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A NEW GOLDEN RATIO ITERATIVE SCHEME FOR SOLVING FIXED POINT PROBLEMS WITH APPLICATION TO IMAGE RESTORATION PROBLEMS

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Abstract. The purpose of this paper is to introduce a new Golden Ratio Relaxed Three-Step Algorithm (GRRITA) for solving fixed point problems in Banach spaces. The proposed iterative scheme uses the golden ratio structure to enhance contraction effects, computational efficiency, and accelerate convergence. In addition, we establish strong convergence under weak assumptions, prove T-stability, data dependence, and error-estimation results, and demonstrate that GRRITA achieves Q-linear convergence with order one. Finally, we present some numerical experiments and an application to image restoration problems. The results validate our theoretical results, that is, the proposed iterative algorithm outperforms some existing iterative algorithms in the literature.

Keywords: fixed point theory; iteration; nonlinear mappings; Banach spaces; rate of convergence; stability

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1. INTRODUCTION

Let T be a self map on a nonempty set X. A point $x \in X$ is called a fixed point of X if x solves the equation

$$x = Tx.$$

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Fixed point and its application form an integral part of nonlinear analysis. It is an important tool for establishing the existence and uniqueness of solution of integral equations, differential equations, optimization problems and so on. It is well-known that most real-life problems cannot be solved analytically, thus a numerical solution is always opted for. In addition, converting a real life problem into a fixed point problem is almost impossible to solve in its closed form, thus, there is a need for an approximate solution.

The first and well-known iterative scheme was introduced by Picard [15]. The Picard iteration is defined by

$$x_{n+1} = Tx_n, \qquad n \geq 0. \tag{1}$$

The Picard iterative scheme is simple and easy to implement on a computer system. However, its convergence only happens when the initial guess is close to the fixed point. In addition, if the nonlinear mapping is not a contraction, the Picard iteration will fail to converge. Due to these drawbacks, researchers have developed different types of iterative processes over the years to provide an affirmative answer to these drawbacks, see [1, 3, 2, 14, 11, 12] and the references therein.

The Mann iteration [13] was introduced and defined as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \qquad 0 \leq \alpha_n \leq 1. \tag{2}$$

Clearly, if $\alpha_n = 1$, we obtain the Picard iteration (1). The Mann iteration can only achieve weak convergence instead of strong convergence. In addition, the weak convergence is achieved under some strict conditions placed on the relaxation parameter. Lastly, it was observed that for some contractive like mappings, the averaging structure may reduce rather than enhance the rate of error reduction.

Motivated by the above drawbacks, researchers have developed several multi-step iterative processes that have been able to achieve strong convergence and improved rate of convergence under suitable contractive-like mappings. For example, the authors in [4] introduced and studied the iterative scheme

$$\begin{cases} z_n & = T[(1-\alpha_n)x_n + \alpha_n T x_n], \\ y_n & = T z_n, \\ x_{n+1} & = T y_n, \quad \forall \quad n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. The authors established strong convergence, T-stability of the iterative scheme (3) and also established that the iterative scheme converges faster than some of the existing iterative schemes in the literature. Furthermore, the concept of AG-iterative scheme was introduced and studied by Okeke et al. in [16]. It was established that the AG-iterative scheme converges faster than some existing iterative schemes in the literature. The AG-iterative scheme is defined as

$$\begin{cases} x_n & = (1-\gamma_n)u_n + \gamma_n T u_n, \\ w_n & = (1-\beta_n)T u_n + \beta_n T x_n, \\ v_n & = T[(1-\alpha_n)w_n + \alpha_n T w_n], \\ u_{n+1} & = T v_n. \end{cases} \tag{4}$$

Another important development is the D-iterative scheme which was introduced and studied by Hussain et al. in [7]. The D-iterative scheme is defined as

$$\begin{cases} u_n & = (1-\alpha_n)x_n + \alpha_n T x_n \\ y_n & = T u_n, \\ v_n & = T[(1-\beta_n)T x_n + \beta_n T y_n], \\ x_{n+1} & = T v_n. \end{cases} \tag{5}$$

The authors established the convergence, T-stability, data dependence, and error-estimation results for the D-iteration. However, it was observed that the results obtained by the authors for the D-iterative scheme [7] were obtained under strong assumptions on the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

The authors in [10] made an important observation regarding the D-iterative scheme and the AG-iterative scheme. They established that both iterative schemes have the same rate of convergence. This observation is significant because it suggests that increasing the complexity of

an iterative scheme does not necessarily result in a corresponding improvement in its asymptotic convergence rate. This important question now gives rise to Remark 1.1 below.

Remark 1.1. Thus, an important question in the design of new fixed point algorithms is whether the additional computational steps introduce a genuinely new acceleration mechanism or merely increase the computational cost of each iteration.

An interesting concept and direction was introduced and studied by Sharma et al. in [17]. The iterative scheme introduced was referred to as SH-I and they established stability and strong convergence and reported linear convergence for their proposed iterative scheme. The SH-I iterative scheme is defined as

$$\begin{cases} z_n &= \frac{\alpha_n x_n + T x_n}{\alpha_n + 1}, \\ y_n &= T z_n, \\ x_{n+1} &= T y_n. \end{cases} \tag{6}$$

where $\alpha_n > 0$. In addition, motivated by the work of Sharma et al. in [17], Nawaz et al. [19] introduced and studied the BK-iteration for generalized contractive mappings, for which both weak and strong convergence and stability were established. The BK iteration is defined as

$$\begin{cases} z_n &= \frac{\alpha_n x_n + T x_n}{\alpha_n + 1}, \\ y_n &= T z_n, \\ w_n &= T y_n, \\ x_{n+1} &= T w_n. \end{cases} \tag{7}$$

where $\alpha_n > 0$. It is easy to see that the BK-iterative scheme is a generalization of the SH-I iterative scheme. Furthermore, Sharma et al. [18] proposed and studied the SH II iterative scheme. The SH II iterative scheme is defined as

$$\begin{cases} z_n &= \frac{1}{\alpha_n}[(1+\alpha_n)Tx_n + T(\frac{\alpha_n x_n + T x_n}{\alpha_n + 1})] \\ y_n &= T[\frac{\beta_n z_n + T z_n}{\beta_n + 1}] \\ x_{n+1} &= T[\frac{\gamma_n y_n + T y_n}{\gamma_n + 1}] \end{cases} \tag{8}$$

where, $\alpha \geq 1, \beta_n > 0$, and $\gamma_n > 0$. The authors established the strong convergence, T-stability, data dependence, and error-estimation results for the SH II.

Remark 1.2. It is important to ask if increasing the number of T-evaluations can improve the theoretical error reduction without necessarily improving computational efficiency to the same extent.

Indeed, a closer examination of the existing literature on iterative algorithms suggests that a very good iterative algorithm should:

(1) converge strongly to a fixed point under appropriate assumptions (weak assumptions);
(2) possess stability property;
(3) have good rate of convergence when compared to other existing iterative schemes;
(4) be sufficiently fast when compared with existing methods;
(5) exhibit data dependence; and
(6) remain computationally effective when implemented on concrete problems.

Thus, strong convergence and good rate of convergence alone are not an adequate measure of the quality of an iterative scheme. Thus an appropriate relaxation structure can be used to control the application of the operator. However, the choice of relaxation alone does not necessarily provide a sufficiently strong acceleration mechanism. This motivates the use of a further averaging structure capable of producing a favorable error propagation factor. The golden ratio

$$\phi = \frac{1+\sqrt{5}}{2} \tag{9}$$

provides a natural parameter for constructing such an averaging mechanism.

Motivated by the above study, in this study, we propose the Golden Ratio Relaxed Three-Step Algorithm (GRITTA). GRRITA provides an affirmative response to the major limitations identified in the existing iterative schemes by using the golden-ratio averaging within a single fixed point iteration. The resulting method possesses not only strong convergence but also stability, data dependence and an improved theoretical error-propagation mechanism. The GRRITTA is not merely another iterative algorithm. The main contributions to knowledge of the paper include but are not limited to the following:

Remark 1.3. (1) strong convergence result was obtained under a weak assumption to a fixed point under appropriate assumptions;

(2) T-stability result was obtained;

(3) it has a good rate of convergence when compared to other existing iterative schemes;

(4) it is sufficiently faster when compared with existing methods;

(5) data dependence was obtained;

(6) The golden-ratio averaging mechanism provides an additional contraction effect through the factors

$$\frac{\phi + k}{\phi + 1}.$$

(7) The theoretical error-propagation factor

$$Q = k^3 \left(\frac{\phi + k}{\phi + 1} \right)^2$$

satisfies

$$Q < k,$$

providing a theoretical basis for the accelerated convergence of GRRITA relative to the basic contraction iteration.

(8) numerical experiments are carried out to further confirm the theoretical advantage by comparing the number of iterations, absolute errors, residuals and computational cost of GRRITA with those of the existing algorithms.

(9) it remains computationally effective when implemented on concrete problems.

The remainder of the paper is organised as follows. In Section 2, we present preliminary definitions and auxiliary results. In Section 3, we present our proposed iterative algorithm and the main findings of this study. In Section 4, we present numerical experiments and an application to the image restoration problem. In the last section, Section 5, we give a conclusion of the study.

2. PRELIMINARIES

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space and C be a closed subspace of X. A self-map $T : C \to C$ is called a contraction if there exists a real number $k \in [0,1)$ such that for every

$x, y \in C$,

$$\|Tx - Ty\| \leq k\|x - y\|.$$

Definition 2.2. Let $\{x_k\}$ be a sequence in a Banach space converging to x^*. We say that $\{x_k\}$ converges *Q-linearly* to x^* if there exists a constant $c \in (0,1)$ such that

$$\|x_{k+1} - x^*\| \leq c\,\|x_k - x^*\|, \quad \text{for all sufficiently large } k.$$

This means the error decreases by at least a fixed ratio c at each iteration, ensuring linear convergence.

Definition 2.3. [5] Let $\{x_k\}_{k=0}^{\infty}$ be a sequence in a Banach space X, and let $x \in X$. We say that $\{x_k\}$ strongly converges to x, written as $x_k \to x \in X$, if

$$\|x_k - x\| \to 0 \quad \text{as } k \to \infty.$$

Definition 2.4. [9] Let X be a real Banach space, $C \subseteq X$ a closed subspace, and $T : C \to C$ a mapping with a fixed point p. Let $\{x_k\}_{k=0}^{\infty}$ be an iterative process defined by

$$x_{k+1} = f(T, x_k),$$

such that the generated sequence $\{x_k\}$ converges to p. Let $\{v_k\}_{k=0}^{\infty}$ be any arbitrary sequence in C and define

$$\varepsilon_k = \|v_{k+1} - f(T, v_k)\|, \quad \forall k \in \mathbb{N}.$$

Then, the iterative process is said to be *T-stable* if

$$\lim_{k\to\infty} \varepsilon_k = 0 \quad \text{if and only if} \quad \lim_{k\to\infty} v_k = p.$$

Definition 2.5. [5, 6] Let $T, S : X \to X$ be operators on an arbitrary set X. Then, S is called an *approximate operator* for T if for all $x \in X$ and for some fixed $\varepsilon > 0$, one has

$$\|Tx - Sx\| \leq \varepsilon.$$

Lemma 2.6. [6] *Let $\{\varepsilon_k\}_{k=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{k\to\infty} \varepsilon_k = 0$, and let $0 \leq \delta < 1$ be a real number. If a sequence of positive numbers $\{u_k\}_{k=0}^{\infty}$ satisfies*

$$u_{k+1} \leq \delta u_k + \varepsilon_k, \quad k = 0, 1, 2, \ldots,$$

then

$$\lim_{k\to\infty} u_k = 0.$$

Lemma 2.7. [16] *Let* $\{\psi_k\}_{k=0}^{\infty}$ *and* $\{\eta_k\}_{k=0}^{\infty}$ *be sequences of non-negative real numbers satisfying*

$$\psi_{k+1} \leq (1-\mu_k)\psi_k + \eta_k,$$

where $\mu_k \in (0,1)$ *for all* $k \in \mathbb{N}$, *with* $\sum_{k=0}^{\infty} \mu_k = \infty$ *and* $\frac{\eta_k}{\mu_k} \to 0$ *as* $k \to \infty$. *Then, it holds that*

$$\lim_{k\to\infty} \psi_k = 0.$$

3. Main Result

In the light of the above findings, we propose a new three-step iterative scheme called the Golden Ratio Relaxed Three-Step Algorithm (GRITTA). This iterative scheme is based on Remark 1.1, Remark 1.2, and some of the findings discussed in the literature and above. The GRRITA is defined as

$$\begin{cases} x_0 & \in C \\ z_n & = T[(1-\lambda_n)x_n + \lambda_n T x_n] \\ y_n & = T[\frac{\phi z_n + T z_n}{\phi+1}] \\ x_{n+1} & = T[\frac{\phi y_n + T y_n}{\phi+1}] \end{cases} \tag{10}$$

where the control sequences satisfy $\phi = \frac{1+\sqrt{5}}{2}$, and $0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < 1$.

Theorem 3.1. *Let* C *be a nonempty closed and convex subspace of a uniformly convex Banach space* X *and let* $T : X \to X$ *be a contraction mapping with constant* k *such that* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be the iterative sequence defined by* (10). *Then* $\{x_n\}$ *converges strongly to the unique fixed point* p *of* T.

Proof. Suppose that $F(T) \neq \emptyset$ and let $p \in F(T)$, we have

$$Tp = p.$$

Using Algorithm (10), and the fact that $Tp = p$, we obtain

$$\|z_n - p\| = \left\|T\left[(1-\lambda_n)x_n + \lambda_n T x_n\right] - Tp\right\|$$

$$\leq k\,\|(1-\lambda_n)(x_n-p)+\lambda_n(Tx_n-Tp)\|$$

$$\leq k\,[(1-\lambda_n)\|x_n-p\|+\lambda_n\|Tx_n-Tp\|]$$

$$\leq k\,[(1-\lambda_n)+\lambda_n k]\,\|x_n-p\|$$

$$= k\big[1-\lambda_n(1-k)\big]\,\|x_n-p\|. \tag{11}$$

Furthermore, using Algorithm (10) and (11), we obtain

$$\|y_n-p\| = \left\|T\left[\frac{\phi z_n+Tz_n}{\phi+1}\right]-Tp\right\|$$

$$\leq k\left\|\frac{\phi(z_n-p)+(Tz_n-Tp)}{\phi+1}\right\|$$

$$\leq \frac{k}{\phi+1}\,[\phi\|z_n-p\|+\|Tz_n-Tp\|]$$

$$\leq \frac{k(\phi+k)}{\phi+1}\|z_n-p\|$$

$$\leq \frac{k^2(\phi+k)}{\phi+1}\big[1-\lambda_n(1-k)\big]\,\|x_n-p\|. \tag{12}$$

Similarly, using Algorithm (10) and (12), we obtain

$$\|x_{n+1}-p\| = \left\|T\left[\frac{\phi y_n+Ty_n}{\phi+1}\right]-Tp\right\|$$

$$\leq \frac{k(\phi+k)}{\phi+1}\|y_n-p\|$$

$$\leq k^3\left(\frac{\phi+k}{\phi+1}\right)^2\big[1-\lambda_n(1-k)\big]\,\|x_n-p\|. \tag{13}$$

From (13) recursively, we have

$$\|x_{n+1}-p\| \leq \left(k^3\left(\frac{\phi+k}{\phi+1}\right)^2\right)^{n+1}\Pi_{i=0}^{n}\big[1-\lambda_i(1-k)\big]^{(n+1)}\|x_0-p\|. \tag{14}$$

Since

$$0\leq\lambda_n\leq 1,$$

we have

$$1-\lambda_n(1-k)\leq 1.$$

Furthermore, since $0\leq k<1$,

$$0<\frac{\phi+k}{\phi+1}<1.$$

Therefore,

$$k^3\left[1-\lambda_n(1-k)\right]\left(\frac{\phi+k}{\phi+1}\right)^2 \leq k^3\left(\frac{\phi+k}{\phi+1}\right)^2 < k^3 < 1.$$

Hence from (14), we obtain

$$\lim_{n\to\infty}\|x_n-p\|=0.$$

Hence, the sequence generated by (10) converges strongly to the unique fixed point p of T.

For uniqueness, suppose p and q are two fixed points of T such that $Tp=p$ and $Tq=q$. It is easy to see that

$$\|p-q\| = \|Tp-Tq\| \leq k\|p-q\| \qquad \text{which implies that} \qquad (1-k)\|p-q\|=0. \tag{15}$$

Thus, $p=q$. Hence the fixed point is unique. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and let $T: X\to X$ be a contraction mapping with constant k such that $F(T)\neq\emptyset$. Then, the iterative schemes* (6)*,* (7)*, and* (8) *converge strongly to the unique fixed point p of T.*

Proof. The proof is similar to the above. For completeness, see [4, 17, 18]. □

Theorem 3.3. *Let C be a nonempty closed and convex subspace of a uniformly convex Banach space X and let $T: X\to X$ be a contraction mapping with constant k such that $F(T)\neq\emptyset$. Let $\{x_n\}$ be the iterative sequence defined by* (10)*. Then, the* (10) *iteration is T-stable.*

Proof. Let $\{v_n\}$ be an arbitrary sequence in C and define

$$\varepsilon_n = \left\| v_{n+1} - T\left(\frac{\phi V_n + TV_n}{\phi+1}\right)\right\|,$$

where

$$V_n = T\left(\frac{\phi Z_n + TZ_n}{\phi+1}\right),$$

and

$$Z_n = T\left((1-\lambda_n)v_n + \lambda_n Tv_n\right),$$

Thus,

$$\varepsilon_n = \|v_{n+1} - F(T,v_n)\|,$$

where F denotes the GRRITA iteration operator. To establish that (10) is T-stable, we need to show that

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{if and only if} \quad lim_{n\to\infty} v_n = p.$$

To start with, we need to show that if

$$\varepsilon_n \longrightarrow 0 \quad \Longrightarrow \quad v_n \longrightarrow p.$$

Using the same approach as in Theorem 3.1, and the fact that

$$\|v_{n+1} - p\| \leq \varepsilon_n + \|F(T, v_n) - p\|.$$

we obtain that

$$\|v_{n+1} - p\| \leq \varepsilon_n + k^3 \left(\frac{\phi + k}{\phi + 1}\right)^2 \big[1 - \lambda_n(1-k)\big] \|v_n - p\|. \tag{16}$$

It has been established above that $0 \leq k^3 \left(\frac{\phi+k}{\phi+1}\right)^2 \big[1 - \lambda_n(1-k)\big] < 1$, Thus applying Lemma 2.6, we have

$$v_n \longrightarrow p.$$

Conversely, suppose that

$$v_n \longrightarrow p.$$

We have that

$$\begin{aligned} \varepsilon_n &= \|v_{n+1} - F(T, v_n)\| \\ &\leq \|v_{n+1} - p\| + \|F(T, v_n) - p\| \\ &\leq \|v_{n+1} - p\| + k^3 (\frac{(\phi + k)}{\phi + 1})^2 \big[1 - \lambda_n(1-k)\big] \|v_n - p\|. \end{aligned} \tag{17}$$

Thus,

$$\lim_{n\to\infty} \varepsilon_n = 0$$

Therefore, the GRRITA iteration (10) is T-stable. □

Theorem 3.4. *Let C be a nonempty closed convex subspace of a Banach space X and let $T,S: C \to C$ be contractions with contraction constant $k \in (0,1)$. Suppose that*

$$\|Tx - Sx\| \leq \varepsilon, \qquad x \in C, \tag{18}$$

for some $\varepsilon > 0$. Let p and q denote the unique fixed points of T and S, respectively. Suppose that the sequences $\{x_n\}$ and $\{\widetilde{x}_n\}$ are generated by (10) *with respect to T and S, respectively, where*

$$A_n = k^3 \left(\frac{\phi + k}{\phi + 1}\right)^2 [1 - \lambda_n(1-k)] \leq A < 1. \tag{19}$$

Then

$$\|p - q\| \leq \frac{B}{1-A}\,\varepsilon, \tag{20}$$

where

$$B_n = k^2 \left(\frac{\phi + k}{\phi + 1}\right)^2 (1 + k\lambda) + \left(1 + \frac{k(\phi + k)}{\phi + 1}\right)\left(1 + \frac{k}{\phi + 1}\right), \tag{21}$$

Proof. Let $\{x_n\}$ and $\{\widetilde{x}_n\}$ be the sequences generated by GRITTA with respect to T and S, respectively. Denote the corresponding auxiliary sequences by

$$z_n, y_n, x_{n+1} \quad \text{and} \quad \widetilde{z}_n, \widetilde{y}_n, \widetilde{x}_{n+1}$$

respectively. Now, using

$$z_n = T[(1 - \lambda_n)u_n + \lambda_n T u_n]$$

and

$$\widetilde{z}_n = S[(1 - \lambda_n)\widetilde{u}_n + \lambda_n S\widetilde{u}_n],$$

using the contraction property of T and S and

$$\|Tx - Sx\| \leq \varepsilon,$$

we obtain

$$\begin{aligned}
\|z_n - \widetilde{z}_n\| &= \|T[(1 - \lambda_n)x_n + \lambda_n T x_n] - S[(1 - \lambda_n)\widetilde{x}_n + \lambda_n S\widetilde{x}_n]\| \\
&\leq \|T[(1 - \lambda_n)x_n + \lambda_n T x_n] - T[(1 - \lambda_n)\widetilde{x}_n + \lambda_n S\widetilde{x}_n]\| \\
&+ \|T[(1 - \lambda_n)\widetilde{x}_n + \lambda_n S\widetilde{x}_n] - S[(1 - \lambda_n)\widetilde{x}_n + \lambda_n S\widetilde{x}_n]\|
\end{aligned}$$

$$\begin{aligned}
&\leq k\|(1-\lambda_n)(x_n-\widetilde{x}_n)+\lambda_n(Tx_n-S\widetilde{x}_n)\|+\varepsilon\\
&\leq k(1-\lambda_n)\|x_n-\widetilde{x}_n\|+k\lambda_n\|Tx_n-T\widetilde{x}_n\|+k\lambda_n\|T\widetilde{x}_n-S\widetilde{x}_n)\|+\varepsilon\\
&\leq k[1-\lambda_n(1-k)]\|x_n-\widetilde{x}_n\|+(1+k\lambda_n)\varepsilon. \qquad (22)
\end{aligned}$$

Using a similar approach as in the above, we obtain

$$\begin{aligned}
\|y_n-\widetilde{y}_n\| &\leq k\frac{\phi+k}{\phi+1}\|z_n-\widetilde{z}_n\|+(1+\frac{k}{\phi+1})\varepsilon\\
&\leq k^2\frac{\phi+k}{\phi+1}[1-\lambda_n(1-k)]\|x_n-\widetilde{x}_n\|+(1+\frac{k}{\phi+1})\varepsilon+(1+k\lambda_n)\varepsilon. \qquad (23)
\end{aligned}$$

Similarly, we have

$$\|x_{n+1}-\widetilde{x}_{n+1}\| \leq k^2(\frac{\phi+k}{\phi+1})^2\|z_n-\widetilde{z}_n\|+(1+\frac{k}{\phi+1})(1+k\frac{\phi+k}{\phi+1})\varepsilon.$$

Using (23), we obtain

$$\begin{aligned}
\|x_{n+1}-\widetilde{x}_{n+1}\| &\leq k^3(\frac{\phi+k}{\phi+1})^2[1-\lambda_n(1-k)]\|x_n-\widetilde{x}_n\|\\
&\quad+\left[k^2(\frac{\phi+k}{\phi+1})^2(1+k\lambda_n)+(1+k\frac{\phi+k}{\phi+1})(1+k(\frac{\phi+k}{\phi+1}))\right]\varepsilon. \qquad (24)
\end{aligned}$$

Suppose that

$$A_n = k^3(\frac{\phi+k}{\phi+1})^2[1-\lambda_n(1-k)]$$

and

$$B_n = k^2(\frac{\phi+k}{\phi+1})^2(1+k\lambda_n)+(1+\frac{k}{\phi+1})(1+k\frac{\phi+k}{\phi+1}).$$

By assumption, there exist constants $A<1$ and $B>0$ such that

$$A_n \leq A, \qquad B_n \leq B.$$

Hence,

$$\|x_{n+1}-\widetilde{x}_{n+1}\| \leq A\|x_n-\widetilde{x}_n\|+B\varepsilon. \qquad (25)$$

Using the fact that x_n converges to p and $\widetilde{x}$ converges to q, and letting $n\to\infty$ in (25), we have

$$\|p-q\| \leq A\|p-q\|+B\varepsilon.$$

Therefore,

$$(1-A)\|p-q\| \leq B\varepsilon,$$

and hence

$$\|p-q\| \leq \frac{B}{1-A}\varepsilon.$$

□

Theorem 3.5. *Let C be a nonempty closed and convex subspace of a uniformly convex Banach space X and let $T : X \to X$ be a contraction mapping with constant k such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the* (10)*. Then the sequence $\{x_n\}$ converges Q-linearly to the unique fixed point p of T. Moreover,*

$$\limsup_{n\to\infty} \frac{\|x_{n+1}-p\|}{\|x_n-p\|} \leq k^3 \left(\frac{\phi+k}{\phi+1}\right)^2 [1-\underline{\lambda}(1-k)] < 1.$$

Hence, the order of convergence of the (10) *iteration is one.*

Proof. From Theorem 3.1, it is easy to see that

$$\|x_{n+1}-p\| \leq Q_n\|x_n-p\|,$$

where

$$Q_n = k^3 \left(\frac{\phi+k}{\phi+1}\right)^2 [1-\lambda_n(1-k)].$$

Dividing both sides by $\|x_n-p\|$, provided $x_n \neq p$, gives

$$\frac{\|x_{n+1}-p\|}{\|x_n-p\|} \leq k^3 \left(\frac{\phi+k}{\phi+1}\right)^2 [1-\lambda_n(1-k)].$$

In addition, since

$$1-\lambda_n(1-k) \leq 1-\underline{\lambda}(1-k).$$

Therefore,

$$\frac{\|x_{n+1}-p\|}{\|x_n-p\|} \leq k^3 \left(\frac{\phi+k}{\phi+1}\right)^2 [1-\underline{\lambda}(1-k)].$$

Taking the upper limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{\|x_{n+1}-p\|}{\|x_n-p\|} \leq k^3 \left(\frac{\phi+k}{\phi+1}\right)^2 [1-\underline{\lambda}(1-k)].$$

Also, since

$$0 < k < 1,$$

$$0 < \frac{\phi + k}{\phi + 1} < 1,$$

and

$$0 < 1 - \underline{\lambda}(1-k) < 1,$$

it follows that

$$0 < k^3 \left(\frac{\phi + k}{\phi + 1}\right)^2 [1 - \underline{\lambda}(1-k)] < 1.$$

Hence,

$$\limsup_{n\to\infty} \frac{\|x_{n+1} - p\|}{\|x_n - p\|} < 1.$$

By the definition of Q-linear convergence, the sequence $\{x_n\}$ converges linearly to p. Therefore, the order of convergence of the (10) iteration is 1.

□

Theorem 3.6. *Let C be a nonempty closed and convex subspace of a uniformly convex Banach space X and let $T : X \to X$ be a contraction mapping with constant k such that $F(T) \neq \emptyset$. Consider the iterative schemes defined by* (6), (7), (8) *and* (10)*, respectively. Then the error estimates of the iterative schemes are given by*

$$\|x_{n+1} - p\| \leq R_{SH-I} \|x_n - p\|,$$

where

$$R_{SH-I} = k^2 \frac{\alpha_n + k}{\alpha_n + 1},$$

$$\|x_{n+1} - p\| \leq R_{BK} \|x_n - p\|,$$

where

$$R_{BK} = k^3 \frac{\alpha_n + k}{\alpha_n + 1},$$

$$\|x_{n+1} - p\| \leq R_{SH-II} \|x_n - p\|,$$

where

$$R_{SH-II} = k^2 \frac{k}{\alpha_n} \left(1 + \alpha_n + \frac{\alpha_n + k}{\alpha_n + 1}\right), \frac{\beta_n + k}{\beta_n + 1} \frac{\gamma_n + k}{\gamma_n + 1},$$

and

$$\|x_{n+1} - p\| \leq R_{GRRITA} \|x_n - p\|,$$

where

$$R_{GRRITA} = k^3 \left(\frac{\phi + k}{\phi + 1} \right)^2 [1 - \lambda_n (1 - k)].$$

Furthermore, if the control parameters satisfy

$$R_{GRRITA} < \min\{R_{SH-I}, R_{BK}, R_{SH-II}\},$$

then the GRRITA iteration converges faster than SH-I, BK and SH-II iterations.

Proof. Using a similar approach as in Theorem 3.1, and the definition of contraction, we have for the (6) iteration

$$\|x_{n+1} - p\| \leq k^2 \frac{\alpha_n + k}{\alpha_n + 1} \|x_n - p\|.$$

Thus,

$$R_{SH-I} = k^2 \frac{\alpha_n + k}{\alpha_n + 1}.$$

In addition, for the (7) iteration, we obtain

$$\|x_{n+1} - p\| \leq k^3 \frac{\alpha_n + k}{\alpha_n + 1} \|x_n - p\|,$$

and therefore,

$$R_{BK} = k^3 \frac{\alpha_n + k}{\alpha_n + 1}.$$

For the (8) iteration, we obtain

$$\|x_{n+1} - p\| \leq k^2 \frac{k}{\alpha_n} \left(1 + \alpha_n + \frac{\alpha_n + k}{\alpha_n + 1} \right) \frac{\beta_n + k}{\beta_n + 1} \frac{\gamma_n + k}{\gamma_n + 1} \|x_n - p\|.$$

Consequently,

$$R_{SH-II} = k^2 \frac{k}{\alpha_n} \left(1 + \alpha_n + \frac{\alpha_n + k}{\alpha_n + 1} \right) \frac{\beta_n + k}{\beta_n + 1} \frac{\gamma_n + k}{\gamma_n + 1}.$$

Finally, for (10), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \frac{k(\phi + k)}{\phi + 1} \|y_n - p\| \\ &\leq k^3 \left(\frac{\phi + k}{\phi + 1} \right)^2 [1 - \lambda_n (1 - k)] \|x_n - p\|. \end{aligned}$$

Thus,

$$R_{GRRITA} = k^3 \left(\frac{\phi + k}{\phi + 1} \right)^2 [1 - \lambda_n (1 - k)].$$

TABLE 1. Numerical comparison of convergence factors for GRRITA, SH-I, BK and SH-II

Example	k	α_n	β_n	γ_n	λ_n	R_{SH-I}	R_{BK}	R_{SH-II}
1	0.5	1	1	1	0.5	0.1875	0.0938	0.1934
2	0.6	1	1	1	0.5	0.2880	0.1728	0.3871
3	0.7	1	1	1	0.6	0.4165	0.2915	0.7063
4	0.4	1	1	1	0.5	0.1120	0.0448	0.0847
5	0.8	2	1	1	0.7	0.5973	0.4779	0.8156

Example	R_{GRRITA}	Comparison
1	0.0614	$R_{GRRITA} < R_{BK} < R_{SH-I} < R_{SH-II}$
2	0.1240	$R_{GRRITA} < R_{BK} < R_{SH-I} < R_{SH-II}$
3	0.2205	$R_{GRRITA} < R_{BK} < R_{SH-I} < R_{SH-II}$
4	0.0266	$R_{GRRITA} < R_{BK} < R_{SH-II} < R_{SH-I}$
5	0.3756	$R_{GRRITA} < R_{BK} < R_{SH-I} < R_{SH-II}$

It is easy to see that

$$R_{GRRITA} < \min\{R_{SH-I}, R_{BK}, R_{SH-II}\},$$

the (10) iteration possesses the smallest error reduction factor and converges faster than the other schemes. □

It is easy to see from the above numerical results that the (10) possesses the smallest error reduction factor. This justifies our claim that the golden ratio averaging structure improves convergence.

4. NUMERICAL EXAMPLE AND APPLICATION

4.1. Numerical Example. In this section, we illustrate the efficiency and computational performance of our proposed iterative scheme (10). Some numerical experiments are presented and compared with the existing (6), (8) and (7).

TABLE 2. Comparison of numerical results for SH-I, BK, SH-II and GRRITA schemes

Method	Number of Iterations	CPU Time	Final Approximation
SH-I	9	0.002764	0.4501836113
BK	7	0.0002419	0.4501836113
SH-II	7	0.0036818	0.4501836113
GRRITA	5	0.0002944	0.4501836113

Example 4.1. Let $T : [0,1] \to [0,1]$ be defined as

$$T(x) = \frac{1}{2}\cos(x), \qquad x \in [0,1]. \tag{26}$$

It is easy to see that the fixed point of the operator T is $p = 0.4501836112948736$.

The iterative schemes were implemented using the initial approximation

$$x_0 = 0.8,$$

with the parameters

$$\alpha_n = 2, \qquad \beta_n = 2, \qquad \gamma_n = 2,$$

and for the GRRITA scheme,

$$\phi = \frac{1+\sqrt{5}}{2},$$

while the control sequence $\{\lambda_n\}$ was selected such that

$$0.2 \leq \lambda_n \leq 0.8.$$

The stopping criterion was defined as

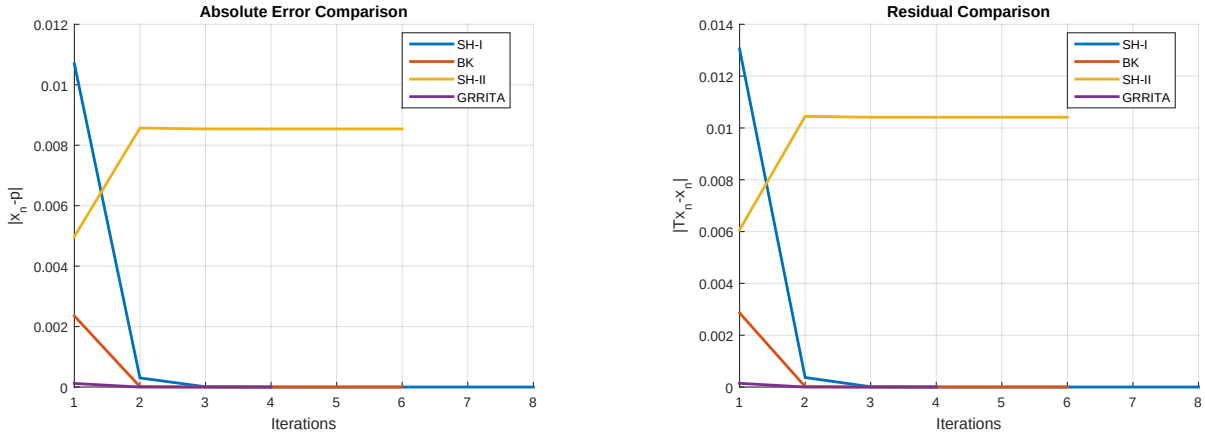
$$\|x_{n+1} - x_n\| < 10^{-12}. \tag{27}$$

The numerical results obtained from the four iterative schemes are displayed in Table 2.

Remark 4.2. The convergence behavior of the iterative schemes was further investigated using the absolute error and residual error plots. The results demonstrate that all the methods converge to the fixed point p, confirming their theoretical convergence properties. However, the

TABLE 3. Comparison of iteration values for SH-I, BK, SH-II and GRRITA

Iter.	SH-I	BK	SH-II	GRRITA
1	0.460878690322	0.447831029376	0.455161852472	0.449902670125
2	0.450485525993	0.450197993111	0.458751864415	0.450184178510
3	0.450192103156	0.450183523296	0.458722248703	0.450183610370
4	0.450183850118	0.450183611833	0.458722493783	0.450183611296
5	0.450183618011	0.450183611292	0.458722491755	0.450183611295
6	0.450183611484	0.450183611295	0.458722491771	—
7	0.450183611300	0.450183611295	0.458722491771	—
8	0.450183611295	—	—	—
9	0.450183611295	—	—	—



(A) Absolute Error Convergence Graph (B) Residual Comparison Graph

FIGURE 1. Absolute error and residual comparison for Example 4.1

(10) method achieves the fastest convergence rate by requiring fewer iterations and producing smaller error values compared with the existing (6),(7), and (8). This indicates that the proposed scheme has improved computational efficiency and convergence performance.

Example 4.3. Let $T : \mathbb{R} \to \mathbb{R}$ be defined as

$$T(x) = e^{-x} + x + \sin x, \qquad x \in \mathbb{R}. \tag{28}$$

It is easy to see that the fixed point of the operator T is $p = 1.234982279$. The iterative schemes were implemented using the initial approximation

$$x_0 = 0.8,$$

with the parameters

$$\alpha_n = 2, \qquad \beta_n = 2, \qquad \gamma_n = 2,$$

and for the GRRITA scheme,

$$\phi = \frac{1+\sqrt{5}}{2},$$

while the control sequence $\{\lambda_n\}$ was selected such that

$$0.2 \leq \lambda_n \leq 0.8.$$

The stopping criterion was defined as

$$\|x_{n+1} - x_n\| < 10^{-12}. \tag{29}$$

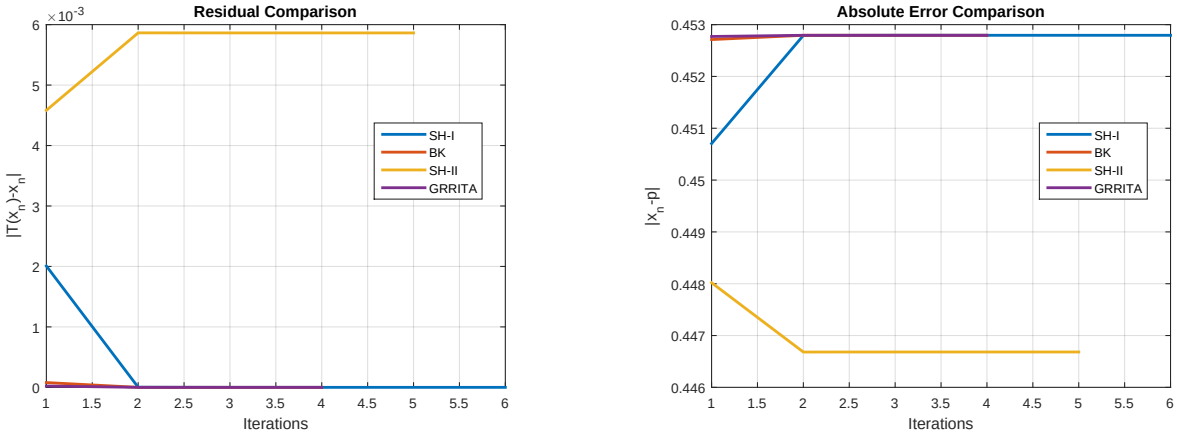
The numerical results obtained from the four iterative schemes are displayed in Table 4.

TABLE 4. Iteration values comparison for SH-I, BK, SH-II and GRRITA

Iter.	**SH-I**	**BK**	**SH-II**	**GRRITA**
1	1.232892647324	1.234899989384	1.230207393465	1.234959608240
2	1.234980125456	1.234982275989	1.228869589298	1.234982279025
3	1.234982277040	1.234982279232	1.228870794267	1.234982279232
4	1.234982279230	1.234982279232	1.228870793171	1.234982279232
5	1.234982279232	–	1.228870793172	–
6	1.234982279232	–	–	–

TABLE 5. Final computational comparison of the iterative schemes

Method	Number of Iterations	CPU Time (s)
SH-I	6	0.0014701
BK	4	0.0003695
SH-II	5	0.0042913
GRRITA	4	0.0001136



(A) Residual Comparison Graph

(B) Absolute Error Comparison Graph

FIGURE 2. Residual and absolute error comparison for Example 4.3

4.2. Application to Image Restoration Problem. This section presents a comparative analysis of the image restoration problem involving (10), (6), (8) and (7). The well-known image restoration problem takes the form

$$\min_x \left\{ \|Ax-b\|_2^2 + \phi \|x\|_1 \right\},$$

where

$$f(x) = \|Ax-b\|_2^2$$

denotes the data fidelity term, and

$$g(x) = \phi \|x\|_1,$$

represents the regularization term, with ϕ being the regularization parameter. The operator A corresponds to the Gaussian blurring operator, while b denotes the observed blurred image. In this numerical experiment, pepper and tire images of size 256×256 are used as the original images. The image is degraded using Gaussian blur with a 7×7 kernel and standard deviation $\rho = 1.5$, followed by additive Gaussian noise of variance 0.00012. The restoration quality is measured using the Peak Signal-to-Noise Ratio (PSNR). The PSNR formula [8] is defined by

$$\text{PSNR}(\kappa_n) = 10\log_{10}\left(\frac{(\text{MAX}_I)^2}{\text{MSE}}\right),$$

where MAX_I denotes the maximum possible pixel intensity value. Since the images are normalized to the interval $[0,1]$, we take $\text{MAX}_I = 1$. Therefore, the PSNR formula reduces to

$$\text{PSNR}(\kappa_n) = 10\log_{10}\left(\frac{1}{\text{MSE}}\right).$$

The mean squared error (MSE) is defined as

$$\text{MSE} = \frac{1}{S}\|\kappa_n - \kappa\|_2^2,$$

where κ denotes the original image and S represents the total number of image pixels. The initial image in this experiment was blurry, and the regularization parameter was set at $\phi = 10^{-5}$. The control parameters

$$\alpha_n = 3, \qquad \beta_n = 2.5, \qquad \gamma_n = 2,$$

and for the (10) scheme,

$$\phi = \frac{1+\sqrt{5}}{2},$$

while the control sequence $\{\lambda_n\}$ was selected such that

$$0.2 \leq \lambda_n \leq 0.8.$$

The maximum number of iterations is taken as $n = 1500$. The numerical results are shown in Figs. 3–5.

Remark 4.4. The above result shows that the iterative processes are very good for image restoration. The results also show that the iterative process (10) is better in image restoration.



(A) Original image (B) Blurred image

FIGURE 3. Original and blurred images



(A) Restored by SH-I (B) Restored by SH-II

FIGURE 4. Images restored by SH-I and SH-II



FIGURE 5. Image restored by GRRITA

5. CONCLUSION

In this study, we have introduced and studied the Golden Ratio Relaxed Three-Step Algorithm (GRRITA) as a novel iterative scheme for fixed point problems. We established the strong convergence, stability, and data dependence results under weak assumptions, while error analysis demonstrates that the proposed iterative scheme (10) achieves faster convergence than the likes of (6), (8), (7) and some existing iterative schemes in the literature. To validate our theoretical results, we also provided some numerical experiments. These numerical experiments show that the iterative scheme (10) outperforms other iterative algorithms.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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