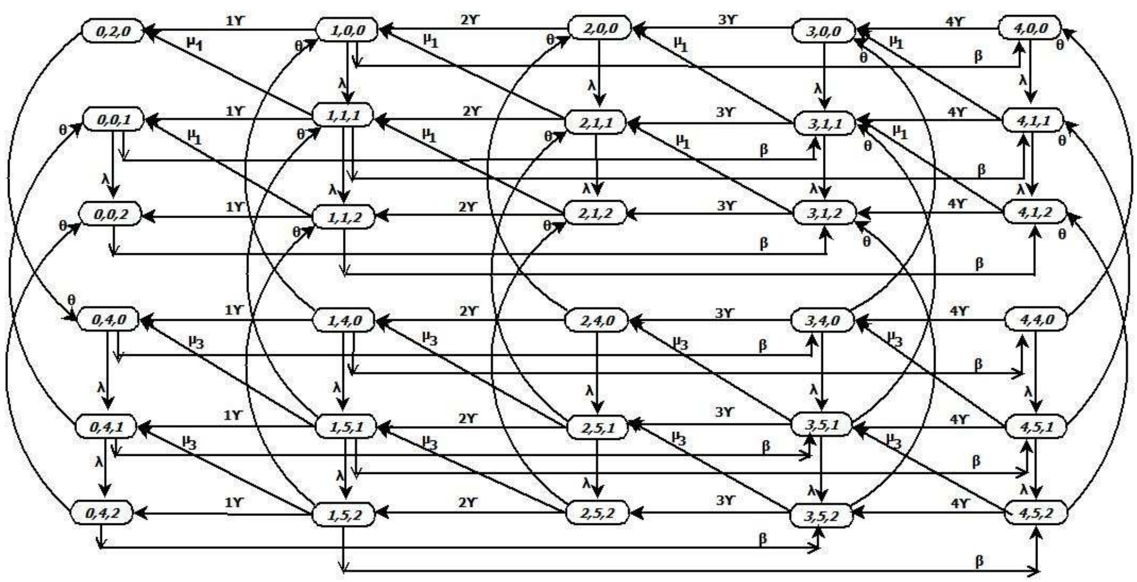

FIGURE 3. Transition diagram of the 1^{st} Stage Working Vacation to the 2^{nd} Stage Working Vacation and 2^{nd} Stage Working Vacation to Regular Period

2.1. Notations:

$$\delta_{ab} = \begin{cases} 1 & \text{if} \quad b = a \\ 0 & \text{otherwise} \end{cases}$$

$$Y(t) = \begin{cases} 0, & \text{if the server is idle in the regular period at time } t, \\ 1, & \text{if the server is busy in the regular period at time } t, \\ 2, & \text{if the server is idle in the first stage working vacation at time } t, \\ 3, & \text{if the server is busy in the first stage working vacation at time } t, \\ 4, & \text{if the server is idle in the second stage working vacation at time } t, \\ 5, & \text{if the server is busy in the second stage working vacation at time } t, \end{cases}$$

$$H(a) = \begin{cases} 1 & \text{if } a \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\bigwedge_{i=j}^{k} x_i = \begin{cases} x_j x_{j-1} \cdots x_k & \text{if } j \geq k \\ 1 & \text{if } j < k \end{cases}$$

$$\bar{\delta}_{ab} = 1 - \delta_{ab}$$

3. Analysis of the Models

Let $L_1(t)$, $L_2(t)$ and $L_3(t)$ respectively, denote the on-hand inventory level(including servicing item), status of the server and the number of customers in the waiting line(including servicing customer) at time t, can be defined as follows:

$$L_1(t) \in \{0, 1, 2, \ldots, s, s+1, \ldots, S\},$$
$$L_2(t) \in \{0, 1, 2, 3, 4, 5\},$$
$$L_3(t) \in \{0, 1, 2, \ldots, M\}$$

Then, the performance of this Markov chain model can be expressed by triplet stochastic process $\{X(t) = (L_1(t), L_2(t), L_3(t)), t \geq 0\}$, with finite discrete state space $F = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$ is given by

$$
\begin{aligned}
E_1 &= \{0, 0, 1, 2, \ldots, M\} \\
E_2 &= \{1, 2, \ldots, S, 0, 0\} \\
E_3 &= \{1, 2, \ldots, S, 1, 1, 2, \ldots, M\} \\
E_4 &= \{0, 2, 0, 1, 2, \ldots, M\} \\
E_5 &= \{1, 2, \ldots, S, 2, 0\} \\
E_6 &= \{1, 2, \ldots, S, 3, 1, 2, \ldots, M\} \\
E_7 &= \{0, 4, 0, 1, 2, \ldots, M\} \\
E_8 &= \{1, 2, \ldots, S, 4, 0\} \\
E_9 &= \{1, 2, \ldots, S, 5, 1, 2, \ldots, M\}
\end{aligned}
$$

As lexicographically, the sets of discrete state space can be arranged in an order as $\{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \ldots, \langle S\rangle\}$, and build its infinitesimal generator matrix denoted by Δ. Without difficulty, this infinitesimal generator matrix can be exhibited as in terms of block partitioned matrix $[\Delta]_{a_1 b_1}$ as given below :

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccc} \langle S\rangle & \langle S-1\rangle & \cdots & \langle Q+1\rangle & \langle Q\rangle & \cdots & \langle s\rangle & \langle s-1\rangle & \cdots & \langle 1\rangle & \langle 0\rangle \end{array} \\
\begin{array}{c} \langle S\rangle \\ \langle S-1\rangle \\ \vdots \\ \langle Q+1\rangle \\ \langle Q\rangle \\ \vdots \\ \langle s\rangle \\ \langle s-1\rangle \\ \vdots \\ \langle 1\rangle \\ \langle 0\rangle \end{array} &
\left(\begin{array}{ccccccccccc}
\Sigma_S & \Omega_S & & & & & & & & & \\
 & \Sigma_{S-1} & \Omega_{S-1} & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & \\
 & & & \Sigma_{Q+1} & \Omega_{Q+1} & & & & & & \\
 & & & & \Sigma_Q & \ddots & & & & & \\
 & & & & & \ddots & \Omega_{s+1} & & & & \\
\Theta & & & & & & \Sigma_s & \Omega_s & & & \\
 & \Theta & & & & & & \Sigma_{s-1} & \Omega_{s-1} & & \\
 & & \ddots & & & & & & \ddots & \ddots & \\
 & & & \Theta & & & & & & \Sigma_1 & \Omega_1 \\
 & & & & \Theta_1 & & & & & & \Sigma_0
\end{array}\right)
\end{array}
$$

where the sub matrices of $\triangle$ are given below,

$$[\Theta_1]_{a_2b_2} = \begin{cases} D_0,\ b_2 = 1,\ a_2 = 0 \\ D_1,\ b_2 = a_2,\ a_2 = 2,4 \\ D_2,\ b_2 = a_2+1,\ a_2 = 2,4 \\ \mathbf{0},\ \text{Otherwise.} \end{cases}$$

where

$$[D_0]_{a_3b_3} = \begin{cases} \beta,\ b_3 = a_3,\ a_3 = 1,2,....M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[D_1]_{a_3b_3} = \begin{cases} \beta,\ b_3 = 0,\ a_3 = 0 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[D_2]_{a_3b_3} = \begin{cases} \beta,\ b_3 = a_3,\ a_3 = 1,2,....M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$\Theta = \beta \times I_{\left(3(M+1)\times 3(M+1)\right)}$$

$$[\Omega_1]_{a_2b_2} = \begin{cases} F_0,\ \ b_2 = 2,\ a_2 = 0,2 \\ F_1,\ \ b_2 = 0,\ a_2 = 1 \\ F_2,\ \ b_2 = 2,\ a_2 = 1 \\ F_3,\ \ b_2 = 2,\ a_2 = 3 \\ F_4,\ \ b_2 = a_2,\ a_2 = 4 \\ F_5,\ \ b_2 = 4,\ a_2 = 5 \\ \mathbf{0},\ \text{Otherwise.} \end{cases}$$

where

$$[F_0]_{a_3b_3} = \begin{cases} \gamma,\ \ b_3 = 0,\ a_3 = 0 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[F_1]_{a_3b_3} = \begin{cases} \gamma,\ b_3 = a_3,\ a_3 = 1,2,....M \\ \mu_1,\ b_3 = a_3-1,\ a_3 = 2,3,...M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[F_2]_{a_3b_3} = \begin{cases} \mu_1,\ b_3 = 0,\ a_3 = 1 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[F_3]_{a_3b_3} = \begin{cases} \gamma,\ b_3 = a_3,\ a_3 = 1,2,....M \\ \mu_1,\ b_3 = a_3 - 1,\ a_3 = 1,2,3,...M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[F_4]_{a_3b_3} = \begin{cases} \gamma,\ \ b_3 = 0,\ a_3 = 0 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[F_5]_{a_3b_3} = \begin{cases} \gamma,\ b_3 = a_3,\ a_3 = 1,2,....M \\ \mu_3,\ b_3 = a_3 - 1,\ a_3 = 1,2,....M \\ 0,\ \text{Otherwise.} \end{cases}$$

For $a_1 = 2,3,....S$

$$[\Omega_{a_1}]_{a_2b_2} = \begin{cases} G_{a_1},\ b_2 = a_2,\ a_2 = 0,2,4 \\ H_1,\ b_2 = 0,\ a_2 = 1 \\ H_2,\ b_2 = 2,\ a_2 = 3 \\ H_3,\ b_2 = 4,\ a_2 = 5 \\ J_{a_1},\ b_2 = a_2,\ a_2 = 1 \\ K_{a_1},\ b_2 = a_2,\ a_2 = 3 \\ L_{a_1},\ b_2 = a_2,\ a_2 = 5 \\ \mathbf{0},\ \text{Otherwise.} \end{cases}$$

where

$$[G_{a_1}]_{a_3b_3} = \begin{cases} a_1\gamma,\ b_3 = 0,\ a_3 = 0 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[H_1]_{a_3b_3} = \begin{cases} \mu_1,\ b_3 = 0,\ a_3 = 1 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[H_2]_{a_3b_3} = \begin{cases} \mu_2,\ b_3 = 0,\ a_3 = 1 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[H_3]_{a_3b_3} = \begin{cases} \mu_3,\ b_3 = 0,\ a_3 = 1 \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[J_{a_1}]_{a_3b_3} = \begin{cases} a_1\gamma,\ b_3 = a_3,\ a_3 = 1,2,....M \\ \mu_1,\ b_3 = a_3 - 1,\ a_3 = 2,3,...M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[K_{a_1}]_{a_3b_3} = \begin{cases} a_1\gamma,\ b_3 = a_3,\ a_3 = 1,2,....M \\ \mu_2,\ b_3 = a_3 - 1,\ a_3 = 2,3,...M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[L_{a_1}]_{a_3b_3} = \begin{cases} a_1\gamma,\ b_3 = a_3,\ a_3 = 1,2,....M \\ \mu_3,\ b_3 = a_3 - 1,\ a_3 = 2,3,...M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[\Sigma_0]_{a_2b_2} = \begin{cases} M_{a_2},\ b_2 = a_2,\ a_2 = 0,2,4 \\ M_1,\ b_2 = 0,\ a_2 = 2,4 \\ M_3,\ b_2 = 4,\ a_2 = 2 \\ \mathbf{0},\ \text{Otherwise.} \end{cases}$$

where

$$[M_0]_{a_3b_3} = \begin{cases} \lambda,\ b_3 = a_3 + 1,\ a_3 = 1,2,\ldots,M-1 \\ -\big(\lambda+\beta\big),\ b_3 = a_3,\ a_3 = 1,2,\ldots,M-1 \\ -\beta,\ b_3 = a_3,\ a_3 = M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[M_2]_{a_3b_3} = \begin{cases} \lambda,\ b_3 = a_3 + 1,\ a_3 = 0,1,2,\ldots,M-1 \\ -\big(\lambda+\beta+\theta\big),\ b_3 = a_3,\ a_3 = 0,1,2,\ldots,M-1 \\ -\big(\beta+\theta\big),\ b_3 = a_3,\ a_3 = M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[M_4]_{a_3b_3} = \begin{cases} \lambda,\ b_3 = a_3 + 1,\ a_3 = 0,1,2,\ldots,M-1 \\ -\big(\lambda+\beta\big),\ b_3 = a_3,\ a_3 = 0 \\ -\big(\lambda+\beta+\theta\big),\ b_3 = a_3,\ a_3 = 1,2,\ldots,M-1 \\ -\big(\beta+\theta\big),\ b_3 = a_3,\ a_3 = M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[M_1]_{a_3b_3} = \begin{cases} \theta,\ b_3 = a_3,\ a_3 = 1,2,\ldots,M \\ 0,\ \text{Otherwise.} \end{cases}$$

$$[M_3]_{a_3b_3} = \begin{cases} \theta, \; b_3 = a_3, \; a_3 = 0 \\ 0, \; \text{Otherwise.} \end{cases}$$

For $a_1 = 1,2,3,\ldots,s,s+1,\ldots,S$

$$[\Sigma_{a_1}]_{a_2b_2} = \begin{cases} N_{a_1}, \; b_2 = 0, \; a_2 = 0 \\ V, \; b_2 = 0, \; a_2 = 2,4 \\ V_1, \; b_2 = a_2+1, \; a_2 = 0,2,4 \\ W_{a_1}, \; b_2 = a_2, \; a_2 = 1 \\ R_{a_1}, \; b_2 = a_2, \; a_2 = 3 \\ T_{a_1}, \; b_2 = a_2, \; a_2 = 5 \\ U_{a_1}, \; b_2 = a_2, \; a_2 = 2,4 \\ X, \; b_2 = 1, \; a_2 = 3,5 \\ \mathbf{0}, \; \text{Otherwise.} \end{cases}$$

where

$$[N_{a_1}]_{a_3b_3} = \begin{cases} -\big(\lambda + H(s-a_1)\beta + a_1\gamma\big), \; b_3 = a_3, \; a_3 = 0 \\ 0, \; \text{Otherwise.} \end{cases}$$

$$[V]_{a_3b_3} = \begin{cases} \theta, \; b_3 = a_3, \; a_3 = 0 \\ 0, \; \text{Otherwise.} \end{cases}$$

$$[V_1]_{a_3b_3} = \begin{cases} \lambda, \; b_3 = 1, \; a_3 = 0 \\ 0, \; \text{Otherwise.} \end{cases}$$

$$[W_{a_1}]_{a_3b_3} = \begin{cases} \lambda, \; b_3 = a_3+1, \; a_3 = 1,2,\ldots,M-1 \\ -\big(\lambda + H(s-a_1)\beta + a_1\gamma + \mu_1\big), \; b_3 = a_3, \; a_3 = 1,2,\ldots,M-1 \\ -\big(H(s-a_1)\beta + a_1\gamma + \mu_1\big), \; b_3 = a_3, \; a_3 = M \\ 0, \; \text{Otherwise.} \end{cases}$$

$$[R_{a_1}]_{a_3b_3} = \begin{cases} \lambda, \; b_3 = a_3+1, \; a_3 = 1,2,\ldots,M-1 \\ -\big(\lambda + H(s-a_1)\beta + a_1\gamma + \theta + \mu_2\big), \; b_3 = a_3, \; a_3 = 1,2,\ldots,M-1 \\ -\big(H(s-a_1)\beta + \theta + a_1\gamma + \mu_2\big), \; b_3 = a_3, \; a_3 = M \\ 0, \; \text{Otherwise.} \end{cases}$$

$$[T_{a_1}]_{a_3b_3} = \begin{cases} \lambda, \ b_3 = a_3+1, \ a_3 = 1,2,\ldots,M-1 \\ -\big(\lambda + H(s-a_1)\beta + a_1\gamma + \theta + \mu_3\big), \ b_3 = a_3, \ a_3 = 1,2,\ldots,M-1 \\ -\big(H(s-a_1)\beta + \theta + a_1\gamma + \mu_3\big), \ b_3 = a_3, \ a_3 = M \\ 0, \ \text{Otherwise.} \end{cases}$$

$$[U_{a_1}]_{a_3b_3} = \begin{cases} -\big(\lambda + H(s-a_1)\beta + a_1\gamma + \theta\big), \ b_3 = 0, \ a_3 = 0 \\ 0, \ \text{Otherwise.} \end{cases}$$

$$[X]_{a_3b_3} = \begin{cases} \theta, \ b_3 = a_3, \ a_3 = 1,2,\ldots,M \\ 0, \ \text{Otherwise.} \end{cases}$$

where the dimension of sub matrices of Δ is given by,

Matrices	Dimensions
Σ_0	$(3M+2)\times(3M+2)$
Ω_1	$3(M+1)\times(3M+2)$
Θ_1	$(3M+2)\times 3(M+1)$
$\Theta, \Sigma_{a_1}\ (a_1 = 1,2,\ldots,S), \Omega_{a_1}\ (a_1 = 1,2,\ldots,S)$	$3(M+1)\times 3(M+1)$
D_2, M_2, M_4, M_3	$(M+1)\times(M+1)$
D_1	$(M+1)\times 1$
F_0, F_4	$1\times(M+1)$
M_1	$(M+1)\times M$
F_2, F_3, F_5	$M\times(M+1)$
$D_0, F_1, J_{a_1}, K_{a_1}, L_{a_1}\ M_0, R_{a_1}, T_{a_1}, W_{a_1}, X$	$M\times M$
H_1, H_2, H_3	$M\times 1$
V_1	$1\times M$
$G_{a_1}, N_{a_1}, U_{a_1}, V$	1×1

3.1. Steady state analysis. It can be observed from the structure of matrix Δ that the time homogeneous Markov chain, $\{X(t) = (L_1(t), L_2(t), L_3(t)) : t \geq 0\}$, with the finite discrete state space F is irreducible, aperiodic and persistent non-null. Therefore the limiting distribution exists

$\Pi^{(a_1,a_2,a_3)} = \lim_{t\to\infty} Pr[\ L_1(t) = a_1, \ L_2(t) = a_2, \ L_3(t) = a_3 |\ L_1(0), \ L_2(0), \ L_3(0)\ ]$ and is independent of the initial state.

Let $\Pi = (\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(S-1)}, \Pi^{(S)})_{1\times(S+1)}$, each vector $\Pi^{(a_1)}$ being partitioned as follows:

For $a_1 = 0$, $\Pi^{(0)}$ is a vector of dimension $1 \times (3M+2)$.

$$\begin{aligned}\Pi^{(0)} &= (\Pi^{(0,0,1)}, \Pi^{(0,0,2)}, \ldots, \Pi^{(0,0,M)}, \Pi^{(0,2,0)}, \Pi^{(0,2,1)}, \Pi^{(0,2,2)}, \ldots, \\ &\qquad \Pi^{(0,2,M)}, \Pi^{(0,4,0)}, \Pi^{(0,4,1)}, \Pi^{(0,4,2)}, \ldots, \Pi^{(0,4,M)})_{1\times(3M+2)}\end{aligned}$$

For $a_1 = 1, 2, \ldots, S$, $\Pi^{(a_1)}$ is a vector of dimension $1 \times 3(M+1)$.

$$\begin{aligned}\Pi^{(a_1)} &= (\Pi^{(a_1,0,0)}, \Pi^{(a_1,1,1)}, \Pi^{(a_1,1,2)}, \ldots, \Pi^{(a_1,1,M)}, \Pi^{(a_1,2,0)}, \Pi^{(a_1,3,1)}, \Pi^{(a_1,3,2)}, \ldots, \\ &\qquad \Pi^{(a_1,3,M)}, \Pi^{(a_1,4,0)}, \Pi^{(a_1,5,1)}, \Pi^{(a_1,5,2)}, \ldots, \Pi^{(a_1,5,M)})_{1\times3(M+1)}\end{aligned}$$

The steady state probability $\Pi^{(a_1,a_2,a_3)}$ for the state (a_1,a_2,a_3) of the continuous time Markov chain satisfies the following equations:

$$\Pi\Delta = 0 \tag{1}$$

$$\Pi\mathbf{e} = \sum\sum\sum_{(a_1,a_2,a_3)} \Pi^{(a_1,a_2,a_3)} = 1, \tag{2}$$

The equation (1) gives the following set of equations

$$\begin{aligned}\Pi^{(a_1)}\Sigma_{a_1} + \Pi^{(a_1+1)}\Omega_{a_1+1} &= 0, \qquad a_1 = 0,1,2,\ldots,Q-1, \\ \Pi^{(a_1-Q)}\Theta_1 + \Pi^{(a_1)}\Sigma_{a_1} + \Pi^{(a_1+1)}\Sigma_{a_1+1} &= 0, \qquad a_1 = Q, \\ \Pi^{(a_1-Q)}\Theta + \Pi^{(a_1)}\Sigma a_1 + \Pi^{(a_1+1)}\Omega_{a_1+1} &= 0, \qquad a_1 = Q+1, Q+2, \ldots, S-1, \\ \Pi^{(a_1)}\Sigma_{a_1} + \Pi^{(a_1-Q)}\Theta &= 0, \qquad a_1 = S,\end{aligned}$$

By solving the above set of equations, we get the required solution. The solution of these equations can be conveniently expressed as:

$$\Pi^{(a_1)} = \Pi^{(Q)}\Lambda_{a_1}, \quad a_1 = 0,1,2,\ldots,S.$$

where

$$
\begin{aligned}
\Lambda_{a_1} &= (-1)^{Q-a_1} \overset{a_1+1}{\underset{j=Q}{\Lambda}} \Omega_j \Sigma_{j-1}^{-1}, && a_1 = 0,1,\ldots,Q-2,Q-1 \\
&= (-1)^{2Q-a_1+1} \sum_{j=0}^{S-a_1} \left[\left(\overset{s+1-j}{\underset{k=Q}{\Lambda}} \Omega_k \Theta_{k-1}^{-1} \right) \Theta \Sigma_{S-j}^{-1} \left(\overset{a_1+1}{\underset{l=S-j}{\Lambda}} \Omega_l \Sigma_{l-1}^{-1} \right) \right], \\
& && a_1 = Q+1, Q+2, \ldots, S-1, S \\
&= I, && a_1 = Q
\end{aligned}
$$

Using the following equations, we get the value of $\Pi^{(Q)}$.

$$
\Pi^{(Q)} \left[\left\{ (-1)^{Q} \sum_{j=0}^{s-1} \left[\left(\overset{s+1-j}{\underset{k=Q}{\Lambda}} \Omega_k \Sigma_{k-1}^{-1} \right) \Theta \Sigma_{S-j}^{-1} \left(\overset{Q+2}{\underset{l=S-j}{\Lambda}} \Omega_l \Sigma_{l-1}^{-1} \right) \right] \right\} \Omega_{Q+1} \right.
$$
$$
\left. + \Sigma_Q + \left\{ (-1)^{Q} \overset{1}{\underset{j=Q}{\Lambda}} \Omega_j \Sigma_{j-1}^{-1} \right\} \Theta_1 \right] = 0,
$$

and

$$
\Pi^{Q} \left[\sum_{a_1=0}^{Q-1} \left((-1)^{Q-a_1} \overset{a_1+1}{\underset{j=Q}{\Lambda}} \Omega_j \Sigma_{j-1}^{-1} \right) + I \right.
$$
$$
\left. + \sum_{a_1=Q+1}^{S} \left((-1)^{2Q-a_1+1} \sum_{j=0}^{S-a_1} \left[\left(\overset{s+1-j}{\underset{k=Q}{\Lambda}} \Omega_k \Sigma_{k-1}^{-1} \right) \Theta \Sigma_{S-j}^{-1} \left(\overset{a_1+1}{\underset{l=S-j}{\Lambda}} \Omega_l \Sigma_{l-1}^{-1} \right) \right] \right) \right] \mathbf{e} = 1.
$$

4. SYSTEM PERFORMANCE MEASURES OF THE MODEL

In this section, we establish some system performance measures in the steady state situations which can be used to estimate the total expected cost rate.

(i) Expected Inventory Level $[I_1]$:

$$
I_1 = \sum_{a_1=1}^{S} a_1 \left(\pi^{(a_1,0,0)} + \pi^{(a_1,2,0)} + \pi^{(a_1,4,0)} \right) + \sum_{a_1=1}^{S} \sum_{a_3=1}^{M} a_1 \left(\pi^{(a_1,1,a_3)} + \pi^{(a_1,3,a_3)} + \pi^{(a_1,5,a_3)} \right)
$$

(ii) Expected Reorder Rate $[I_2]$:

$$I_2 = (s+1)\gamma\left(\pi^{(s+1,0,0)} + \pi^{(s+1,2,0)} + \pi^{(s+1,4,0)}\right) + \sum_{a_3=1}^{M}\Bigg[\Big(\big(s+1\big)\gamma + \mu_1\Big)\pi^{(s+1,1,a_3)} + \Big(\big(s+1\big)\gamma + \mu_2\Big)\pi^{(s+1,3,a_3)} + \Big(\big(s+1\big)\gamma + \mu_3\Big)\pi^{(s+1,5,a_3)}\Bigg]$$

(iii)Expected Perishable Rate $[I_3]$:

$$I_3 = \sum_{a_1=1}^{S} a_1\gamma\left(\pi^{(a_1,0,0)} + \pi^{(a_1,2,0)} + \pi^{(a_1,4,0)}\right) + \sum_{a_1=1}^{S}\sum_{a_3=1}^{M} a_1\gamma\left[\pi^{(a_1,1,a_3)} + \pi^{(a_1,3,a_3)} + \pi^{(a_1,5,a_3)}\right]$$

(iv)Expected Number of Customers in the Waiting Hall $[I_4]$:

$$I_4 = \sum_{a_3=1}^{M} a_3\left(\pi^{(0,0,a_3)} + \pi^{(0,2,a_3)} + \pi^{(0,4,a_3)}\right) + \sum_{a_1=1}^{S}\sum_{a_3=1}^{M} a_3\left(\pi^{(a_1,1,a_3)} + \pi^{(a_1,3,a_3)} + \pi^{(a_1,5,a_3)}\right)$$

(v)Expected Number of Customers arrival into the Waiting Hall $[I_5]$:

$$I_5 = \sum_{a_3=1}^{M-1}\lambda\left(\pi^{(0,0,a_3)}\right) + \sum_{a_3=0}^{M-1}\lambda\left(\pi^{(0,2,a_3)} + \pi^{(0,4,a_3)}\right) + \sum_{a_1=1}^{S}\lambda\left(\pi^{(a_1,0,0)} + \pi^{(a_1,2,0)} + \pi^{(a_1,4,0)}\right) + \sum_{a_1=1}^{S}\sum_{a_3=1}^{M-1}\lambda\left(\pi^{(a_1,1,a_3)} + \pi^{(a_1,3,a_3)} + \pi^{(a_1,5,a_3)}\right)$$

(vi)Expected Waiting Time of Customers $[I_6]$:

$$I_6 = \frac{\text{Expected number of customers in the Waiting Hall at time } t}{\text{Effective arrival rate at time } t}$$

(vii)Expected Number of Customers lost $[I_7]$:

$$I_7 = \lambda\left(\pi^{(0,0,M)}+\pi^{(0,2,M)}+\pi^{(0,4,M)}\right)+\sum_{a_1=1}^{S}\lambda\left(\pi^{(a_1,1,M)}+\pi^{(a_1,3,M)}+\pi^{(a_1,5,M)}\right)$$

(viii)Fraction of Time the Server is on First Stage Working Vacation $[I_8]$:

$$I_8=\sum_{a_3=0}^{M}\pi^{(0,2,a_3)}+\sum_{a_1=1}^{S}\pi^{(a_1,2,0)}+\sum_{a_1=1}^{S}\sum_{a_3=1}^{M}\pi^{(a_1,3,a_3)}$$

(ix)Fraction of Time the Server is on Second Stage Working Vacation $[I_9]$:

$$I_9 = \sum_{a_3=0}^{M}\pi^{(0,4,a_3)}+\sum_{a_1=1}^{S}\pi^{(a_1,4,0)}+\sum_{a_1=1}^{S}\sum_{a_3=1}^{M}\pi^{(a_1,5,a_3)}$$

5. COST ANALYSIS AND NUMERICAL EVIDANCES

The following costs are involved in this model which are used to estimate the total expected cost rate per unit of time.

- ch = inventory holding cost/item
- cr = reorder cost/reorder
- cp = cost per item of perishable unit
- cw = cost of waiting per customer
- cl = cost due to loss of customers per unit time

By fixing all parameters except s and S, the objective function of the total expected cost per unit of time can be minimized and it can be easily verified to

$$TC(S,s) = ch\times EI + cr\times ER + cp\times EP + cw\times EW + cl\times EL$$

Due to the complex form of the limiting distribution, it is hard to study the qualitative behaviour of the cost function TC analytically. Hence, a complete computational study of the expected cost function is carried out in the next section.

5.1. Numerical Evidances. For a given finite set values of s, S and M, we have searched a numerical procedure to determine the optimal values of the decision variables s and S. By this procedure, fix any one of the given variables at constant level and the other two variables are allowed to vary for corresponding studies which lead to obtain the optimum values of the two variables. Under these considerations we study the behaviour of cost function $TC(S,s)$ and the other system parameters on the optimal values and the result agreed with what one would expect. Some of our results are presented in Tables 2 through 10 where the lower entry in each cell gives the optimal expected cost rate and the upper entries the corresponding S^* and s^*. The parameters and cost values are fixed for the following evidences from 1 to 5.

$$\lambda = 1,\ \beta = 1,\ \gamma = 0.4,\ \theta = 1,\ \mu_1 = 7,\ \mu_2 = 6,\ \mu_3 = 3,$$
$$c_h = 0.01,\ c_r = 9,\ c_p = 0.06,\ c_w = 9,\ c_l = 7. \tag{3}$$

Evidence 1 : Existence of Optimal Expected Total Costs

In Table 1, the existence of optimal expected total cost and the corresponding optimal maximum inventory level S^* and the reorder level s^* obtained by numerical search technique.

TABLE 1. Total expected cost rate as a function of S and s

S \ s	2	3	4	5	6
71	3.988366	3.929238	**3.924538**	3.950193	3.993909
72	3.987954	3.929042	**3.924311**	3.949768	3.993164
73	3.987699	<u>3.929010</u>	**<u>3.924258</u>**	3.949531	3.992625
74	<u>3.987596</u>	3.929133	**3.924370**	<u>3.949474</u>	3.992282
75	3.987638	3.929406	**3.924642**	3.94 9589	<u>3.992126</u>
76	3.987820	3.929824	**3.925066**	3.949869	3.992151

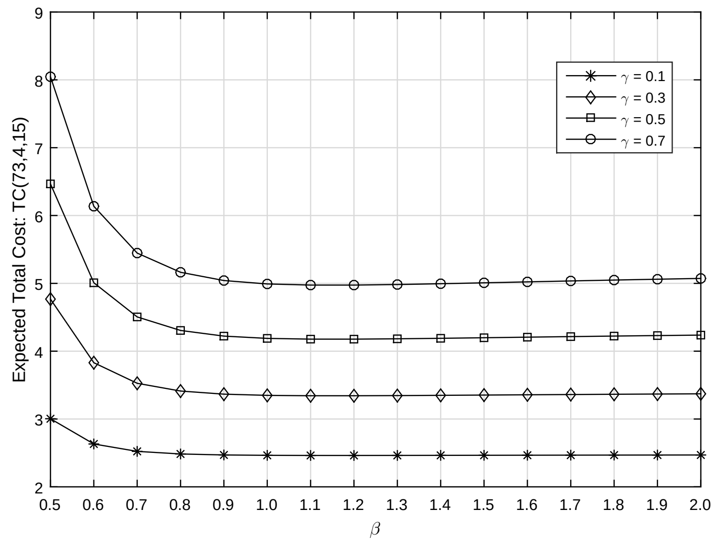

FIGURE 4. TC vs β for different values of γ

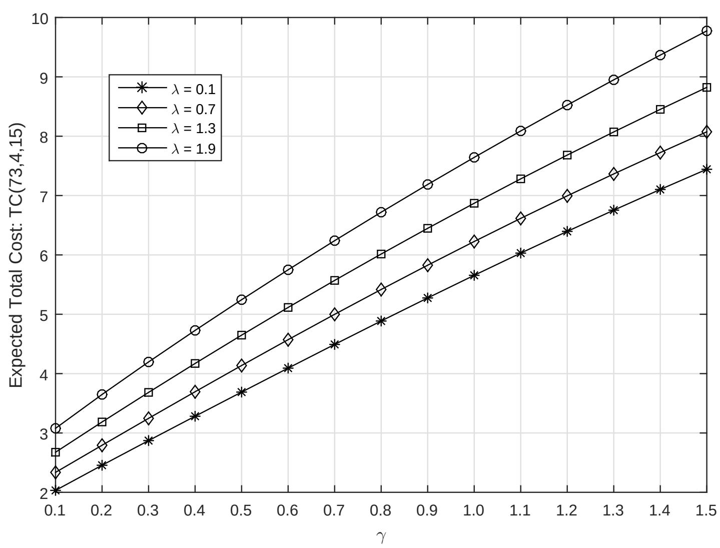

FIGURE 5. TC vs γ for different values of λ

From the Table 1, we attain the minimum value of each row, which is represented in **bold** form, and the minimum value of each column, which is represented underlined. After attaining these values, we can exhibit the local minimum value of the cost function of this table.
The intersection of the row minimum and the column minimum is known as the optimal expected total cost (TC^*) and the corresponding values of S and s are known as the optimal S^* and s^* respectively. In this example, under the given fixed costs and values of parameters, the optimal expected total cost (TC^*) is 3.924258 and the corresponding optimal inventory level S^*

and the reorder level s^* are 73 and 4 respectively.

Evidence 2 : Optimal Expected Total cost Vs Cost values

Here, we study the impact of the holding cost c_h, the setup cost(or)the reorder cost c_r, the perishable cost c_p, the waiting cost c_w, cost due to loss of customers c_l on the optimal values (s^*, S^*) and the corresponding total expected cost rate TC^*.

TABLE 2. Variation in optimal values for different values of c_h and c_r

c_h \ c_r	6		9		12		15		18	
0.01	60	4	73	4	85	3	96	3	106	2
	3.473039		3.924258		4.330799		4.715145		5.077669	
0.02	50	4	61	3	71	3	80	2	88	2
	3.652956		4.128696		4.555325		4.961876		5.332689	
0.03	44	4	53	3	62	3	69	2	76	2
	3.813618		4.305524		4.753682		5.168161		5.556100	
0.04	40	4	48	3	55	3	61	2	68	2
	3.960454		4.466042		4.933314		5.353967		5.757113	
0.05	37	4	43	3	50	2	55	2	61	2
	4.096907		4.614189		5.092286		5.524319		5.941157	

(1) As we anticipated, the rate of expected total cost increases as c_r, c_h, c_p, c_l and c_w increases.

(2) There must be, increase in the maximum inventory level (S) whenever the ordering cost per order(c_r) increases that will reduce the total ordering cost. Hence we obtain values from Tables $(2-4)$ and 6 that the optimal maximum inventory level S^* increases as c_r increases, but the optimal reorder level s^* decreases as c_r increases because the items are of perishable nature.

TABLE 3. Variation in optimal values for different values of c_w and c_r

c_r / c_w	6		9		12		15		18	
7	57	4	70	3	81	2	91	2	103	2
	3.054389		3.492041		3.896060		4.263262		4.618122	
8	58	4	71	3	83	3	94	2	104	2
	3.263893		3.710708		4.115027		4.495178		4.848056	
9	60	4	73	4	85	3	96	3	106	2
	3.473039		3.924258		4.330799		4.715145		5.077669	
10	61	5	74	4	86	3	97	3	108	3
	3.675538		4.130068		4.546272		4.928757		5.297984	
11	62	5	75	4	87	4	99	3	109	3
	3.875329		4.335665		4.754877		5.142172		5.509932	

(3) Whereas Tables (5, 6 and 10) indicates that there is no change in the optimal inventory level S^* and the optimal reorder level s^* as the cost of a customer lost c_l increases. The reason is that the expected number of customers lost is unaltered whenever we increase the level of S and s.

(4) There must be, a decrease in the maximum inventory level (S) as well as the reorder level(s) whenever the holding cost per item(c_h) increases that will reduce the total holding cost. Hence we obtain values from Tables (2, 7 and 8) that the optimal maximum inventory level S^* and the optimal reorder level s^* decreases as c_h increases.

(5) Whenever the cost of perishability of an item (c_p) per unit time increases, decrease the maximum inventory level (S) as well as the reorder level (s) which will reduce the total perishability cost in the long run. Hence we obtain values from Tables $(4,5,$ 8 and 9) that the optimal maximum inventory level S^* and the optimal reorder level s^* decreases as c_p increases.

TABLE 4. Variation in optimal values for different values of c_p and c_r

<table>
<tr><td>c_r
c_p</td><td colspan="2">6</td><td colspan="2">9</td><td colspan="2">12</td><td colspan="2">15</td><td colspan="2">18</td></tr>
<tr><td rowspan="2">0.01</td><td>108</td><td>5</td><td>135</td><td>4</td><td>160</td><td>4</td><td>183</td><td>3</td><td>205</td><td>3</td></tr>
<tr><td colspan="2">2.984408</td><td colspan="2">3.368480</td><td colspan="2">3.716111</td><td colspan="2">4.045758</td><td colspan="2">4.354912</td></tr>
<tr><td rowspan="2">0.06</td><td>60</td><td>4</td><td>73</td><td>4</td><td>85</td><td>3</td><td>96</td><td>3</td><td>106</td><td>2</td></tr>
<tr><td colspan="2">3.473039</td><td colspan="2">3.924258</td><td colspan="2">4.330799</td><td colspan="2">4.715145</td><td colspan="2">5.077669</td></tr>
<tr><td rowspan="2">0.11</td><td>44</td><td>4</td><td>53</td><td>3</td><td>62</td><td>3</td><td>69</td><td>2</td><td>76</td><td>2</td></tr>
<tr><td colspan="2">3.813618</td><td colspan="2">4.305524</td><td colspan="2">4.753682</td><td colspan="2">5.168161</td><td colspan="2">5.556100</td></tr>
<tr><td rowspan="2">0.16</td><td>37</td><td>4</td><td>43</td><td>3</td><td>62</td><td>3</td><td>55</td><td>2</td><td>61</td><td>2</td></tr>
<tr><td colspan="2">4.096907</td><td colspan="2">4.614189</td><td colspan="2">4.753682</td><td colspan="2">5.524319</td><td colspan="2">5.941157</td></tr>
<tr><td rowspan="2">0.21</td><td>32</td><td>3</td><td>37</td><td>3</td><td>42</td><td>2</td><td>47</td><td>2</td><td>52</td><td>2</td></tr>
<tr><td colspan="2">4.332500</td><td colspan="2">4.882931</td><td colspan="2">5.372878</td><td colspan="2">5.830646</td><td colspan="2">6.271873</td></tr>
</table>

TABLE 5. Variation in optimal values for different values of c_p and c_l

<table>
<tr><td>c_l
c_p</td><td colspan="2">7</td><td colspan="2">14</td><td colspan="2">21</td><td colspan="2">28</td><td colspan="2">35</td></tr>
<tr><td rowspan="2">0.01</td><td>135</td><td>4</td><td>135</td><td>4</td><td>135</td><td>4</td><td>135</td><td>4</td><td>135</td><td>4</td></tr>
<tr><td colspan="2">3.368480</td><td colspan="2">3.368487</td><td colspan="2">3.368494</td><td colspan="2">3.368501</td><td colspan="2">3.368507</td></tr>
<tr><td rowspan="2">0.06</td><td>73</td><td>4</td><td>73</td><td>4</td><td>73</td><td>4</td><td>73</td><td>4</td><td>73</td><td>4</td></tr>
<tr><td colspan="2">3.924258</td><td colspan="2">3.924266</td><td colspan="2">3.924274</td><td colspan="2">3.924283</td><td colspan="2">3.924291</td></tr>
<tr><td rowspan="2">0.11</td><td>53</td><td>3</td><td>53</td><td>3</td><td>53</td><td>3</td><td>53</td><td>3</td><td>53</td><td>3</td></tr>
<tr><td colspan="2">4.305524</td><td colspan="2">4.305536</td><td colspan="2">4.305548</td><td colspan="2">4.305561</td><td colspan="2">4.305573</td></tr>
<tr><td rowspan="2">0.16</td><td>43</td><td>3</td><td>43</td><td>3</td><td>43</td><td>3</td><td>43</td><td>3</td><td>43</td><td>3</td></tr>
<tr><td colspan="2">4.614189</td><td colspan="2">4.614202</td><td colspan="2">4.614215</td><td colspan="2">4.614229</td><td colspan="2">4.614242</td></tr>
<tr><td rowspan="2">0.21</td><td>37</td><td>3</td><td>37</td><td>3</td><td>37</td><td>3</td><td>37</td><td>3</td><td>37</td><td>3</td></tr>
<tr><td colspan="2">4.882931</td><td colspan="2">4.882945</td><td colspan="2">4.882959</td><td colspan="2">4.882974</td><td colspan="2">4.882988</td></tr>
</table>

TABLE 6. Variation in optimal values for different values of c_r and c_l

c_r \ c_l	7		14		21		28		35	
6	60	4	60	4	60	4	60	4	60	4
	3.473039		3.473048		3.473057		3.473066		3.473075	
9	73	4	73	4	73	4	73	4	73	4
	3.924258		3.924266		3.924274		3.924283		3.924291	
12	85	3	85	3	85	3	85	3	85	3
	4.330799		4.330810		4.330820		4.330830		4.330841	
15	96	3	96	3	96	3	96	3	96	3
	4.715145		4.715154		4.715164		4.715174		4.715184	
18	106	2	106	2	106	2	106	2	106	2
	5.077669		5.077682		5.077695		5.077709		5.077722	

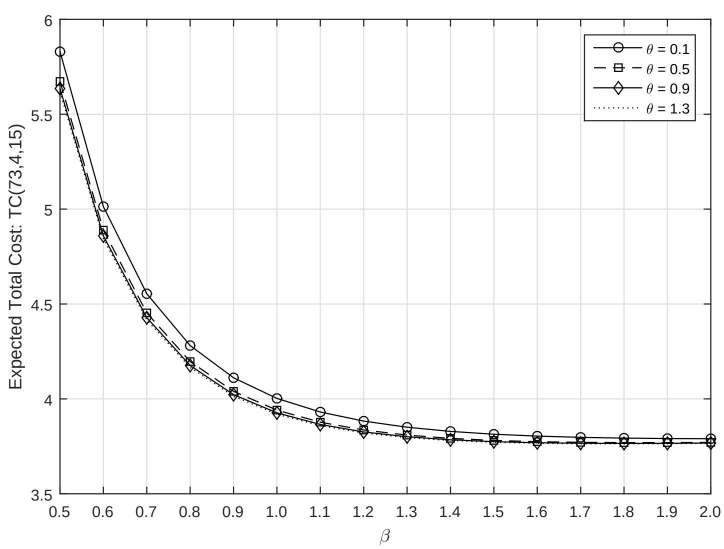

FIGURE 6. TC vs β for different values of θ

(6) Since the total expected cost of customers lost and the total expected cost of waiting time of a customer decreases as S and s increases. Therefore we obtain values from

TABLE 7. Variation in optimal values for different values of c_h and c_w

c_h \ c_w	7		8		9		10		11	
0.01	70	3	71	3	73	4	74	4	75	4
	3.492041		3.710708		3.924258		4.130068		4.335665	
0.02	59	3	60	3	61	3	62	4	63	4
	3.685211		3.907153		4.128696		4.342480		4.550592	
0.03	51	3	52	3	53	3	54	3	55	4
	3.856320		4.081147		4.305524		4.529470		4.741723	
0.04	45	2	47	3	48	3	49	3	50	4
	4.004106		4.239143		4.466042		4.692482		4.915813	
0.05	41	2	43	3	43	3	44	3	45	3
	4.138339		4.385029		4.614189		4.842843		5.071047	

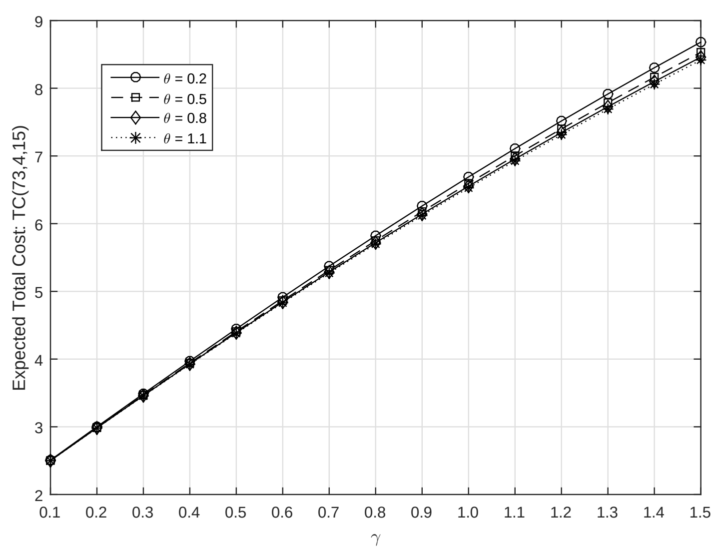

FIGURE 7. TC vs γ for different values of θ

Tables (3, 7, 9 and 10) that the optimal maximum inventory level S^* and the optimal reorder level s^* increases as the cost of waiting time of a customer c_w increases.

Evidence 3 : Expected Total Cost Vs Parameters

TABLE 8. Variation in optimal values for different values of c_h and c_p

c_p / c_h	0.01		0.06		0.11		0.16		0.21	
0.01	135	4	73	4	53	3	43	3	37	3
	3.368480		3.924258		4.305524		4.614189		4.882931	
0.02	92	4	61	3	48	3	40	3	35	3
	3.678838		4.128696		4.466042		4.752490		5.006619	
0.03	73	4	53	3	43	3	37	3	33	2
	3.924258		4.305524		4.614189		4.882931		5.120536	
0.04	61	3	48	3	40	3	35	3	31	2
	4.128696		4.466042		4.752490		5.006619		5.226050	
0.05	53	3	43	3	37	3	33	2	30	2
	4.305524		4.614189		4.882931		5.120536		5.327374	

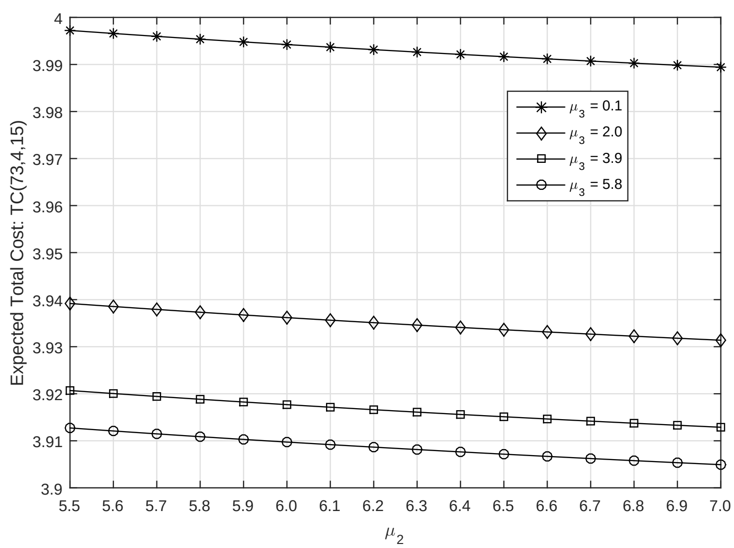

FIGURE 8. TC vs μ_2 for different values of μ_3

Here, we have studied the change of values of parameters λ, β, γ, θ, μ_1, μ_2 and μ_3 that impact the total expected cost $TC(73,4,15)$ according to the fixed cost values as mentioned above [i.e.,(3)] which are illustrated in Figures 4 to 9. As we anticipated, the expected total

TABLE 9. Variation in optimal values for different values of c_p and c_w

c_w / c_p	7		8		9		10		11	
0.01	130	4	133	4	135	4	137	5	139	5
	2.970556		3.169602		3.368480		3.560235		3.751844	
0.06	70	3	71	3	73	4	74	4	75	4
	3.492041		3.710708		3.924258		4.130068		4.335665	
0.11	51	3	52	3	53	3	54	3	55	4
	3.856320		4.081147		4.305524		4.529470		4.741723	
0.16	41	2	43	3	43	3	44	3	45	3
	4.138339		4.385029		4.614189		4.842843		5.071047	
0.21	35	2	36	2	37	3	38	3	39	3
	4.380793		4.638009		4.882931		5.115594		5.347836	

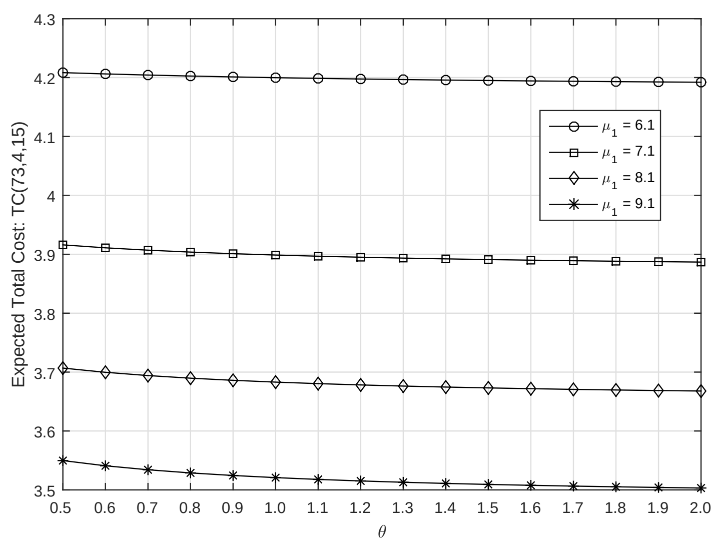

FIGURE 9. TC vs θ for different values of μ_1

cost decreases as increasing the rate of replenishment β, the rate of working vacation θ, the rate of service in regular period, the first stage working vacation and the second stage working vacation [ie.,μ_1, μ_2, μ_3] respectively, and decreasing the rate of perishability of an item γ, the

TABLE 10. Variation in optimal values for different values of c_w and c_l

<table>
<tr><td>c_l
c_w</td><td colspan="2">7</td><td colspan="2">14</td><td colspan="2">21</td><td colspan="2">28</td><td colspan="2">35</td></tr>
<tr><td rowspan="2">7</td><td>70</td><td>3</td><td>70</td><td>3</td><td>70</td><td>3</td><td>70</td><td>3</td><td>70</td><td>3</td></tr>
<tr><td colspan="2">3.492041</td><td colspan="2">3.492052</td><td colspan="2">3.492063</td><td colspan="2">3.492074</td><td colspan="2">3.492085</td></tr>
<tr><td rowspan="2">8</td><td>71</td><td>3</td><td>71</td><td>3</td><td>71</td><td>3</td><td>71</td><td>3</td><td>71</td><td>3</td></tr>
<tr><td colspan="2">3.710708</td><td colspan="2">3.710719</td><td colspan="2">3.710730</td><td colspan="2">3.710741</td><td colspan="2">3.710752</td></tr>
<tr><td rowspan="2">9</td><td>73</td><td>4</td><td>73</td><td>4</td><td>73</td><td>4</td><td>73</td><td>4</td><td>73</td><td>4</td></tr>
<tr><td colspan="2">3.924258</td><td colspan="2">3.924266</td><td colspan="2">3.924274</td><td colspan="2">3.924283</td><td colspan="2">3.924291</td></tr>
<tr><td rowspan="2">10</td><td>74</td><td>4</td><td>74</td><td>4</td><td>74</td><td>4</td><td>74</td><td>4</td><td>74</td><td>4</td></tr>
<tr><td colspan="2">4.130068</td><td colspan="2">4.130076</td><td colspan="2">4.130084</td><td colspan="2">4.130092</td><td colspan="2">4.130101</td></tr>
<tr><td rowspan="2">11</td><td>75</td><td>4</td><td>75</td><td>4</td><td>75</td><td>4</td><td>75</td><td>4</td><td>75</td><td>4</td></tr>
<tr><td colspan="2">4.335665</td><td colspan="2">4.335674</td><td colspan="2">4.335682</td><td colspan="2">4.335690</td><td colspan="2">4.335698</td></tr>
</table>

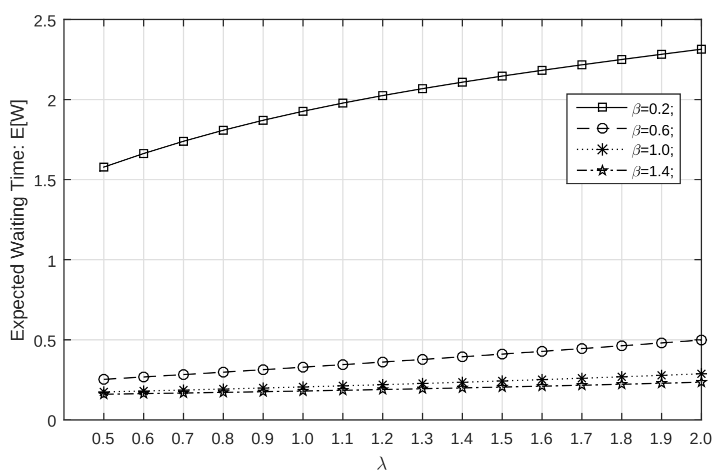

FIGURE 10. $E[W]$ vs λ for different values of β

rate of arrival of customers λ. Additionally, we notice the following significant consequences from the Figures 4 to 9.

(1) As γ increases and θ decreases, the rate of expected total cost is significant at the lower value of β [Figs.4 and 6].

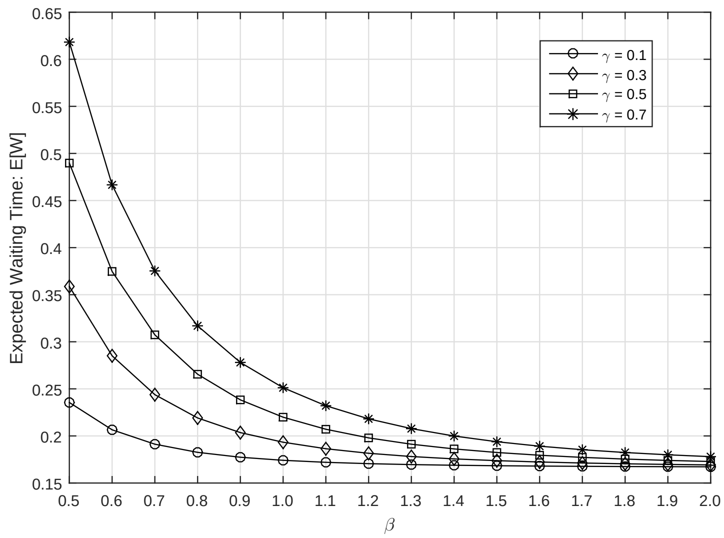

FIGURE 11. $E[W]$ vs β for different values of γ

(2) As γ increases, the rate of expected total cost is significant at the higher value of λ and the lower value of θ [Figs.5 and 7].

(3) With reference to any given values of μ_2, the rate of expected total cost is significant at the lower values of μ_3 [Fig.8].

(4) Similarly, with reference to any given values of θ, the rate of expected total cost is significant at the lower values of μ_1 [Fig.9].

(5) In this discussion, β is more an effective parameter. Hence we optimize the effect of any parameter on expected total cost by making sufficient number of orders.

Evidence 4 : Expected Waiting Time Vs Parameters

Investigating the measure of the total expected waiting time $E[W]$ which is the impact on any values of the parameters λ, β, γ, μ_1, μ_2 and μ_3 over the given range. As usual, the expected waiting time of a customer does not decrease as λ, γ increases and β, θ, μ_1, μ_2, μ_3 decreases. Additionally, we notice the following significant consequences that has been observed from the Figures 10 to 15.

(1) As λ and γ increases, the rate of expected waiting time of a customer $E[W]$ is significant at the lower value of β [Figs.10 and 11].

(2) As γ increases, the rate of expected waiting time of a customer $E[W]$ is significant at the higher value of λ and the lower value of θ [Figs.12 and 13].

(3) As λ increases, the rate of expected waiting time of customer $E[W]$ is significant at the lower value of μ_1 [Fig.14].

(4) With reference to any given values of μ_2 and μ_3, the rate of expected waiting time of customer $E[W]$ is significant at the lower values of μ_2 as well as μ_3 [Fig.15].

(5) The comparisons made from all parameters, β is more significant in effect on $E[W]$. Hence we optimize the effect of any parameter on $E[W]$ by making sufficient number of orders.

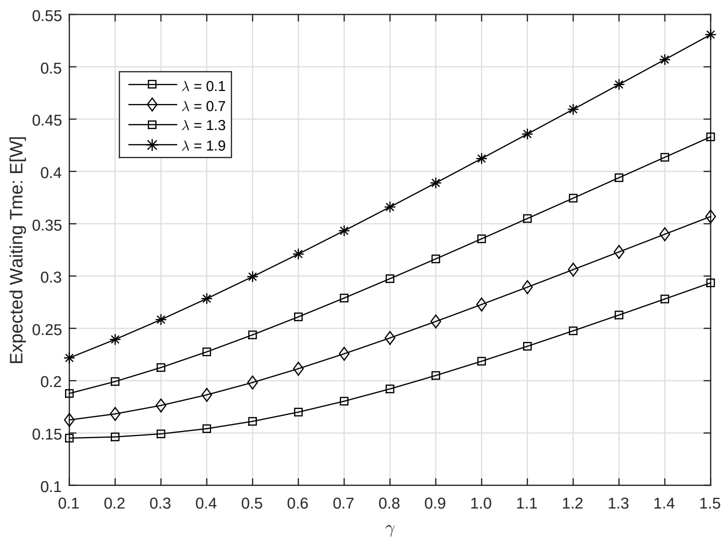

FIGURE 12. $E[W]$ vs γ for different values of λ

Evidence 5 : Expected number of customers lost Vs Parameters

Investigating the measure of the total expected number of customers lost $E[L]$ which is the impact of any values of the parameters λ, β, γ, μ_1, μ_2 and μ_3 over the given range. As usual, the expected number of customers does not decrease as λ, γ increases and $\beta,\ \theta,\ \mu_1,\ \mu_2,\ \mu_3$ decreases. Additionally, we notice the following significant consequences have been observed from Figures 16 to 21.

(1) As λ and γ increases, the rate of expected number of customers lost $E[L]$ is significant at the lower value of β [Figs.16 and 17].

(2) With reference to any value of θ, the rate of expected number of customers lost $E[L]$ is significant at the lower value of β [Fig.18]. As γ increases, the rate of expected number of customers lost $E[L]$ is significant at the lower value of θ [Fig.19].

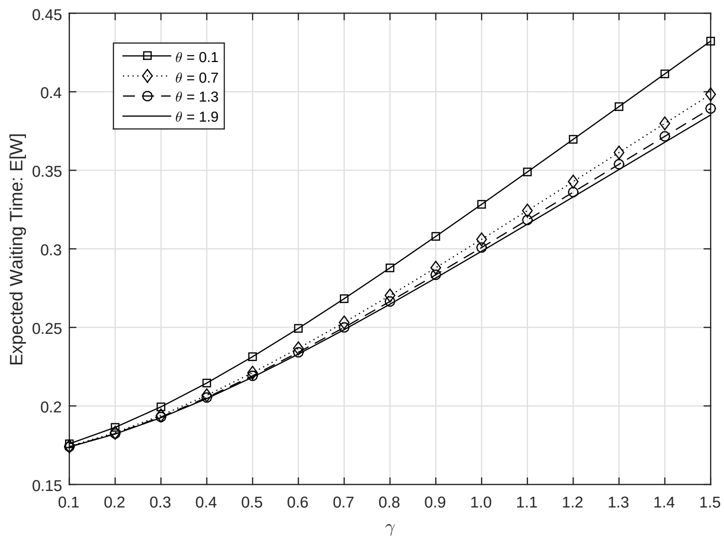

FIGURE 13. $E[W]$ vs γ for different values of θ

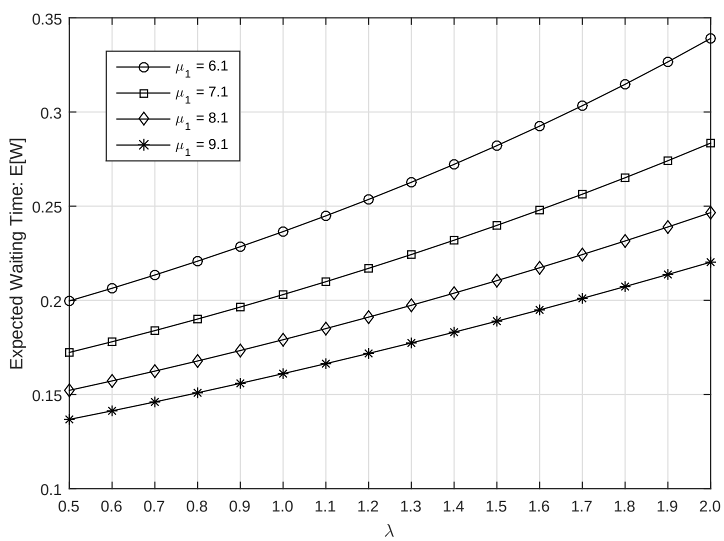

FIGURE 14. $E[W]$ vs λ for different values of μ_1

(3) With reference to any value of μ_2, the rate of expected number of customers lost $E[L]$ is significant at the lower value of μ_3 [Fig.20].

(4) With reference to any value of θ, the rate of expected number of customers lost $E[L]$ is significant at the lower value of μ_1 [Fig.21].

(5) In comparing all parameters, β is more significant effect on $E[L]$. Hence we optimize the effect of any parameter on $E[L]$ by making sufficient number of orders.

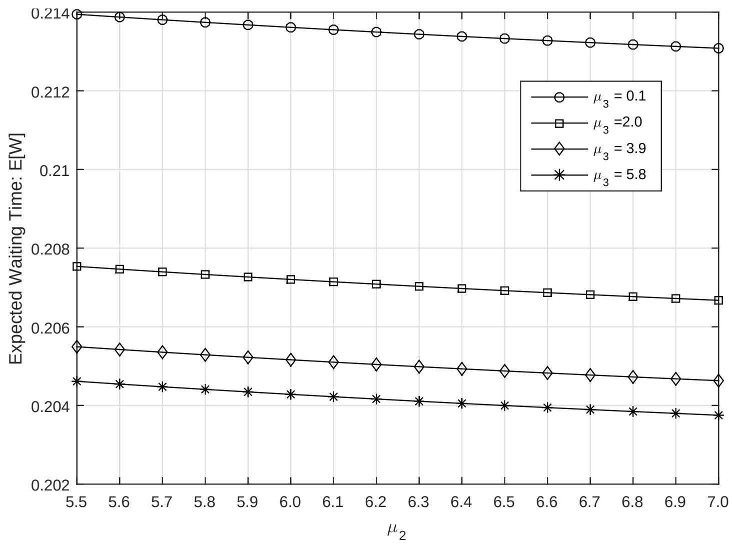

FIGURE 15. $E[W]$ vs μ_2 for different values of μ_3

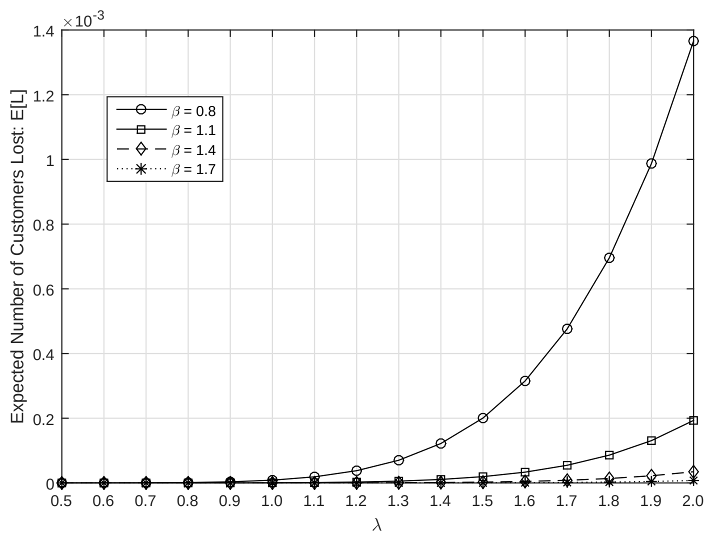

FIGURE 16. $E[L]$ vs λ for different values of β

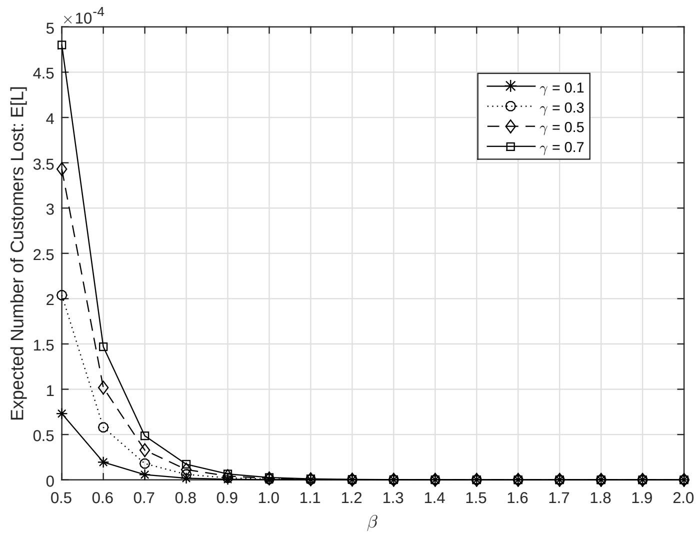

FIGURE 17. $E[L]$ vs β for different values of γ

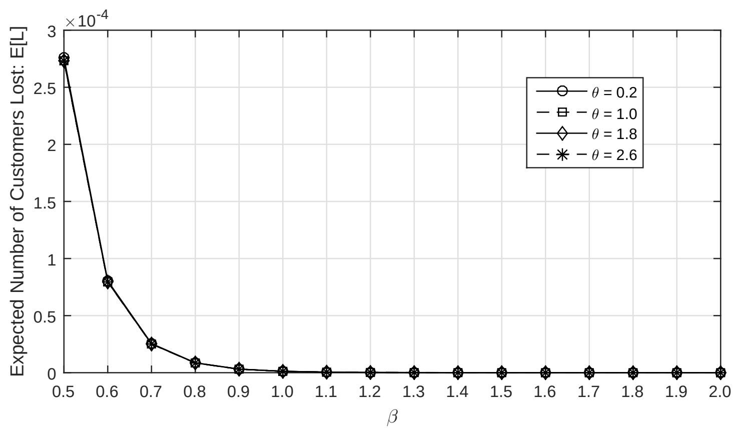

FIGURE 18. $E[L]$ vs β for different values of θ

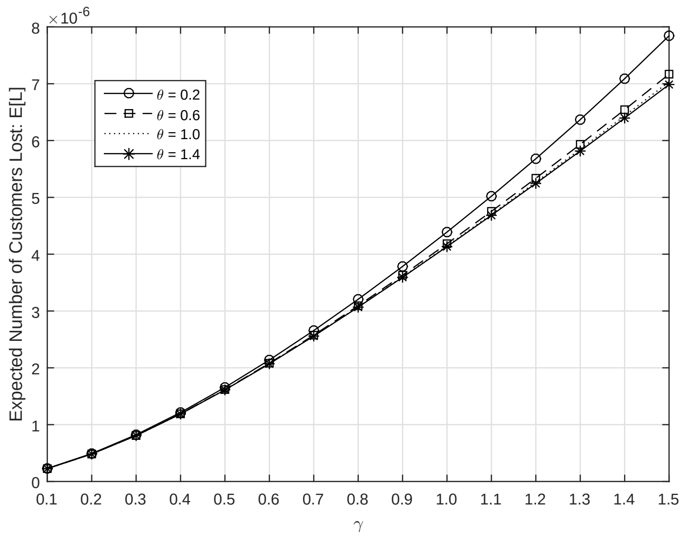

FIGURE 19. $E[L]$ vs γ for different values of θ

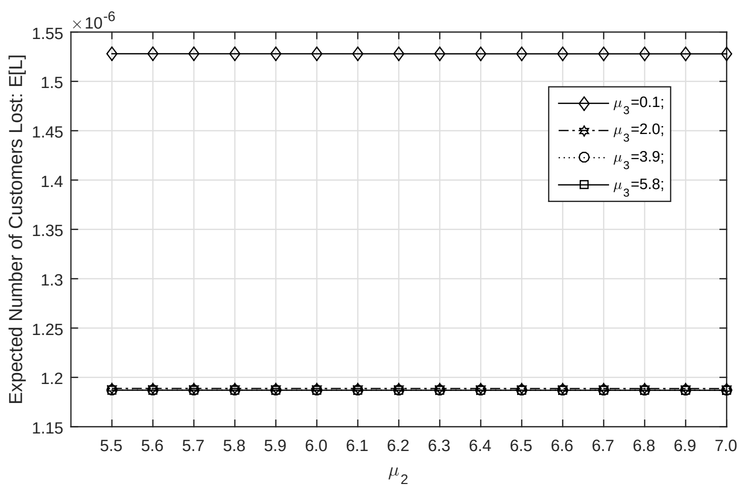

FIGURE 20. $E[L]$ vs μ_2 for different values of μ_3

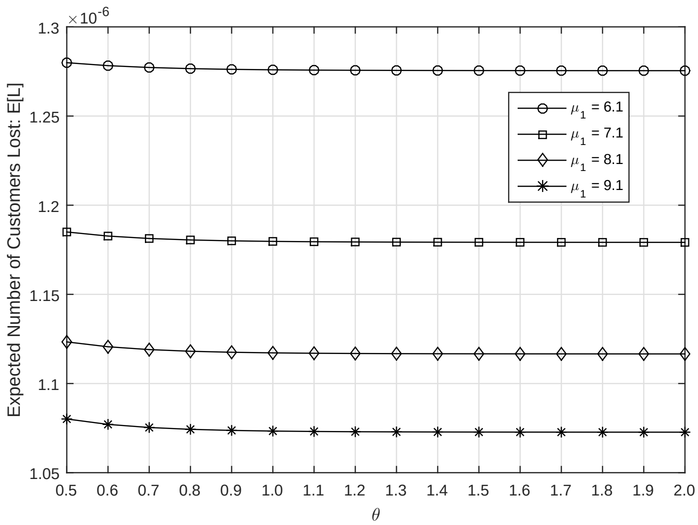

FIGURE 21. $E[L]$ vs θ for different values of μ_1

6. CONCLUDING REMARKS:

This paper aims at the discussion of a two stage working vacations with a perishable inventory queuing system. In a realistic situation, any server is required to fulfill its own needs at different times. In this situation, the server can adapt different rate of service at different working vacation based on the preference offered. The preference may depend on the rate of customer flow, item flow and nature of servers' own business, etc. Using matrix analytical technique, we derive the limiting distribution of all random activities in the system as stationary. From the numerical case studies, the expected total cost function is optimized. Also, we study the optimal behaviour of maximum inventory level and reorder level with various cost structure. For varying different parameters' values, the nature of expected number of customers lost and expected waiting time of a customer are also noticed. In future, we study the policy of a two stage working vacations of a single server on (s,Q) perishable inventory system with finite queue and a retrial orbit in which the $1st$ stage working vacation is considered for regular and the $2nd$ stage working vacation is optional.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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