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PORTFOLIO OPTIMIZATION USING SECOND ORDER CONIC PROGRAMMING APPROACH

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Abstract. In this paper, we examine the framework to estimate financial risk called conditional-value-at-risk (CVaR) and examine models to optimize portfolios by minimizing CVaR. We note that total risk can be a function of multiple risk factors combined in a linear or nonlinear forms. We demonstrate that, when using CVaR, several common nonlinear models can be expressed as second order cone programming problems and therefore efficiently solved using modern algorithms. This property is not shared with the more classical estimation of financial risk based on value-at-risk.

Keywords: total risk; risk measure; risk factors; conditional-value-at-risk; value-at-risk; second order cone programming.

2010 AMS Subject Classification: 90C25, 91G10, 91G40.

1. INTRODUCTION

Financial institutes make investments in different assets to grow their business but the variability of returns gives rise to risk. So risk, along with returns, is a major consideration for capital budgeting decisions. Thus, it becomes pertinent to measure risk.

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There are several types of risk, for example, interest rate risk, equity risk, currency risk, commodity risk, credit spread risk, default risk, operational risk etc [13]. Various risk measures have been proposed to quantify these risks, for example, Market beta, Standard deviation, Downside deviation, Mean absolute deviation, Semi-mean absolute deviation etc. The most popular and widely used risk measure is Value-at-Risk (VaR), see for example, Jorion [14], Linsmeier and Pearson [23], Basak and Shapiro [6], Glasserman et al. [12], Gençay and Selçuk [11], Adrian and Shin [1], and many more. VaR is often approximated by linear approximation of the risk portfolios which assumes normal distribution of losses. As a result, VaR becomes unstable when losses are not normally distributed (which is often the case) [22]. In these cases, historical or Monte-Carlo simulations are often used in the presence of non-linear instruments [2, 10, 19].

Although, VaR is a very popular risk measure, it lacks subadditivity and convexity [5]. As an alternative measure of risk, Conditional-Value-at-Risk (CVaR) has better properties than VaR and is proved to be a coherent and convex risk measure [18, 22]. It provides us convenient way of estimating risk in the presence of linear or non-linear instruments. It has been used in many applications, for example, Bucay and Rosen [8], Uryasev [25], Andersson et al. [4], Rockafellar and Uryasev [21,22], Quaranta and Zaffaroni [20], Zhu and Fukushima [26], Noyan and Rudolf [17], Dai et al. [9] and many more.

In this paper, we demonstrate a novel advantage of CVaR over VaR. Specifically, we show that when using CVaR it is possible to combine multiple risk factors in a number of nonlinear models and still produce a final optimization problem that is a second order cone program (SOCP). Since modern algorithms can efficiently solve SOCPs, this allows for a broad class of risk optimization model with tractable solve times. In this paper, we define a model to minimize the total risk as a function of different risk factors and discuss the methodology to solve the same.

The remaining paper is organized as follows. In Section 2, we consider different risk factors that contribute to the total risk of any financial firm and total risk is defined as a non-linear function of these risk factors assuming different risk factors follow different distributions of losses. The model to minimize the total risk, using CVaR to measure risk, is formulated and

CVaR is also known as average value-at-risk, mean access loss, expected shortfall, or tail VaR [21].

transformed into an SOCP in Section 3. Finally, in Section 4 conclusion is made and future directions are discussed.

2. DESCRIPTION OF THE MODEL

Assuming N different assets, the investment decision vector x is expressed as:

$$(1) \quad x = (x_1, x_2, \dots, x_N),$$

where x_i , $i = 1, 2, \dots, N$ represents the proportion of budget to invest in i^{th} asset. These investment proportions x_i are assumed to be non-negative and must satisfy the unity budget constraint

$$(2) \quad \sum_{i=1}^N x_i = 1, x_i \geq 0 \text{ for all } i = 1, 2, \dots, N.$$

The returns of each asset are random and so are the losses (being negative of returns).

Our goal is to minimize total risk. There are several types of risk in market: interest rate risk, equity risk, currency risk, commodity risk, credit spread risk, default risk, operational risk etc [13]. Each asset can have a distinct combination of risk factors and these risk factors may follow different distributions of losses. Here we consider m distinct risk factors, so total risk is the function of these risk factors.

$$(3) \quad \text{Total Risk (TR)} = f(\gamma_1, \gamma_2, \dots, \gamma_m),$$

where γ_k is the distribution of losses for k^{th} risk factor. We classify the assets into three categories as follows

$$(4) \quad TR = f \left(\bigcup_{k \in A} (\gamma_k), \bigcup_{k \in B} (\gamma_k), \bigcup_{k \in C} (\gamma_k) \right),$$

$$(5) \quad TR = \sum_{k \in A} \beta_k \text{CVaR}_{\alpha_k}(x, \gamma_k) + \sum_{k' \in B} \beta_{k'} \|\{(\text{CVaR}_{\alpha_k}(x, \gamma_k)) \mid k \in B_{k'}\}\|_2 + \sum_{k' \in C} \beta_{k'} \|\{(\text{CVaR}_{\alpha_k}(x, \gamma_k)) \mid k \in C_{k'}\}\|_2^2,$$

where

- (1) The collection $\gamma_k, k \in A$ represents the distributions of losses for the linear risk factors.

- (2) The collection $\gamma_k, k \in B$ represents the distributions of losses for the risk factors which have non-linear contribution to the total risk and their non-linearity is defined by the l_2 norm.
- (3) The collection $\gamma_k, k \in C$ represents distributions of losses for the risk factors which have non-linear contribution to the total risk and their non-linearity is defined by the l_2 norm squared.
- (4) The distribution of losses for the k^{th} risk factor γ_k , across the N assets is written as

$$\gamma_k = (\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{Nj}) \in \mathbb{R}^N, 1 \leq j \leq Q_k,$$

where γ_{ij} represents the j^{th} scenario of the i^{th} asset while considering Q_k number of historical scenarios for k^{th} risk factor.

To minimize the total risk, there is need to calculate the risk in all the distributions of losses for different risk factors. The random variable $X(x, \gamma_k)$ represents the losses of each asset for k^{th} risk factor. These depend on the random losses and the investment in each asset. The risk measure $\text{CVaR}_{\alpha_k}(X(x, \gamma_k))$, for a given parameter $0 < \alpha_k < 1$, can be calculated as [22]

$$(6) \quad \text{CVaR}_{\alpha_k}(X(x, \gamma_k)) = \min_{l_k \in \mathbb{R}} \left\{ l_k + \frac{1}{1 - \alpha_k} E[(X(x, \gamma_k) - l_k)^+] \right\},$$

$$(7) \quad = \min_{l_k \in \mathbb{R}} \left\{ l_k + \frac{1}{1 - \alpha_k} \left[\sum_{j=1}^{Q_k} \left(\sum_{i=1}^N \gamma_{ij} x_i - l_k \right)^+ \rho_j \right] \right\}, k \in \{1, 2, \dots, m\},$$

where $(a)^+ = \text{maximum}\{0, a\}$ and l_k represents VaR that is obtained as a by product while optimizing CVaR [22]. The function $E[\cdot]$ represents the expected loss/risk of the loss distribution with joint probability distribution function $p(\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{Nj}) = \rho_j, j \in \{1, 2, \dots, Q_k\}$ for k^{th} risk factor.

2.1. Risk Estimation across the Linear Risk Factors. The set A provides indices of the risk factors that are treated linearly. The risk across these risk factors is calculated as a linear combination as follows:

$$(8) \quad \sum_{k \in A} \beta_k \text{CVaR}_{\alpha_k}(x, \gamma_k),$$

where $\beta_k, k \in A$ are assumed to be any non-negative real numbers.

2.2. Risk Estimation across the Non-linear Risk Factors. The sets B and C provide indices that are treated non-linearly. Set B corresponds to risk factors that are combined through an l_2 norm. For each $k' \in B$, we generate a set $B_{k'}$ that links the appropriate risk measures together. The resulting risk across risk factors in B is calculated as:

$$\sum_{k' \in B} \beta_{k'} \left(\sum_{k \in B_{k'}} (\text{CVaR}_{\alpha_k}(x, \gamma_k))^2 \right)^{\frac{1}{2}}.$$

where $\beta_{k'}, k' \in B$ are assumed to be any non-negative real numbers.

Set C corresponds to risk factors that are combined through an l_2 norm squared. For each $k' \in C$, we generate a set $C_{k'}$ that links the appropriate risk measures together. The resulting risk across risk factors in C is calculated as:

$$\sum_{k' \in C} \beta_{k'} \left(\sum_{k \in C_{k'}} (\text{CVaR}_{\alpha_k}(x, \gamma_k))^2 \right).$$

where $\beta_{k'}, k' \in C$ are assumed to be any non-negative real numbers.

The total risk across all the linear and non-linear risk factors can be defined as follows

$$(9) \quad TR = \sum_{k \in A} \beta_k \text{CVaR}_{\alpha_k}(x, \gamma_k) + \sum_{k' \in B} \beta_{k'} \left(\sum_{k \in B_{k'}} (\text{CVaR}_{\alpha_k}(x, \gamma_k))^2 \right)^{\frac{1}{2}} + \sum_{k' \in C} \beta_{k'} \left(\sum_{k \in C_{k'}} (\text{CVaR}_{\alpha_k}(x, \gamma_k))^2 \right)$$

where $\beta_k, 1 \leq k \leq m$ and $\beta_{k'}, 1 \leq k' \leq m$ are any non-negative real numbers.

3. MODEL FORMULATION AND TRANSFORMATION

The model to minimize total risk (9) subject to constraints (2) is formulated as follows

$$(10) \quad \text{Minimize } \sum_{k \in A} \beta_k \text{CVaR}_{\alpha_k}(x, \gamma_k) + \sum_{k' \in B} \beta_{k'} \left(\sum_{k \in B_{k'}} (\text{CVaR}_{\alpha_k}(x, \gamma_k))^2 \right)^{\frac{1}{2}} + \sum_{k' \in C} \beta_{k'} \left(\sum_{k \in C_{k'}} (\text{CVaR}_{\alpha_k}(x, \gamma_k))^2 \right)$$

subject to

$$\sum_{i=1}^N x_i = 1,$$

$$x_i \geq 0, i = 1, 2, \dots, N.$$

Now, we transform this non-linear model (10) into an equivalent SOCP. Assuming

$r_k = \text{CVaR}_{\alpha_k}(X(x, \gamma_k))$ for $1 \leq k \leq m$, the model (10) is transformed into

$$\begin{aligned} & \text{Minimize } \sum_{k \in A} \beta_k r_k + \sum_{k' \in B} \beta_{k'} \left(\sum_{k \in B_{k'}} r_k^2 \right)^{\frac{1}{2}} + \sum_{k' \in C} \beta_{k'} \left(\sum_{k \in C_{k'}} r_k^2 \right) \\ & \text{subject to} \\ (11) \quad & r_k = \text{CVaR}_{\alpha_k}(X(x, \gamma_k)), \quad 1 \leq k \leq m, \\ & \sum_{i=1}^N x_i = 1, \\ & x_i \geq 0, \quad i = 1, 2, \dots, N. \end{aligned}$$

Using equation (7), this can be rewritten as

$$\begin{aligned} & \text{Minimize } \sum_{k \in A} \beta_k r_k + \sum_{k' \in B} \beta_{k'} \left(\sum_{k \in B_{k'}} r_k^2 \right)^{\frac{1}{2}} + \sum_{k' \in C} \beta_{k'} \left(\sum_{k \in C_{k'}} r_k^2 \right) \\ & \text{subject to} \\ (12) \quad & r_k = l_k + \frac{1}{1 - \alpha_k} \left[\sum_{j=1}^{Q_k} \left(\sum_{i=1}^N \gamma_{ij} x_i - l_k \right)^+ \rho_j \right], \quad 1 \leq k \leq m, \\ & \sum_{i=1}^N x_i = 1, \\ & x_i \geq 0, \quad i = 1, 2, \dots, N. \end{aligned}$$

The non-linear constraints of problem (12) can be transformed into linear constraints [22] using dummy variables $s_j, 1 \leq j \leq Q_k$,

$$\begin{aligned} & \text{Minimize } \sum_{k \in A} \beta_k r_k + \sum_{k' \in B} \beta_{k'} \left(\sum_{k \in B_{k'}} r_k^2 \right)^{\frac{1}{2}} + \sum_{k' \in C} \beta_{k'} \left(\sum_{k \in C_{k'}} r_k^2 \right) \\ & \text{subject to} \\ (13) \quad & r_k = l_k + \frac{1}{1 - \alpha_k} \left[\sum_{j=1}^{Q_k} s_j \rho_j \right], \quad 1 \leq k \leq m, \\ & \sum_{i=1}^N \gamma_{ij} x_i - l_k - s_j \leq 0, \quad 1 \leq k \leq m, 1 \leq j \leq Q_k, \\ & \sum_{i=1}^N x_i = 1, \\ & x_i \geq 0, \quad i = 1, 2, \dots, N, \\ & s_j \geq 0, \quad 1 \leq j \leq Q_k. \end{aligned}$$

Defining $t_{k'} = (\sum_{k \in B_{k'}} r_k^2)^{\frac{1}{2}}, k' \in B$ and $t_{k'} = (\sum_{k \in C_{k'}} r_k^2)^{\frac{1}{2}}, k' \in C$, the problem (13) can be transformed into the following SOCP

$$\begin{aligned}
 & \text{Minimize } \sum_{k \in A} \beta_k r_k + \sum_{k' \in B} \beta_{k'} t_{k'} + \sum_{k' \in C} \beta_{k'} t_{k'}^2 \\
 & \text{subject to} \\
 & r_k = l_k + \frac{1}{1 - \alpha_k} \left[\sum_{j=1}^{Q_k} s_j \rho_j \right], \quad 1 \leq k \leq m, \\
 & \sum_{i=1}^N \gamma_{ij} x_i - l_k - s_j \leq 0, \quad 1 \leq k \leq m, 1 \leq j \leq Q_k, \\
 & \sum_{k \in B_{k'}} r_k^2 \leq t_{k'}^2, \quad k' \in B, \\
 & \sum_{k \in C_{k'}} r_k^2 \leq t_{k'}^2, \quad k' \in C, \\
 & \sum_{i=1}^N x_i = 1, \\
 & x_i \geq 0, \quad i = 1, 2, \dots, N, \\
 & s_j \geq 0, \quad 1 \leq j \leq Q_k, \\
 & t_{k'} \geq 0, \quad k' \in B, \\
 & t_{k'} \geq 0, \quad k' \in C.
 \end{aligned} \tag{14}$$

The quadratic constrains $\sum_{k \in B_{k'}} r_k^2 \leq t_{k'}^2, k' \in B$ and $\sum_{k \in C_{k'}} r_k^2 \leq t_{k'}^2, k' \in C$ in problem (14) represents the Lorentz cones, so the model can be rewritten as follows

$$\begin{aligned}
 & \text{Minimize } \sum_{k \in A} \beta_k r_k + \sum_{k' \in B} \beta_{k'} t_{k'} + \sum_{k' \in C} \beta_{k'} t_{k'}^2 \\
 & \text{subject to} \\
 & r_k = l_k + \frac{1}{1 - \alpha_k} \left[\sum_{j=1}^{Q_k} s_j \rho_j \right], \quad 1 \leq k \leq m, \\
 & \sum_{i=1}^N \gamma_{ij} x_i - l_k - s_j \leq 0, \quad 1 \leq k \leq m, 1 \leq j \leq Q_k, \\
 & \sum_{i=1}^N x_i = 1,
 \end{aligned}$$

The constraint $\sum_{k \in B_{k'}} r_k^2 \leq t_{k'}^2, k' \in B$, is equivalent to $\sum_{k \in B_{k'}} r_k = t_{k'}, k' \in B$ due to the fact that we are minimizing

$t_{k'}, k' \in B$. An Analogous comment holds for the $\sum_{k \in C_{k'}} r_k^2 \leq t_{k'}^2, k' \in C$ constraint.

$$\begin{aligned}
(15) \quad & x_i \geq 0, \quad i = 1, 2, \dots, N, \\
& s_j \geq 0, \quad 1 \leq j \leq Q_k, \\
& t_{k'} \geq 0, \quad k' \in B, \\
& t_{k'} \geq 0, \quad k' \in C, \\
& (r_{k'}, t_{k'}) \in \mathcal{L}^{k'}, \quad k' \in B, \\
& (r_{k'}, t_{k'}) \in \mathcal{L}^{k'}, \quad k' \in C,
\end{aligned}$$

where $\mathcal{L}^{k'} = \{(r_{k'}, t_{k'}) \in \mathbb{R}^{k'} \times \mathbb{R} \mid \sum_{k'} r_{k'}^2 \leq t_{k'}^2\}$.

The model (15) is an SOCP. As such, a number of algorithms exist that guarantee converging to a solution to this problem [3, 15, 16, 24]. For example, it can be solved by using any of the solvers: QUADPROG, SeDuMi, CPLEX, Gurobi, or MOSEK.

Remark 1. If set $B = \phi$ and $C = \phi$ as well, then Problem (15) will be a linear model that can also be solved by using Simplex algorithm.

4. CONCLUSION AND FUTURE DIRECTIONS

In portfolio design, there generally exist many risk factors across the assets. The total risk depends on all the risk factors simultaneously. This total risk, being a function of different risk factors, can include linear or non-linear forms. We have studied the problem by considering total risk as a non-linear function where some risk factors are assumed to be linear, some follow the l_2 norm and others follow the l_2 norm squared. We demonstrate that, when risk is measured using CVaR, the resulting model is an SOCP and therefore solvable by a number of modern algorithms.

In this paper we have presented the methodology to transform a specific non-linear model into a tractable SOCP. Our next step is to perform some experiments and check the applicability of the proposed method by implementing it to some real life data. Moreover, this method is applicable to a specific class of problems where non-linearity can be defined as a quadratic form. However, total risk could have any kind of non-linearity so there is still scope for advancements in the proposed method that can support the more general forms of non-linearity.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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