Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 2, 254-262
ISSN: 1927-6303

ON THE TOPOLOGICAL DEGREE THEORY AND FIXED POINT THEOREMS IN PROBABILISTIC NORMED SPACES

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Abstract. In this paper, we study some properties of the Leray-Schauder topological degree in the probabilistic linear normed spaces, and obtain some fixed point theorems applying the topological degree theory.

Keywords: Leray-Schauder topological degree; Probabilistic normed space; Compact continuous operator; Menger-PN space; Fixed points theorem.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

A function $f : R \to R$ is called a distribution function if it is non-decreasing, left-continuous and has inf 0 and sup 1. Let $\mathscr{D}$ be the set of all such distribution functions and $H(t)$ denote the specific distribution function defined by

$$H(t) = \begin{cases} 1 & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Definition 1.1. [1] A function $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if for all $a, b, c, d \in [0,1]$ satisfying the following conditions:

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Received February 16, 2015

(1) $\Delta(a,1)=a$;

(2) $\Delta(a,b)=\Delta(b,a)$;

(3) $\Delta(c,d)\geqslant\Delta(a,b), \quad \forall c\geqslant a, d\geqslant b$;

(4) $\Delta(\Delta(a,b),c)=\Delta(a,\Delta(b,c))$.

Definition 1.2. [1] A Menger probabilistic linear normed space (Menger PN-space for short) is a triplet $(E,\mathscr{F},\Delta)$, where E is a real linear space, Δ is a t-norm and $\mathscr{F}:E\to\mathscr{D}$ is a mapping (we shall denote the distribution function $\mathscr{F}(x)$ by f_x) satisfying the following conditions:

PN-(1) $f_x(0)=0$;

PN-(2) $f_x(t)=H(t),\forall t\in R$, if and only if $x=\theta$;

PN-(3) $f_{\alpha x}(t)=f_x(t/|\alpha|)$, for all real number $\alpha\neq 0$;

PN-(4) (Menger triangle inequality) for any $x,y\in E$

$$f_{x+y}(t_1+t_2)\geqslant\Delta(f_x(t_1),f_y(t_2)), \quad \forall t_1,t_2\geqslant 0.$$

More if the t-norm Δ satisfied the condition: $\Delta(t,t)\geqslant t,\forall t\in[0,1]$, then as Schweizer and Sklar [2] showed, $(E,\mathscr{F},\Delta)$ is a Hausdorff topological vector space in the topology τ induced by the family of

$$\{U_x(\varepsilon,\lambda):x\in E,\varepsilon>0,\lambda>0\}$$

where $U_x(\varepsilon,\lambda)=\{y:y\in E,f_{y-x}(\varepsilon)>1-\lambda\}$. This topology τ is locally convex and $\{U_\theta(\varepsilon,\lambda):\varepsilon>0,\lambda>0\}$ is a local base at θ.

Zhang [1] established the Leray-Schauder topological degree theory in the probabilistic linear normed space and gave out some applications on the fixed point theory. Zhu [3,4,5] obtained some fixed point theorems based on the topological degree theory. This paper was motivated by Zhang [1], we studied properties of the Leray-Schauder topological degree and obtained some new fixed point theorems in the probabilistic linear normed space. More the Leray-Schauder topological degree theory in the probabilistic linear normed space by Zhang [1] is a specification of the degree theory of compact perturbations of identity in locally convex spaces by Nagumo [6], but we still focus our attention on the probabilistic linear normed space and some results can be generalized to locally convex topological vector spaces.

Through this paper, let $(E,\mathscr{F},\Delta)$ be a Menger PN-space with $\Delta(t,t)\geqslant t,\forall t\in[0,1]$. Let Ω be a bounded open set of E, and $T:\overline{\Omega}\to E$ is a compact continuous mapping. Let $S=I-T$ and $p\overline{\in}S(\partial\Omega)$. Let $Deg(S,\Omega,p)$ be the Leray-Schauder topological degree defined in Zhang [1, Definition 2].

2. The topological degree in Menger PN-spaces

Lemma 2.1. *Suppose $(E,\mathscr{F},\Delta)$ be a Menger PN-space with $\Delta(t,t)\geqslant t,\forall t\in[0,1]$. Let Ω be a bounded open set of E, and $T:\overline{\Omega}\to E$ is a compact continuous mapping. Let $S=I-T$, then*

(1) If $C\subset E$ is compact, then $S^{-1}(C)$ is compact;

(2) S is a closed mapping, i.e. $S(C)$ is closed if $C\subset\overline{\Omega}$ is a closed subset;

(3) If $B\subset\overline{\Omega}$ is bounded, then $S(B)$ is bounded.

Proof.

(1) Suppose the net $\{x_\delta\}_\delta\subset S^{-1}(C)$. We have $x_\delta\in\overline{\Omega}$ and $y_\delta=S(x_\delta)=x_\delta-Tx_\delta\in C$. Since T is a compact mapping and $\overline{\Omega}$ is bounded, there exist a subnet of $\{x_\delta\}_\delta$ such that $Tx_\delta\to z\in E$ (Without loss of generality, we assume the subnet is the net itself). Since C is a compact set, there exist a subnet of $\{y_\delta\}_\delta$ such that $y_\delta\to y\in C$. Hence $x_\delta=Sx_\delta+Tx_\delta\to y+z=x\in\overline{\Omega}$. Since T is continuous we have $y=Sx$ and $x\in S^{-1}(y)\subset S^{-1}(C)$. This means that $S^{-1}(C)$ is compact.

(2) (See Zhang [1, Lemma 1.2], where the details of proof are omitted) Suppose $C\subset\overline{\Omega}$ is a closed subset and $z\in\overline{S(C)}$. Then there exist a net $\{x_\delta\}_\delta\subset C$ such that $S(x_\delta)\to z$. Since T is a compact mapping and $\overline{\Omega}$ is bounded, there exist a subnet of $\{x_\delta\}_\delta$ such that $Tx_\delta\to y\in E$. Notice C is closed, $x_\delta=Tx_\delta+Sx_\delta\to y+z=x\in C$. Since T is continuous we have $z=x-Tx=Sx\in S(C)$. This means that $S(C)$ is closed.

(3) Suppose B is a bounded set. Since T is a compact mapping, $\overline{T(B)}$ is compact. Hence $\overline{T(B)}$ is bounded and so is $T(B)$. Hence $S(B)=(I-T)(B)\subset B-T(B)$ is bounded.

Lemma 2.2. [1, Theorem 2.1] The topological degree has the following properties:

(I) $Deg(I,\Omega,p)=1,\forall p\in\Omega$;

(II) $Deg(S,\Omega,p)\neq 0$, then the equation $Sx=p$ has a solution in Ω;

(III) if $H(t,x)$ is a compact continuous mapping defined on $[0,1]\times\overline{\Omega}$ and $p\overline{\in}(I-H(t,\cdot))(\partial\Omega),\forall t\in[0,1]$, then $Deg(I-H(t,\cdot),\Omega,p)$ is independent of $t\in[0,1]$;

(IV) if Ω_1 and Ω_2 are two disjointed open subsets of Ω and $p\overline{\in}S(\overline{\Omega}\backslash(\Omega_1\cup\Omega_2))$, then $Deg(S,\Omega,p)=Deg(S,\Omega_1,p)+Deg(S,\Omega_2,p)$;

(V) if Ω_0 is an open subset of Ω and $p\overline{\in}S(\overline{\Omega}\backslash\Omega_0)$, then $Deg(S,\Omega,p)=Deg(S,\Omega_0,p)$;

(VI) if $p\overline{\in}S(\partial\Omega)$, then $Deg(S,\Omega,p)=Deg(S-p,\Omega,\theta)$.

Theorem 2.3. *The topological degree has the following properties:*

(1) if p belongs to an unbounded connected component of $E\backslash S(\partial\Omega)$, then

$$Deg(S,\Omega,p)=0.$$

(2) If there is a $y_0\in E$, $y_0\neq\theta$, such that

$$S(x)\neq p+ry_0,\quad \forall x\in\partial\Omega, r\geqslant 0.$$

Then $Deg(S,\Omega,p)=0$.

Proof.

(1) Suppose $Deg(S,\Omega,p)\neq 0$. Let C be the unbounded connected component of $E\backslash S(\partial\Omega)$ that contains p. From Lemma 2.1, we know $S(\overline{\Omega})$ is bounded. Hence $S(\overline{\Omega})\subset MU_\theta(\varepsilon,\lambda)$ for sufficient large M, here $U_\theta(\varepsilon,\lambda)=\{x\in E: f_x(\varepsilon)>1-\lambda\}$. But C is unbounded, there exist $p_1\in C\backslash MU_\theta(\varepsilon,\lambda)$. Thus from Zhang [1, Theorem 2.2], we have $Deg(S,\Omega,p_1)=Deg(S,\Omega,p)\neq 0$. Hence from Lemma 2.2 we know the equation $Sx=p_1$ has a solution in Ω. This contradicts to $p_1\in C\backslash MU(\varepsilon,\lambda)\subset C\backslash S(\Omega)$. Therefore $Deg(S,\Omega,p)=0$.

(2) Let C be the connected component of $E\backslash S(\partial\Omega)$ that contains p. Let $M=\{p+ry_0: r\in[0,+\infty)\}$. Obviously $M\subset C$. More M is unbounded and so is C . From (1) we have $Deg(S,\Omega,p)=0$.

Definition 2.4. [5, Definition 2] Suppose $(E,\mathscr{F},\Delta)$ is a Menger PN-space, Δ is a t-norm satisfying $\Delta(t,t)\geqslant t,\forall t\in[0,1]$, Ω is a nonempty bounded open subset of E, $T:\overline{\Omega}\to E$ is a compact continuous mapping and $S=I-T$. x_0 is an isolated zero point of S in Ω, this means there is a neighbourhood $U_{x_0}(\varepsilon,\lambda)=\{x: f_{x-x_0}(\varepsilon)>1-\lambda\}\subset\Omega$ of x_0 and x_0 is the only zero point of S

in $\overline{U_{x_0}(\varepsilon,\lambda)}$. We can define

$$ind(I-T,x_0) = Deg(I-T,U_{x_0}(\varepsilon,\lambda),\theta).$$

$ind(I-T,x_0)$ is called the index of $I-T$ at the zero point x_0. From Lemma 2.2, for any open set W satisfying $x_0 \in W \subset U_{x_0}(\varepsilon,\lambda)$ we have

$$Deg(I-T,W,\theta) = ind(I-T,x_0).$$

Theorem 2.5. *Let Ω be a bounded open set of E. $F:\overline{\Omega}\to E$ is a compact continuous mapping and $f=I-F$. $G:f(\overline{\Omega})\to E$ is a compact continuous mapping and $g=I-G$. x_0 is an isolated zero point of f and θ is an isolated zero point of g, then x_0 is an isolated zero point of $g\cdot f$ and more*

$$ind(g\cdot f,x_0) = ind(f,x_0)\cdot ind(g,\theta).$$

Proof. Since θ is an isolated zero point of g, there is a neighbourhood U of θ such that θ is the only zero point of g in $\overline{U}$. Moreover, g is continuous and x_0 is an isolated zero point of f, so there exists a neighbourhood V of x_0 such that $f(\overline{V})\subset U$ and x_0 is the only zero point of f in $\overline{V}$. It is easy to verify x_0 is the only zero point of $g\cdot f$ in $\overline{V}$. More we have

$$ind(f,x_0) = Deg(f,V,\theta), \tag{2.1}$$

$$ind(g,\theta) = Deg(g,U,\theta), \tag{2.2}$$

$$ind(g\cdot f,x_0) = Deg(g\cdot f,V,\theta). \tag{2.3}$$

Let $\{D_\alpha\}_\alpha$ be the set of all connected component of $E\backslash f(\partial V)$. From the Product Theorem of topological degree (see [6, Theorem 9]), we have

$$Deg(g\cdot f,V,\theta) = \sum_{D_\alpha\subset f(V)} Deg(f,V,D_\alpha)\cdot Deg(g,D_\alpha,\theta). \tag{2.4}$$

Without loss of generality, we can suppose D_{α_0} be the component containing θ. Next we divide our proof into the following two cases.

(1) $D_{\alpha_0} \subset f(V)$:

For any $D_\alpha \subset f(V)(\alpha \neq \alpha_0)$, we have

$$g(y) \neq \theta, \quad \forall y \in D_\alpha.$$

From Lemma 2.2, we have $Deg(g, D_\alpha, \theta) = 0$. This means

$$Deg(g, D_\alpha, \theta) = 0, \quad \forall \alpha \neq \alpha_0, D_\alpha \subset f(V). \tag{2.5}$$

On the other hand, since $\theta \in D_{\alpha_0} \subset U$, we have

$$Deg(g, D_{\alpha_0}, \theta) = Deg(g, U, \theta). \tag{2.6}$$

Combine (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6), we have

$$\begin{aligned} ind(g \cdot f, x_0) &= Deg(g \cdot f, V, \theta) \\ &= Deg(f, V, D_{\alpha_0}) \cdot Deg(g, D_{\alpha_0}, \theta) \\ &= Deg(f, V, \theta) \cdot Deg(g, D_{\alpha_0}, \theta) \\ &= Deg(f, V, \theta) \cdot Deg(g, U, \theta) \\ &= ind(f, x_0) \cdot ind(g, \theta). \end{aligned}$$

(2) $D_{\alpha_0} \nsubseteq f(V)$:

Since $D_{\alpha_0} \nsubseteq f(V)$, there exists $p_0 \in D_{\alpha_0} \backslash f(V)$. Immediately we have $Deg(f, V, p_0) = 0$. Hence

$$Deg(f, V, \theta) = Deg(f, V, p_0) = 0. \tag{2.7}$$

On the other hand, for any $D_\alpha \subset f(V)$, we have $D_\alpha \cap D_{\alpha_0} = \emptyset$. Therefore $\theta \overline{\in} D_\alpha$ and more $\theta \overline{\in} g(D_\alpha)$. Hence

$$Deg(g, D_\alpha, \theta) = 0. \tag{2.8}$$

Combine (2.1), (2.2), (2.3), (2.4), (2.7) and (2.8)

$$ind(g \cdot f, x_0) = ind(f, x_0) \cdot ind(g, \theta) = 0.$$

This complete the proof.

Remark 2.6. We can generalize Definition 2.3 and Theorem 2.2 to locally convex topological vector spaces with the topological degree defined as Nagumo M [6]. Here we omit the details.

3. Fixed point theorems in Menger PN-spaees

In this section, we would study some new fixed point theorems for mapping in Menger PN-space based on the topological degree theory.

Theorem 3.1 *Let $(E,\mathscr{F},\Delta)$ be a locally bounded Menger PN-space with $\Delta(t,t)\geqslant t,\forall t\in[0,1]$. $T:E\to E$ is a compact continuous mapping. If the set $V=\{x\in E:f_x(t)=f_{\mu Tx}(t),0<\mu<1,t>0\}$ is bounded. Then T has fixed points in E.*

Proof. Since V is bouned and $(E,\mathscr{F},\Delta)$ is locally bounded, we can choose a bounded open set Ω, such that $V\subset\Omega$ and $\theta\in\Omega$. Without loss of generality, we can suppose T has no fixed points in $\partial\Omega$ (Otherwise the conclusion is proved). Define $H(s,x)=sTx,\forall s\in[0,1],x\in\overline{\Omega}$. Clearly $H:[0,1]\times\overline{\Omega}\to E$ is a compact continuous mapping. We prove that $x-H(s,x)\neq\theta,\forall s\in[0,1],x\in\partial\Omega$. In fact, if there exists $x_0\in\partial\Omega,s_0\in[0,1]$ such that $x-H(s,x)=\theta$. That is $x_0=s_0Tx_0$. Thus $s_0\neq 0$ (Otherwise $x_0=\theta\in\Omega$, which is in contradiction to $x_0\in\partial\Omega$) and we obtain that $f_{Tx_0}(t)=f_{x_0/s_0}(t)$. Notice $s_0\neq 1$ (Otherwise $x_0=Tx_0$, which is contradiction to T has no fixed point in $\partial\Omega$), it follows $x_0\in V\subset\Omega$. This is contradiction to $x_0\in\partial\Omega$. Hence $x-H(s,x)\neq\theta,\forall s\in[0,1],x\in\partial\Omega$. By the homotopy invariance of the topological degree in Lemma 2.2, we obtain $Deg(I-T,\Omega,\theta)=Deg(I,\Omega,\theta)=1$. Thus T has fixed points in Ω. The proof is complete.

Theorem 3.2. *Let $(E,\mathscr{F},\Delta)$ be a Menger PN-space with $\Delta(t,t)\geqslant t,\forall t\in[0,1]$. Let Ω be an bounded open set and $T:\overline{\Omega}\to E$ is a compact continuous mapping. If there exists an $e\in\Omega$ such that:*

$$f_{Tx-x}(t)\leqslant f_{Tx-e}(t),\forall x\in\partial\Omega,t>0.$$

Then $Deg(I-T,\Omega,\theta)=1$ and T has fixed points in Ω.

Proof. First we prove T has no fixed points in $\partial\Omega$. In fact if there exists $x_0 \in \partial\Omega$ such that $Tx_0 = x_0$, it follows $f_{Tx_0-e}(t) \geqslant f_{Tx_0-x_0}(t) = H(t)$. Immediately we have $Tx_0 = e$. This means $x_0 = e \in \Omega$, which is contradiction to $x_0 \in \partial\Omega$. Hence T has no fixed point in $\partial\Omega$.

Define $H(s,x) = (1-s)Tx + se, \forall s \in [0,1], x \in \overline{\Omega}$. Clearly $H : [0,1] \times \overline{\Omega} \to E$ is a compact continuous mapping. More we conclude

$$x - H(s,x) \neq \theta, \forall s \in [0,1], x \in \partial\Omega. \tag{4.1}$$

In fact if (4.1) is false, there exists $s_0 \in [0,1], x_0 \in \partial\Omega$ such that

$$x_0 - H(s_0,x_0) = \theta. \tag{4.2}$$

Obviously $s_0 \neq 1$ (Otherwise $x_0 = e \in \Omega$, which is contradiction to $x_0 \in \partial\Omega$) and $s_0 \neq 0$ (Otherwise $x_0 = Tx_0$, which is contradiction to T has no fixed points in $\partial\Omega$). This means $0 < s_0 < 1$. From (4.2) we obtain $Tx_0 - x_0 = s_0(Tx_0 - e)$. Hence

$$f_{Tx_0-x_0}(t) = f_{s_0(Tx_0-e)}(t) = f_{Tx_0-e}(t/s_0) \geqslant f_{Tx_0-x_0}(t/s_0).$$

Thus $f_{Tx_0-x_0}(t) \geqslant f_{Tx_0-x_0}(t/s_0^n), n = 1,2,3,\cdots, t > 0$. Take the limit ($n \to \infty$), notice $0 < s_0 < 1$ and the definition of distribution function, we obtain

$$f_{Tx_0-x_0}(t) = H(t), \forall t > 0.$$

It follows $Tx_0 - x_0 = \theta$, which is contradiction to T has no fixed points in $\partial\Omega$. Hence (4.1) is true. By the homotopy invariance of the topological degree in Lemma 2.2, we obtain $Deg(I - T, \Omega, \theta) = Deg(I - H(0,\cdot), \Omega, \theta) = Deg(I - H(1,\cdot), \Omega, \theta) = Deg(I - e, \Omega, \theta) = Deg(I, \Omega, e) = 1$ and T has fixed points in Ω.

Conflict of Interests

The authors declare that there is no conflict of interests.

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