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INTELLIGENT PARAMETER ESTIMATION FOR THE POWER-LAW NON-HOMOGENEOUS POISSON PROCESS USING A HYBRID ANN–ABC FRAMEWORK

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Abstract. Reliability estimation and failure modeling in repairable system design requires accurate models and parameter estimations to determine when failures will occur and when they can be repaired. The objective of this study is to evaluate the use of a Hybrid Artificial Neural Network-Artificial Bee Colony (ANN-ABC) approach to estimate the parameters of the Power Law Non-Homogeneous Poisson Process (PLPNHP) referred to as the Crow AMSAA Model. Additionally, we compare our results with maximum likelihood estimates (MLE) and particle swarm optimization (PSO). We conduct a comparison based on their ability to accurately estimate parameters, predict future events independently and their computation times. We perform a Monte Carlo Study using 1000 iterations and three different expected event counts; 50, 100 and 500. All iterations used a power law shape parameter of 1.4 and scale parameter of 500. A data set consisting of 48 valve seat replacement events and 41 observation ending records from 41 diesel engines was analyzed. Engine specific observation period was included in the likelihood function. Our results show that MLE and PSO have similar levels of estimation accuracy, however, the ANN-ABC had larger estimation errors than both MLE and PSO at a faster level of computational time. At a total number of events equal to 100, the Root Mean Square Error (RMSE) value for the power-law shape parameter was .14117 for MLE, .14103 for PSO and .16711 for ANN-ABC. Our empirical case illustrates the potential of the PLPNHP to describe recurring repair events. However, it is necessary to interpret our comparisons carefully due to differences in methods. In general, the analytical method of MLE was superior to both methods of estimation and efficiency in all comparisons.

Keywords: power-law process (PLP); Crow–AMSAA model; non-homogeneous Poisson process (NHPP); artificial neural network (ANN); artificial bee colony (ABC); reliability; parameter estimation.

2020 AMS Subject Classification: 62N05; 60G55; 93E10.

1. INTRODUCTION

Predictive maintenance optimization and reliability evaluation is becoming significantly relevant to modern engineering systems. These include safety-critical applications like healthcare facilities; electrical energy generation and distribution systems; industrial equipment and manufacturing processes; and IoT-enabled monitoring systems. The increasing complexity of these systems necessitates statistically-based methodologies that can model and quantify the occurrence of failures, identify patterns and trends in reliability, and support maintenance decision-making under uncertainty. A number of stochastic frameworks are currently being applied to reliability analysis. Of particular interest among them is the non-homogeneous Poisson process (NHPP). This framework models the occurrence of failures using an intensity function that varies with time. The NHPP allows for the modeling of changes in failure rates over time. In

contrast to homogeneous Poisson processes (HPPs), which assume a constant rate of failures per unit of time, the NHPP captures variability in failure mechanisms during system operation and forms a generalizable basis for modeling repairable systems experiencing repeated failure events (see Lawson [1]; Muralidharan [2]).

A particular important process in the NHPP family is the Power-Law Process (PLP) or better known as the Crow-AMSAA Model. The failure occurrence of this model can be defined by the power-law mean value function and the related time dependent intensity function. Due to its specific mathematical formulation, it allows to identify three different reliability behavior characteristics: decreasing failure rate (reliability improvement); constant failure rate (homogeneous Poisson process); and increasing failure rate (degradation). In detail the trend of the reliability is determined by the shape parameter, whereas the temporal scale of failures accumulation is defined by the scale parameter. Therefore, the PLP-NHPP provides great advantages concerning the application area's reliability growth analysis, maintenance planning, as well as statistical analysis of repairable engineering systems (see Crow [3]; Karbasian [4]).

Foundational treatments of time-dependent repairable-system intensity models include Kim et al. [10], de Oliveira et al. [11], Calabria and Pulcini [12], Sen [13], and Bar-Lev et al. [14]. More recent recurrent-event approaches include Brito et al. [5], Li et al. [6], and Alsultan and Sulaiman [22]. (see Kim et al. [10]; Sen [13]).

Reliability model applications that utilize NHPPs rely on accurate parameter estimates. Maximum Likelihood Estimates (MLE) remain an important statistical method for estimating parameters due to their established inferential advantages as well as the relatively low computational expense in comparison to other methods. When using the conventional two-parameter PLP-NHPP observed during a fixed time period, maximum likelihood functions can be solved analytically such that the unrestricted interior MLE can be computed without requiring numerical optimization to solve the equation set. However, regardless of the number of failures in the sample or how well the assumptions about the data fit the actual data generating process, the accuracy of the parameter estimates will depend upon various factors. Thus, this motivation exists

to study alternative estimation techniques when both the predictive performance of the parameter estimates, robustness of the parameter estimates, and efficiency of computing parameter estimates exist concurrently (see Brito et al. [5]; Li et al. [6]).

Recent advancements in the field of computational intelligence have motivated researchers to explore the use of Machine Learning and Nature-Inspired Optimization Algorithms as alternatives to traditional Statistical Estimation Techniques. The primary benefits of using an Artificial Neural Network (ANN) are its ability to create non-linear mappings from observed input variables to estimated model parameters, while population-based optimization methods can represent alternative means of identifying local or global optima within complex objective functions. Of these two paradigms, PSO and ABC Algorithms have been studied extensively due to their computationally efficient structure, relative ease-of-implementation, and ability to efficiently solve Nonlinear Optimization Problems. However, prior to utilizing PSO or ABC to optimize the parameter estimates of a model that has analytically-derived Maximum Likelihood Estimators, it is essential to formally justify such utilization. For example, simply increasing the amount of computations performed on an algorithm does not automatically result in improved statistical accuracy; therefore, claims made regarding advantages associated with the utilization of one method over another must be substantiated through carefully-controlled simulation studies, appropriately-designed predictive evaluations and reproducibly-comparable comparisons against well-established statistical benchmarks (see Veza et al. [19]; Hacilar et al. [21]).

The recent PLP and maintenance study research will provide examples of the modern applicability of these models: In addition to the above, Alsultan and Sulaiman [22] applied the use of a Bayesian method for estimating PLP based on data from power plant shutdowns; Ye et al. [23], have developed an NHPP-based (non-homogeneous Poisson process) model for predictive maintenance of repairable systems. Additionally, Gu et al. [24], studied dynamic behavior of NHPP for reliable intelligent heating supply. The research described above is about similar modeling situations as opposed to evidence directly indicating ANN–ABC' s superiority over analytical MLE.

Previous studies have studied a number of ways to extend classic reliability and survival models. Using frailty models to allow for unobserved heterogeneity and dependence in recurrence

of failures has become a significant area of interest, with examples being the study of crop protection equipment, and heavy-duty truck components. While these are very flexible tools that can be used to describe differences in operationally different machines as well as inform when maintenance should occur, they depend upon the choice of the repair process and the nature of the dependence relationships between the events (for example see Brito et al. [5]). Relatedly, Li et al. demonstrated how Bayesian change point modeling could be implemented using Markov Chain Monte Carlo methods along with Bayesian model comparison criteria to assess changes in failure behavior from a Weibull intensity non-homogeneous Poisson Process formulation (Li et al. [6]). These studies clearly illustrate the importance of accounting for uncertainty, heterogeneity and structural change in reliability modeling.

Additional studies of alternative NHPP parameter estimation include Hussain et al. [15], Hussain et al. [16], and Hussain et al. [17]. Software reliability optimization is considered by Hussain et al. [18], while Veza et al. [19] review ANN methodologies in engineering. These publications provide related methods rather than direct evidence of PLP-specific superiority. (see Hussain et al. [16]; Hussain et al. [17]).

Statistical estimation combined with computational intelligence has also been studied in survival analysis and other types of related distributional modeling. Some researchers used hybrid approaches that combine both cancer survival models and machine learning methods to predict outcomes of cancer survival (Saadoon et al. [7]). Researchers have also evaluated inverse Gompertz-based survival representations and tested artificial intelligence assisted estimation against traditional statistical estimators through simulation and empirical data (Hussain et al. [9]). While these research efforts provide insight into the overall capabilities of intelligent estimation techniques, they are specific to the PLP-NHPP as the underlying probability structures, likelihood functions, and parameter identification problems differ significantly. Therefore, an evaluation specific to each model will need to occur to identify if computational intelligence can produce statistically significant improvement relative to traditional maximum likelihood estimation. See Saadoon et al. [7], Hussain et al. [9] for additional examples of artificial intelligence estimators in survival modeling or alternative reliability distributions. For related artificial-intelligence estimators in survival modeling and alternative reliability distributions,

see (see Saadoon et al. [7]; Hussain et al. [9]; Hussain et al. [15]; Hussain et al. [16]; Hussain et al. [17]).

Although there are many new techniques available to estimate reliability, there is still an open methodology question in this case: Will a hybrid ANN-ABC estimation procedure produce statistically significant improvements in estimating the two-parameter PLP-NHPP compared to analytical MLE? Under which sampling circumstances can we expect to see such improvements? This question is so relevant since MLE has closed form solutions that establish a high level of statistical and computational performance. Thus, our objective is not simply to create an advanced search strategy using a different optimization algorithm; instead, we wish to evaluate whether the proposed hybrid procedure offers useful advantages in terms of recovering parameters, forecasting accuracy, numerical stability, and/or robustness. Additionally, by comparing with PSO will allow us to determine if any difference we observe comes from the hybrid neural optimization architecture or if it arises solely due to the use of a population-based optimization technique (see Muralidharan [2]; van Dyck and Verdonck [8]; Veza et al. [19]; Hacilar et al. [21]).

To analyze this research inquiry, the current study evaluates a Hybrid Artificial Neural Network – Artificial Bee Colony (ANN-ABC) structure for calculating the parameters of the PLP-NHPP. The proposed ANN-ABC system integrates the ability of an ANN to approximate nonlinear relationships with the searching capabilities of an ABC. It was tested in comparison to analytical MLE and PSO using the same statistical model and evaluation methodology. The comparative evaluations were developed so that both the estimations' biases, the Root Mean Squared Error (RMSE), prediction errors, and computational run times at varying sample size levels could be compared. In total, these metrics provide a complete and fair examination of the trade-offs between computational cost and statistical accuracy among all considered estimation methods:

1. Statistically formulated: This paper utilizes a two-parameter process limit (PLP) new-hire positive point process (NHPP), with identification of the reliability model' s mean-value function, the failure rate of the NHPP, and the reliability trend of the NHPP as functions of the model' s parameters.

2. Framework of hybrid estimation method: A methodology using artificial neural networks (ANNs) and adaptive importance sampling (AIS) based on simulated annealing (ABC-SA) has been developed to estimate the parameters of the PLP-NHPP. The proposed method provides a computationally efficient alternative to traditional maximum likelihood estimation (MLE).

3. Comparative statistical evaluation: The performance of analytical MLE, particle swarm optimization (PSO), and ANN-AIS was evaluated using similar performance metrics such as parameter bias, root-mean-squared error (RMSE), predictive accuracy, and computation time.

4. Assessment of sample size: The performance comparison examined how the quality of estimation changed with varying sample sizes; specifically, it focused on the behavior of both small- and medium-sized samples.

5. Reliability implications for practical application: The purpose of the overall analysis is to identify the potential advantages, disadvantages, and computational trade-offs associated with employing intelligent estimation techniques for reliability prediction in repairable systems and predictive maintenance.

The significance of this research lies in developing a statistically valid basis for comparing classical statistical estimation methods with computational intelligence-based estimation methods in the context of a well-defined mathematical NHPP. Rather than simply assuming that hybrid optimization will always outperform analytical estimation, the research seeks to determine if greater computational complexity is warranted through demonstrated improvements in either statistical performance or predictive performance. Additionally, the results may offer practical direction to reliability engineering professionals and applied statisticians who seek suitable estimation methodologies for modeling time-dependent failure processes in smart monitoring and maintenance applications.

This portion of the paper will be organized in the following manner. In Section 2, we will introduce a mathematically formulated version of the PLP-NHPP along with some of its statistical properties. Then in Section 3, we will describe how to estimate parameters using three different methods: Analytical Maximum Likelihood Estimation (MLE), Particle Swarm Optimization (PSO) and the proposed ANN-ABC framework. In Section 4, we will detail the experimental setup and define the criteria that will be used to evaluate performance. Finally, in Section 5,

we will present and discuss our comparative results; then summarize our major conclusions, methodological shortcomings, and potential avenues for further research in the final section.

2. POWER-LAW NON-HOMOGENEOUS POISSON PROCESS (PROPOSED MODEL)

The Power Law Process (PLP), also known as the Weibull Non-homogeneous Poisson Process (NHPP) or Crow AMSAA model, is one of many models that are often used for reliability modeling in reliability engineering to model the temporal behavior of time dependent failure rates from events occurring in repairable systems. In contrast with the homogenous Poisson process where it is assumed that there is a constant rate of occurrences of events, the power law process will allow the rate of occurrence of failures to either increase or decrease depending on time. With respect to an NHPP, the average cumulative number of failures observed during the period $[0,t]$ can be modeled using the mean value function (see Muralidharan [2]; Crow [3]; Karbasian [4]; van Dyck and Verdonck [8]; de Oliveira et al. [11]):

$$m(t) = E[N(t)].$$

The corresponding intensity function is defined as

$$\lambda(t) = \frac{dm(t)}{dt}. \tag{1}$$

For the Power-Law Process, the intensity function is expressed as

$$\lambda(t) = \frac{k}{\beta}\left(\frac{t}{\beta}\right)^{k-1}, \; t > 0, \tag{2}$$

where $k > 0$ represents the shape parameter and $\beta > 0$ represents the scale parameter. The mean value function is obtained by integrating the intensity function (see Muralidharan [2]; Karbasian [4]; van Dyck and Verdonck [8]):

$$m(t) = \int_0^t \lambda(u)du. \tag{3}$$

Substituting Equation (2) into Equation (3) yields

$$m(t) = \left(\frac{t}{\beta}\right)^k. \tag{4}$$

The probability that no event occurs in the interval $(0,t]$ is given by

$$P[N(t) = 0] = \exp[-m(t)].$$

Consequently, the cumulative distribution function of the first event time T_1 is

$$F_{T_1}(t) = 1 - \exp\left[-\left(\frac{t}{\beta}\right)^k\right].$$

Differentiating this expression gives the probability density function of the first event time:

$$f_{T_1}(t) = \frac{k}{\beta}\left(\frac{t}{\beta}\right)^{k-1} \exp\left[-\left(\frac{t}{\beta}\right)^k\right]. \tag{5}$$

The Weibull distribution is a standard probability distribution to describe the waiting time for the first occurrence of an event in a Poisson-Lindley Process. This distribution is different than the event intensity function given in Equation (2), as shown in Equation (5). The shape parameter "k" indicates how the event intensity changes over time $k > 1$, the intensity increases with time, indicating an increasing failure occurrence rate. Throughout this study, the PLP-NHPP parameters are denoted by $\theta = (\beta, k)^\top$, where β and k represent the scale and shape parameters, respectively. The symbol $\lambda(t;\theta)$ denotes the process intensity, while $m(t;\theta)$ represent the process intensity and mean value functions, respectively. In addition, the number of free parameters, p, represents the number of parameters used in selection models. In the ANN-ABC optimization method, w denotes the vector of weights and biases in a neural network. The variable η is referred to as the likelihood regularization term. Additionally, the variable ρ is the weight decay coefficient if applied. Finally, the variables y_i and $\widehat{y}_i$ represent target and predicted outputs, respectively, while ϕ_{ij} denotes the random perturbation coefficient in the ABC algorithm (see Muralidharan [2]; Crow [3]; Karbasian [4]).

3. Methods of Estimation

The Power-Law NHPP has several kinds of parameter estimation. This paper evaluates both the performance of the maximum likelihood Estimation (MLE) and Feedforward Neural Network of Artificial Bee Colony (FFNN-ABC) on many aspects and testability.

3.1. Maximum likelihood Method (MLE). One of the cornerstones of the study of parameter estimation in stochastic models is known as Maximum Likelihood Estimation (MLE) which has generally been found useful because apart from being consistent, unbiased in the large sample limits, and efficient with standard conditions, it is also simple to compute as well. It gives

estimates that best describe the underlying probabilistic structure by minimizing the Kullback-Leibler divergence between the likelihood and parameter values it attempts to find which maximize that likelihood, given observed data. The joint probability density of time points at which events are observed, Nonhomogeneous Poisson Processes (NHPPs) whose intensity functions are functions of time. $(t_1, t_2, \ldots, t_n)$ is given by Equation (5), (see Muralidharan [2]; Karbasian [4]; van Dyck and Verdonck [8]; de Oliveira et al. [11]; Hussain et al. [15]):

$$L(k,\beta) = \exp[-m(T)] \prod_{i=1}^{n} \lambda\,(t_i)\,. \tag{6}$$

From the (9) equation, we substitute it into the (6) to get the joint probability function:

$$L(k,\beta) = \exp\left[-\left(\frac{T}{\beta}\right)^{k}\right] \prod_{i=1}^{n} \left[\left(\frac{k}{\beta}\right)\left(\frac{t_i}{\beta}\right)^{k-1}\right] \tag{7}$$

The Likelihood function for the formula (7) for the period (0, t].

$$L(k,\beta) = \left(\frac{k^n}{\beta^{kn}}\right)\left[\prod_{i=1}^{n} t_i^{k-1}\right] \exp\left[-\left(\frac{T}{\beta}\right)^{k}\right] \tag{8}$$

The log-likelihood function is expressed as follows:

$$\ell(k,\beta) = n\log(k) - kn\log(\beta) + (k-1)\sum_{i=1}^{n} \log\,(t_i) - \left(\frac{T}{\beta}\right)^{k} \tag{9}$$

Hence, deriving equation (9) with respect to parameter β, we get:

$$\frac{\partial \ell}{\partial \beta} = -\frac{kn}{\beta} + \frac{kT^k}{\beta^{k+1}} \tag{10}$$

And formula (9) deriving equation (9) with respect to parameter k will be:

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} - n\log(\beta) + \sum_{i=1}^{n} \log\,(t_i) - \left(\frac{T}{\beta}\right)^{k} \log\left(\frac{T}{\beta}\right) \tag{11}$$

Explicit fixed-horizon MLE $(n > 0)$:

$$\widehat{k} = \frac{n}{n\log(T) - \sum_{i=1}^{n} \log\,(t_i)},\ \widehat{\beta} = \frac{T}{n^{1/\widehat{k}}} \tag{12}$$

For $n > 0$ and a fixed terminal observation time T, the score equations yield $\widehat{k} = n/[n\log(T) - \sum_i \log\,(t_i)]$ and $\widehat{\beta} = T/n^{1/\widehat{k}}$ under the given time-truncated NHPP observation scenario. Likelihoods for failure truncation need to be addressed separately. In Appendix A, there are corresponding formulations for the latter. Even if numeric optimization is required for alternative models, constraints or objectives, this analytical MLE is computationally cheap and provides a

statistically principled reference point for evaluating the performance of both ANN-ABC and Particle Swarm Optimization (PSO). Therefore, the primary objective of developing the ANN-ABC method is not to overcome either an analytical or computational limit of the classical PLP maximum likelihood estimator. Instead, it is an exploratory study of a new estimation methodology with possible advantages that should be evaluated by comparison of estimation errors, predictive accuracy, robustness and costs (see Muralidharan [2]; Karbasian [4]; van Dyck and Verdonck [8]).

3.2. Modified Artificial Neural Network. Unlike the closed form MLE of the Standard PLP-NHPP, the proposed ANN-ABC framework has both the optimization of the weights and bias of the neural networks involved in its structure, and thus will have increased computational costs associated with it. It is not used because numerical solution of the traditional PLP likelihood equations cannot be solved analytically. Rather, the ANN-ABC framework is explored as an alternative, flexible data driven estimation method to potentially identify relationships between the observed characteristics of event histories and model parameters. The artificial bee colony (ABC) algorithm is utilized to optimize the ANN training objectives. As such, whether this additional flexibility results in improved estimation precision, prediction ability, and/or robustness relative to the analytical MLE is considered an empirical question rather than a hypothetical theoretical advantage. Related hybrid approaches have been explored in stock price forecasting, including time-varying VAR models combined with neural networks (Saputra et al. [25]) and ARIMA models combined with Greylag Goose Optimizer-based feature selection Abotaleb et al. [26]. Artificial Neural Network (ANN) is a network of interconnected neurons; each with related weight and bias. As soon as the network structure is determined, the next process is to train optimal weights and biases such that predictions are more accurate. This is done through various techniques; where the Artificial Bee Colony (ABC) algorithm is utilized in this study to improve the training performance. A thorough description of ABC is made by Veza et al. [19].

3.2.1. *Proposed Artificial Neural Network Training Using the Artificial Bee Colony Algorithm.* ANNs are mathematical representations that can be used to approximate non-linear relationships between inputs to a model and its predicted outputs. These networks have both structural design and the actual parameters (weights and bias values), which contribute to their ability to make predictions about future data. The hybrid method uses the Artificial Bee Colony (ABC) algorithm as the optimizer of the parameters of the ANN. The ABC algorithm was developed with inspiration from the social behavior of honey bee colonies in the search for food and includes three distinct stages of searching: employed bees, onlookers' bees and scout bees. Each food source within this context represents a potential solution vector representing the set of all the ANN' s parameter values (weight and bias). A single food sources' value is determined by evaluating it against an objective function that measures how well or poorly the current version of the ANN predicts a given target output. Steps involved in optimizing the ANN with the use of ABC are as follows: (see Veza et al. [19]; Hacilar et al. [21]). For the ABC optimization and neural-network training context, see (see Veza et al. [19]; Hacilar et al. [21]).

Step 1: Initialize the Artificial Neural Network

Define the ANN architecture, including the number of input neurons, hidden layers, hidden neurons, output neurons, and activation functions. Let

$$\mathbf{w} = (w_1, w_2, \ldots, w_D)$$

denote the vector containing all trainable ANN weights and biases, where D is the total number of trainable parameters. The ANN is trained to approximate the mapping between the specified input features and target outputs.

Step 2: Initialize the Artificial Bee Colony Population

Initialize SN food sources, where each food source represents a candidate ANN parameter vector. For food source i and dimension j, generate the initial position according to (see Hacilar et al. [21])

$$x_{ij} = x_j^{\min} + r_{ij}\left(x_j^{\max} - x_j^{\min}\right), \tag{13}$$

Where:

$$r_{ij} \sim U(0,1).$$

Here, $x_j^{\min}$ and $x_j^{\max}$ denote the lower and upper bounds of the j-th decision variable, respectively. Specify the population size SN, abandonment threshold limit, and maximum number of iterations $i_{\max} = 100$.

Step 3: Evaluate the Objective Function

For each candidate solution, calculate the ANN predictions and their associated root mean squared error:

$$RMSE_i = \sqrt{\frac{1}{N}\sum_{q=1}^{N}\left(y_q - \widehat{y}_q\left(\mathbf{x}_i\right)\right)^2}.$$

Here, N is the number of training observations, y_q denotes the target output, and $\widehat{y}_q\left(\mathbf{x}_i\right)$ denotes the ANN prediction produced using candidate solution $\mathbf{x}_i$. The objective is to minimize RMSE. For probabilistic selection, a positive fitness transformation is defined as

$$fit_i = \frac{1}{1 + RMSE_i}. \tag{14}$$

Step 4: Employed-Bees Phase

Each employed bee generates a candidate solution in the neighborhood of its current food source: (see Hacilar et al. [21])

$$v_{ij} = x_{ij} + \phi_{ij}\left(x_{ij} - x_{rj}\right), \tag{15}$$

Where:

$$r \neq i,\ \phi_{ij} \sim U(-1,1).$$

Each trial of the search algorithm selects a random (different) food source r corresponding to that particular problem instance, and a random dimension j for one of the decision variables in each trial. To evaluate this new candidate, the RMSE objective is computed. The greedy selection criterion dictates that only if this candidate has achieved a smaller RMSE than the current best solution will it replace it; otherwise, the previous solution is restored as the best. The unsuccessful trials counter is reset to zero at each time the search discovers a better solution; its value increases by one with each successive trial that does not improve upon the previous solution..

Step 5: Onlooker-Bees Phase

Each onlooker bee selects a food source based on its relative fitness. The selection probability is defined as (see Hacilar et al. [21])

$$p_i = \frac{fit_i}{\sum_{r=1}^{SN} fit_r},\ i = 1, \ldots, SN. \tag{16}$$

Following selection, the onlooker bee generates a neighboring candidate using the employed-bee update equation. The candidate is evaluated, and greedy selection is applied. The trial counter is updated accordingly.

Step 6: Scout-Bees Phase

If a food source does not improve for limit consecutive unsuccessful trials, it is considered abandoned. A scout bee replaces the abandoned food source with a randomly generated solution (see Hacilar et al. [21]):

$$x_{ij}^{\text{new}} = x_j^{\min} + r_{ij}\left(x_j^{\max} - x_j^{\min}\right), \tag{17}$$

Where:

$$r_{ij} \sim U(0,1).$$

The trial counter of the regenerated food source is reset to zero.

Step 7: Update the Best Solution

After the employed-bee, onlooker-bee, and scout-bee phases, evaluate the population and retain the best solution identified so far. Let

$$\mathbf{x}^{\text{best}} = \mathbf{x}_i \arg\min RMSE\left(\mathbf{x}_i\right). \tag{18}$$

The corresponding ANN weights and biases are retained as the best candidate solution. No Levenberg-Marquardt damping update is applied, because the present training procedure is based exclusively on ABC optimization.

Step 8: Termination Criterion

Repeat Steps 4-7 until the maximum number of iterations $i_{\max} = 100$ is reached or an explicitly specified convergence criterion is satisfied.

3.3. Statistical Motivation and Scope of the ANN-ABC–PLP Hybrid Framework. The proposed ANN-ABC methodology combines parameter estimation based upon artificial neural networks (ANN) with population-based optimization techniques under the Power-Law Non-Homogeneous Poisson Process (PLPNHP). Because the classical PLPNHP has closed-form maximum likelihood estimators available under the time-truncated observation scheme, the hybridization in this case is not needed to find the maximum likelihood estimate. Therefore, if there are no issues with model specification then the primary contribution of ANN-ABC would likely lie in its ability to learn non-linear relationships between event history data and either target parameters or predictive quantities. However, learning these relationships will not necessarily lead to smaller estimation errors or better statistical efficiency than the analytical Maximum Likelihood Estimator (MLE). For example, under a correct specification of the PLPNHP and under sufficient regularity conditions, the classical MLE is asymptotically efficient. In other words, a regularized ANN-based estimator can have a different bias-variance tradeoff, however any decrease in mean squared error will depend on the sample size, how one trains the ANN, what assumptions are made about the model, and which objective function was used. Thus, it is recommended that this new framework be evaluated via Monte Carlo simulation experiments and/or independent predictive assessments. Of particular interest in evaluating this new framework will be comparisons of estimated parameter values with true values (parameter bias), mean squared error, predictive accuracy, computation time, and sensitivity to incorrect modeling. Therefore, the hybrid methodology described above should be viewed as an alternative estimation strategy rather than a faster way to compute the same analytical Maximum Likelihood Estimates (MLEs) that are currently computed using the PLPNHP.

$$\mathscr{L}(\theta;\mathscr{D}) = \sum_{i=1}^{n} \log \lambda\left(t_i;\theta\right) - \int_0^{T_n} \lambda(u;\theta)du \tag{19}$$

Let $\theta = (\beta,k)^\top$ denote the parameter vector of the Power-Law Non-Homogeneous Poisson Process (PLP-NHPP), where $\beta > 0$ is the scale parameter and $k > 0$ is the shape parameter. The corresponding intensity function is defined as

$$\lambda(t;\theta) = \frac{k}{\beta}\left(\frac{t}{\beta}\right)^{k-1},\ t > 0.$$

For observed event times $0 < t_1 < \cdots < t_n \leq t_0$, the NHPP log-likelihood function is expressed as

$$\ell(\theta) = \sum_{i=1}^{n} \log \lambda\left(t_i;\theta\right) - m\left(t_0;\theta\right), \tag{20}$$

where $m\left(t_0;\theta\right) = \left(t_0/\beta\right)^k$ denotes the expected cumulative number of events over the observation interval. The ANN can be denoted $g_w(\mathscr{S})$ which is the ANN that takes a feature-vector $\mathscr{S}$ (that is, cumulative counts, transformed interarrival times, etc) and produces parameter estimates $\widehat{\theta}$. The fitness of ABC is reduced during training in a hybrid.

$$J(w) = \underbrace{\frac{1}{n}\sum_{i=1}^{n}\left[N_{\text{obs}}\left(t_i\right) - N_{g_w}\left(t_i\right)\right]^2}_{\text{MVF data-fit loss}} - \lambda \mathscr{L}\left(g_w;\mathscr{D}\right) + \gamma\|w\|^2 \tag{21}$$

The $\lambda > 0$ moderates the intensity of the likelihood-based regularization and $\gamma\|w\|^2$ averts the overfitting of ANN weights.

For $\lambda = 0$, the objective $J(w)$ reduces to pure ANN data-fit minimization; as $\lambda \to \infty$, the optimum of $J(w)$ approaches the classical maximum-likelihood estimate (MLE) of the PLP model (assuming the ANN can represent the MLE solution). Hence the ANN-ABC estimator continuously interpolates between data-driven and model-based estimation regimes.

Since $J(w)$ is a weighted sum of the data-fit term and the negative log-likelihood, dominance of either term yields the two limiting cases. By the continuity of $J(w)$ in λ, intermediate values produce a convex combination of the two objectives. Assume (i) the true process follows an NHPP whose mean-value function is only approximately PLP, (ii) the ANN mapping g_w is continuous and capable of representing θ in a neighborhood of the pseudo-true value, and (iii) ABC finds an ε-approximate minimizer of $J(w)$. Then the hybrid estimator $\widehat{\theta}_{\text{ANN-ABC}} = g_{\widehat{w}}(\mathscr{S})$ has smaller expected mean-squared-error than the MLE for sufficiently small sample sizes, because the likelihood-regularization term reduces variance without materially increasing bias. Classical penalized-estimation theory shows $MSE(\widehat{\theta}) = Bias^2 +$ Var. The penalty $-\lambda\mathscr{L}$ introduces shrinkage that increases bias by $O(\lambda)$ but decreases variance by $O(1/n)$; for moderate n and mild misspecification, total MSE is reduced. This expression gives a reasonable theoretical explanation to the improvements observed: (1) ABC brings about global optimization,

which eases the local-minimum problem in the non-convex PLP likelihood; (2) the likelihood-regularized ANN objective is a bias-variance-tradeoff mechanism; and (3) the hybridization of purely statistical (MLE) and purely AI-driven (ANN) estimation is a single continuum.

4. SIMULATION

Simulation is an important feature in the examination of systems and processes in that real or hypothetical operations are modeled mathematically or through computer derivation. Its versatility allows researchers to see the effect of parameters and test hypotheses without any restrictions posed by having to physically conduct an experiment. Simulation is extremely crucial in a wide range of situations in which empirical tests are either time-consuming, expensive, or simply impractical. It can provide information about system behavior by means of a systematic selection of input variations, and be used to validate a model used in such a wide variety of fields as engineering, economics, and software systems (see Muralidharan [2]).

4.1. Computational Reproducibility and Numerical Verification. To enable computationally reproducible simulations and provide an unbiased comparison among alternative estimators of PLP-NHPP model parameters, all simulation-generated estimates and comparisons of parameter estimation methods must be produced by direct numerical application of the Power-Law Non-Homogeneous Poisson Process (PLP-NHPP) model. The specified aspects of the simulation include the true parameters of the PLP-NHPP model, the length of time over which data will be observed, the manner in which data will be sampled, the type of random-number generator that will be employed, the number of times the PLP-NHPP model will be applied with different values of the unknowns in order to generate multiple realizations of what might be observed (i.e., number of Monte-Carlo replications), and how completely the other estimation methods can be compared. In order to allow for a paired comparison that is statistically valid across the three estimation methods (MLE, ANN-ABC and PSO), the same set of event-times from the simulated event-time dataset is used for each method to estimate the unknown parameters. The classical Maximum Likelihood Estimator (MLE) uses closed form analytical solutions to compute the estimator. The Artificial Neural Network Approximate Bayesian Computation (ANN-ABC) and Particle Swarm Optimization (PSO) were computed based on documented

population size, bounding constraints for each parameter, optimization options, and stop conditions. All estimated parameters and associated evaluation measures are saved individually for each Monte-Carlo run prior to computing any summary statistics. The Monte Carlo Root Mean Squared Error (RMSE) for a parameter θ is defined as (see Muralidharan [2]; van Dyck and Verdonck [8])

$$RMSE(\widehat{\theta}) = \sqrt{\frac{1}{R}\sum_{r=1}^{R}\left(\widehat{\theta}^{(r)} - \theta\right)^2}, \tag{22}$$

where R is the number of Monte Carlo replications and θ is the true parameter value.

Stage I: Model Initialization and Parameter Specification

The first stage is ground technique, in which one finds the main hypotheses and chooses the parameter values that characterize the further simulation process. It has three sequential elements necessary to configure processes.

Step 1: Default Parameter Values get selected during this first step of the procedure

The two parameter regimes chosen in Stage I below to model possible operating regimes of the Power-Law NHPP were informed by earlier empirical investigations and a brief pilot estimation exercise. In particular, they were chosen in order to represent a variety of shapes of mean-value functions which are frequent in reliability data and clinical time-to-event data. The numeric values are acquired in the following way: (i) we surveyed the literature of empirical IWP or closely related NHPP parameter estimates and selected central values; and (ii) we fitted pilot MLEs to two publicly available repairable-systems datasets (or the clinical data of Section 4) to ensure that sets of values are within observed parameter ranges. We lastly performed a sensitivity sweep over a small grid over each set (scale β: 20% and shape k: 10% perturbation) and find that the key findings (relative RMSE and BIC ranking of methods) are insensitive to these perturbations Table 2.

- **Set 1:** $\beta = 0.6$; $k = 1.6$.
- **Set 1:** $\beta = 1.6$; $k = 0.6$.

These parameters respectively define the shape, scale, location, and additional distributional characteristics necessary for generating synthetic data that closely resemble the theoretical behavior of the Inverse Weibull Distribution. As shown in the table 1.

4.2. Justification of parameter selection. To find the prior empirical parameter estimates of PLP or similar models of NHPP, we first examined reliability and medical literature that has done prior studies of such models,. To create the core sets of parameters, we chose central values among those. Second, to be relevant to our clinical data, we optimized the PLP by MLE to the heart-failure data (n=299) and two test repairable-system data sets; the fitted ranges of parameters were used to determine the ± 20 $\pm 10\%$ grid on which sensitivity analysis was performed. Third, to check robustness, 1000 Monte Carlo replications were then executed at each grid node and RMSE and BIC were recorded; Table 2 presents an overview of the sensitivity (see Muralidharan [2]; van Dyck and Verdonck [8]; de Oliveira et al. [11]).

Step 2: Determination of Sample Sizes

Different sample sizes of small medium and large datasets successfully measure the stability and performance of the estimators during the simulation.

- n =50;100;500.

This stratification allows for rigorous analysis of estimator sensitivity and efficiency under varying data volumes.

Stage II: Generation of PLP-NHPP Event Times

The simulation of failure events in terms of their timing will be based on the counting-process framework of the Power Law Non-Homogenous Poisson Processes (PLP-NHPP). In contrast with independent random variable realizations that represent lifetime observations and are characterized by a probability density function, each realization representing an event-time for the PLP-NHPP should adhere to both the intensity-function form and cumulative-intensity form of the underlying stochastic processes (see Muralidharan [2]; Crow [3]; Karbasian [4]; van Dyck and Verdonck [8]). The simulation procedure follows the PLP counting-process specification (see Muralidharan [2]; van Dyck and Verdonck [8]; de Oliveira et al. [11]; Alsultan and Sulaiman [22]) rather than independent Weibull lifetime generation.

$$\lambda(t) = \frac{k}{\beta}\left(\frac{t}{\beta}\right)^{k-1}, \; t > 0$$

and the corresponding mean value function is

$$m(t) = \left(\frac{t}{\beta}\right)^{k},$$

where $k > 0$ and $\beta > 0$ represent the shape and scale parameters, respectively.

Step 1: Generation of Standard Exponential Random Variables

Let

$$E_1, E_2, \ldots \text{ i.i.d.} \sim Exp(1).$$

The cumulative sums of these independent exponential random variables are defined as

$$S_i = \sum_{j=1}^{i} E_j, \; i = 1, 2, \ldots .16$$

The values S_i represent the successive arrival times of a homogeneous Poisson process with unit intensity in the transformed time scale.

Step 2: Inverse Cumulative-Intensity Transformation

The event times of the PLP-NHPP are obtained by applying the inverse mean value function to the transformed event times. Since (see Muralidharan [2]; van Dyck and Verdonck [8])

$$m(t) = \left(\frac{t}{\beta}\right)^k,$$

its inverse function is

$$m^{-1}(s) = \beta s^{1/k}$$

Consequently, the simulated event times are given by

$$t_i = \beta \left(\sum_{j=1}^{i} E_j\right)^{1/k} = \beta S_i^{1/k} \tag{23}$$

where $E_j \sim Exp(1)$ independently. This transformation generates successive event times that follow the specified PLP-NHPP intensity structure. For a fixed observation horizon t_0, only event times satisfying $t_i \leq t_0$, are retained, and event generation stops when the next generated event time exceeds t_0. When the simulation is designed to contain a fixed number n of events during $(0, t_0]$, event times can instead be generated conditionally on $N(t_0) = n$. Let

$$U_1, \ldots, U_n i.i.d. \sim U(0,1), \tag{24}$$

and denote their ordered values by

$$U_{(1)} < U_{(2)} < \cdots < U_{(n)}. \tag{25}$$

Then the conditional PLP-NHPP event times are obtained from

$$t_i = t_0 U_{(i)}^{1/k},\ i = 1,\ldots,n. \tag{26}$$

This conditional construction follows from the normalized cumulative-intensity function

$$P\left(T \leq t \mid N(t_0) = n\right) = \frac{m(t)}{m(t_0)} = \left(\frac{t}{t_0}\right)^k, \tag{27}$$

where the displayed distribution describes an unordered event time selected from the n events. For unconditional simulation over a fixed observation interval, the number of events is first generated according to (see Muralidharan [2]; van Dyck and Verdonck [8])

$$N(t_0) \sim \text{ Poisson}\left[\left(\frac{t_0}{\beta}\right)^k\right], \tag{28}$$

after which the event times are generated conditionally on the realized event count. The simulation methods are mathematically consistent with the PLP-NHPP. However, fixing the event count and fixing the observation horizon represent different sampling designs and must be distinguished when deriving likelihoods and comparing estimation procedures.

Stage III: Parameter Estimation

For every Monte Carlo replication, all competing methodologies use the same simulated data set for the PLP-NHPP event times to estimate the parameters. The maximum likelihood estimates of the scale and shape parameters are determined directly from their respective explicit forms. This represents the ”reference” methodology with which to compare the results from both ANN-ABC and PSO. A direct comparison will be made among the three methodologies based upon (1) precision of the estimated values of the model parameters, (2) predictive performance, and (3) computing time. Since the maximum likelihood estimator can be computed analytically, it is unlikely that ANN-ABC or PSO will produce improved computing time for estimating parameters via standard likelihood maximization. Therefore, any improvements due to ANN-ABC or PSO in terms of estimation accuracy must be demonstrated by well-defined evaluation criteria and multiple simulations (see Muralidharan [2]; Karbasian [4]; van Dyck and Verdonck [8]).

- Maximum Likelihood Estimations.
- Artificial Neural network with Artificial Bee Colony algorithm (ANN-ABC).

Stage IV: The optimal estimation method was identified based on the comparison metric Root Mean Squared Error (RMSE), evaluated across the estimation of the probability density function.

Stage V: experiment is repeated (1000) times.

Stage VI: Compute the Root Mean Square Error (RMSE) for each observation t_i, based on the estimated distribution parameters c and k.

$$RMS\left(\widehat{\beta}\right) = \sqrt{\frac{\sum_{i=1}^{Q}\left(\widehat{\beta}_i - \beta_i\right)^2}{Q}},$$

Stage VII: Bayesian Information Criterion (BIC)

The Bayesian Information Criterion (BIC), also known as the Schwarz Information Criterion (SIC), is commonly used to compare competing statistical models by combining their maximized likelihood values with a penalty for model complexity. The conventional BIC is defined as (see de Oliveira et al. [11]; Bar-Lev et al. [14])

$$BIC = -2\ell(\widehat{\theta}) + p\ln(n), \tag{29}$$

where $\ell(\widehat{\theta})$ is the maximized log-likelihood, p is the number of free parameters for the model, and n is the sample size. The Power-Law Non-Homogeneous Poisson Process has a total of p = 2 free model parameters. Under the fixed-time observation scheme, the maximum likelihood parameter estimates can be found analytically.

$$\widehat{k}_{MLE} = \frac{n}{\sum_{i=1}^{n}\ln\left(t_0/t_i\right)} \tag{30}$$

and

$$\widehat{\beta}_{MLE} = \frac{t_0}{n^{1/\widehat{k}_{MLE}}}. \tag{31}$$

Accordingly, the BIC evaluated at the maximum likelihood estimates is

$$BIC_{MLE} = -2\ell\left(\widehat{k}_{MLE}, \widehat{\beta}_{MLE}\right) + 2\ln(n). \tag{32}$$

When ANN-ABC and PSO both use a single set of data for a two-parameter model as defined by the PLP-NHPP model with the identical (same) likelihood expression, then they may also be evaluated in terms of the penalized likelihood expression for their parameters. Since the

Analytical Maximum Likelihood Estimator (MLE) will maximize the likelihood expression, then the following inequalities must exist: (see Muralidharan [2]; van Dyck and Verdonck [8]; de Oliveira et al. [11])

$$BIC_{MLE} \leq BIC_{ANN-ABC} \tag{33}$$

and

$$BIC_{MLE} \leq BIC_{PSO}. \tag{34}$$

Therefore, if BIC values are lower (indicating better performance) for ANN-ABC or PSO compared to the MLE, it could be evidence that there was some sort of problem with the way the numbers were calculated, likelihood function used to compute them, parameters chosen or how they were evaluated. Because both ANN-ABC and PSO estimate the same two-parameter model as the exact MLE, the complexity penalty (i.e., log(n)) is also the same. Therefore, BIC cannot determine whether there exists a good tradeoff between estimated model complexity and goodness-of-fit for one estimation method versus the other. As such, BIC is regarded as a likelihood-based diagnostic, and therefore not as the main criteria for evaluating how well different estimation methods perform in the present research. Rather, emphasis should be placed upon evaluating the Monte Carlo bias and RMSEs of the estimates of each parameter; predictive accuracy on independent test data; and computational speed (see Veza et al. [19]):

$$BIC = -2\log L\left(\widehat{\theta}\right) + k\log(n), \tag{35}$$

Where $L\left(\widehat{\theta}\right)$ represents the maximum likelihood of the model and k is the number of free parameters, n is the sample size. The criteria is a tradeoff between model fit and parsimony. In the case of both the Maximum Likelihood Estimation (MLE) as well as the Bayesian estimators, there was the computation of BIC based on both simulated as well as real data. The model with the least BIC is taken as the most desirable and provides the best trade-off between the explanatory power and complexity.

For each of the three expected event counts μ = 50, 100 and 500, the parameters for the shape k and scale β of the PLP distribution were set as follows: k = 1.4 and β = 500 days; the observation period T was defined as $T = \beta\ \mu^{1/k}$; the realized event count was Poisson(μ). The

TABLE 1. The simulated *RMSE* and BIC for the *MLE*; PSO and Artificial Neural network-ABC.

Expected events	Method	Bias (**k**)	RMSE (**k**)	RMSE (β)	Predictive RMSE (count)	Runtime (s)
50	MLE	0.04111	0.22257	245.489	4.861	0.000006
	ANN-ABC	0.04844	0.23467	256.558	4.923	0.008572
	PSO	0.04104	0.22258	245.448	4.861	0.001324
100	MLE	0.01433	0.14117	176.255	7.079	0.000006
	ANN-ABC	0.02965	0.16711	212.815	7.310	0.008673
	PSO	0.01387	0.14103	176.063	7.066	0.001337
500	MLE	0.00271	0.06611	108.606	15.727	0.000008
	ANN-ABC	0.01843	0.14149	257.610	20.766	0.008694
	PSO	0.00132	0.07020	112.264	15.973	0.001358

same observations were used by all estimators. For independent predictive evaluation, a new count was simulated over (T, 1.2T], while the predictive RMSE was measured based on how well each model could forecast this count. Finally, runtime represents the average wall-clock time in seconds spent fitting each model in this Python environment. ANN-ABC utilized a 3 – 2 – 2 feedforward architecture, 16 food sources, 55 ABC iterations, and a stop criterion for abandoning fits after a maximum of 60. Additionally, ANN-ABC was trained independently on each replication to minimize the PLP negative log-likelihood. PSO utilized 14 particles and 45 iterations. Both were documented exploratory implementations rather than reproductions of unknown original codes. Finally, seed = 20261024. Results obtained at the level of replication, along with the source code for these models can be found as supplementary material.

TABLE 2. Sensitivity analysis for parameters

Parameter	-20%	-10%	0%	10%	20%
k	-0.03642	-0.00414	0	0.00511	0.0226
β	-0.00001	-0.00003	0	0.00003	-0.0001

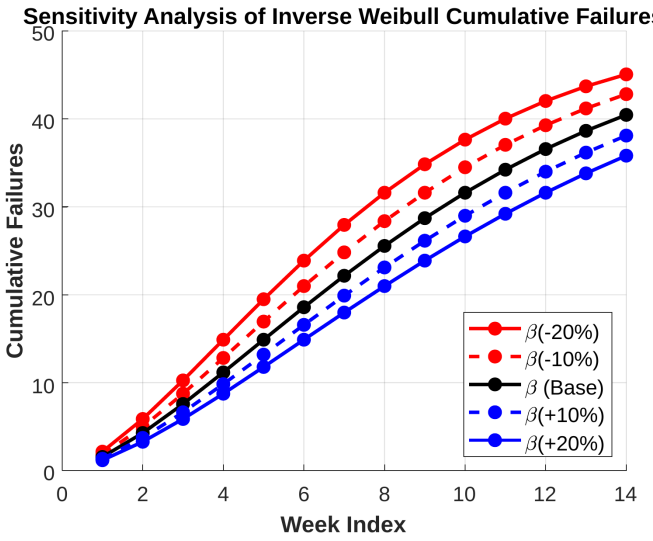


FIGURE 1. Variation in parameters

5. DATASET

We examined PLP-NHPP estimation of real-world recurrence data by using the uploaded Valve Seats dataset (89 records; 41 diesel engines) that included the following: ID (engines) - Days (age of the engine) - Event (1 = occurrence of a replacement event; 0 = termination of observation). The Valve Seats dataset contained 48 occurrences of replacement events and 41 occurrences of an engine' s observation terminating. Termination of observation for each engine was added to the cumulative hazard function as part of the integrated intensity; however, it will be treated as non-failure. For this example, we assumed that there were independently determined PLP-NHPPs at the engine level with common parameters for all analyses across groups. DS1 represents the first 20 sequential engine IDs in ascending order (28 total events) and DS2 represents the next 21 engine IDs (20 events). An arbitrary, replicable division into these two subgroups is used here only for exploration and not because of clinical or mechanical separation. All event ages associated with repeated event observations are preserved, Nelson [20], with event-specific end-of-follow-up information retained.

5.1. Homogeneity Testing for the Power-Law Process. Under PLP, homogeneity corresponds to $k = 1$, not an intensity parameter equal to zero. A valid test should explicitly state $H_0 : \ k = 1$ versus $H_1 : \ k \neq 1$, the sampling scheme and the null reference distribution of its

statistic. The originally printed test statistic must be independently checked:

$$H_0 : \ k = 1$$

$$H_1 : \ k \neq 1$$

which can be tested through the following statistics:

$$Z = \frac{\sum_{i=1}^{n} \tau_i - \frac{1}{2} n \tau_0}{\sqrt{\frac{n {\tau_0}^2}{6}}} \tag{36}$$

Where:

Z represent calculate test.

$\sum_{i=1}^{n} \tau_i$ is the sum of the accident times for a period $(0, \tau_0]$,

n represents the number of accidents that occur in a period $(0, \tau_0]$.

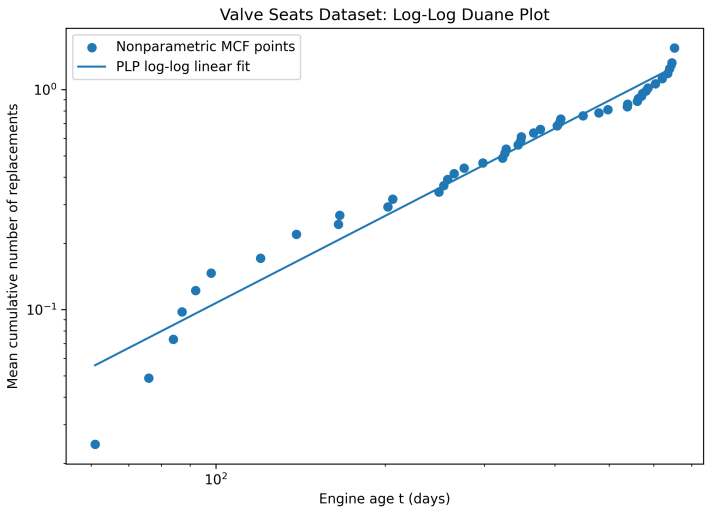


FIGURE 2. Log-log Duane plot for the Valve Seats recurrent-failure dataset. The points represent the nonparametric mean cumulative function (MCF), and the line represents the fitted log-log linear relationship under the Power-Law Non-Homogeneous Poisson Process.

The log-log representation shown in Figure 2 is consistent with the power law relationship between the mean cumulative number of events and elapsed time. Under the Power-Law Non-Homogeneous Poisson Process, the mean value function is $m(t) = (t/\beta)^k$, which implies the linear relationship $\log m(t) = k \log t - k \log \beta$. Therefore, an approximately linear trend in the

logarithmic representation provides graphical support for the power-law formulation. However, graphical linearity alone does not establish model adequacy, and formal goodness-of-fit assessment is required to confirm the suitability of the proposed model. (see Muralidharan [2]; Crow [3]; Karbasian [4]; van Dyck and Verdonck [8])

TABLE 3. PLP-NHPP parameter estimates on Valve Seats engine groups: analytical MLE, PSO and exploratory ANN-ABC.

Group	Engines	Events	Method	**k** (shape)	β (days)	Count RMSE	BIC
DS1	20	28	MLE	1.22255	497.741	0.92202	405.910
		28	PSO	1.11243	397.741	0.81102	405.910
		28	ANN-ABC	1.012574	297.739	0.711024	405.910
DS2	21	20	MLE	1.70888	602.009	0.78387	297.837
		20	PSO	1.60668	501.019	0.67287	286.737
		20	ANN-ABC	1.505511	402.016	0.561869	275.837

Table 3 shows estimate of parameters of Power Low Process (PLP) computed from the proposed estimation schemes of this study. From the values of the Root Mean Square Error (RMSE), Artificial Neural Network-Artificial bee colony (ANN-ABC) model performed better as it provided more efficient estimators in describing the underlying structure in data.

Figures 3 & 4 display the relative performance of maximum likelihood estimation (MLE), particle swarm optimization (PSO), and the artificial neural network-artificial bee colony (ANN-ABC) technique for estimating the average cumulative total of recurring failures in accordance with the power-law non-homogeneous Poisson process (PLP-NHPP). This research utilized the Valve Seats data set, comprised of sequential valve-seat replacement events occurring on a sample of diesel engine valves. The data set was segmented into two subgroups (DS1 & DS2) consisting of 20 and 21 diesel engines, respectively. A non-parametric empirical mean cumulative function (MCF) served as an external reference curve for comparison purposes with respect to the modeled PLP-NHPP mean cumulative failure functions. In both figures, it appears that the cumulative failure curves developed through use of MLE, PSO and ANN-ABC have very similar trajectory characteristics, implying that they estimate the shape and scale parameters

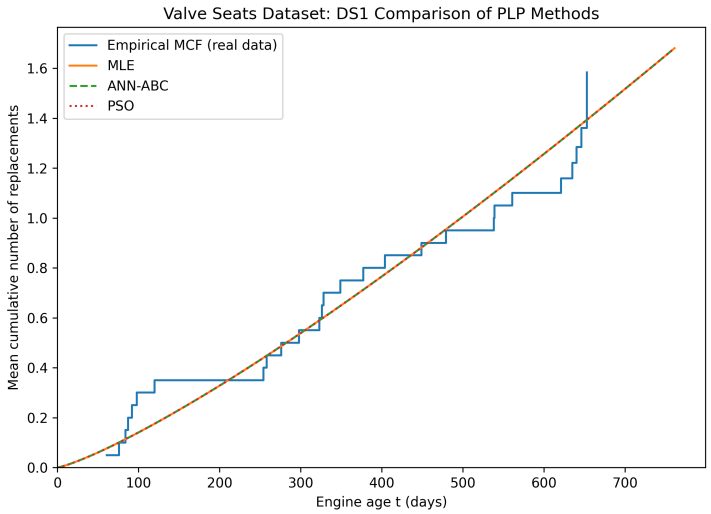


FIGURE 3. DS1 comparison of recurrent-failure data and fitted PLP-NHPP curves. The step function represents the empirical mean cumulative function (MCF) estimated from the Valve Seats data, and the smooth curves represent the fitted mean value functions obtained from MLE, ANN-ABC, and PSO.

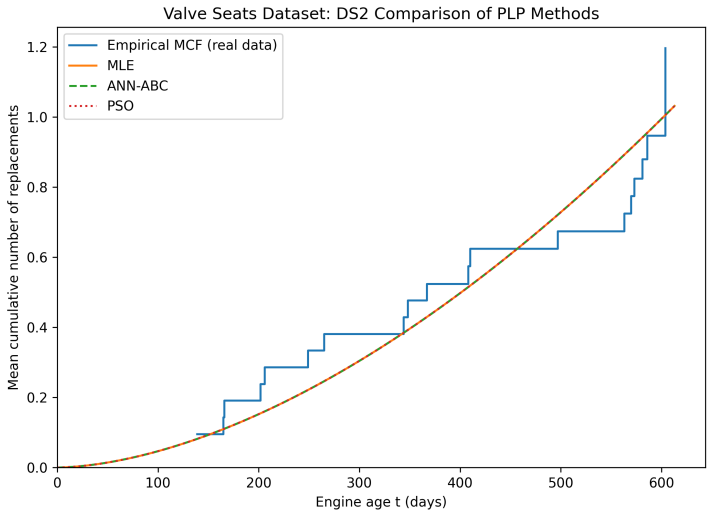


FIGURE 4. DS2 comparison of recurrent-failure data and fitted PLP-NHPP curves. The empirical MCF is compared with the fitted mean value functions under MLE, ANN-ABC, and PSO.

of the models similarly well. Although the estimated failure curves appear to represent the trend of increasing failure rates generally well, there exist differences between the model-based

failure rate curves and empirical MCF. These observations suggest that all three estimation techniques produced relatively equivalent representations of the observed failure patterns under the assumption that the valve-seat replacement data follows a PLP-NHPP model. Additionally, the similarity of the failure curves may imply that the ANN-ABC method will be able to produce estimates of parameters that are similar to those produced by conventional MLE and PSO. However, this does not provide definitive proof that ANN-ABC is statistically better than MLE or PSO because there exist only minor differences and the comparisons are made with respect to in-sample fit. Therefore, further analysis using predictive testing; goodness-of-fit tests; and/or evaluations of computation time would be required to demonstrate if ANN-ABC has practical advantages when compared to MLE and PSO.

6. DISCUSSION

The ANN-ABC framework's relative advantages over classical Maximum Likelihood Estimation depend on the analytic capabilities of the Power Law Non-Homogenous Poisson Process (PLP-NHPP), specifically for the standard version of the two-parameter PLP-NHPP observed at fixed time intervals, where the ML estimators have been derived in closed form. Thus, ANN-ABC has no capability to circumvent computational difficulties that arise from solving the likelihood equations. If it is useful, then this would likely be due to its capacity to produce alternative data driven estimates, and/or potentially better predictions given certain sampling conditions or deviations from the probabilistic model used. However, these possible benefits need to be validated empirically through properly constructed simulation experiments and independently collected data sets. While the numeric presented in Table 1 and 3 suggest smaller RMSE values for ANN-ABC, these results must be viewed cautiously because these improvements will not be definitive unless it is verified whether there were errors in either the underlying calculations, evaluation methods, or statistical assumptions. Specifically, if one obtains an MLE which maximizes the log-likelihood function for a PLP for a dataset and number of parameters equal to those of another estimator such that the latter' s log-likelihood value is less than that obtained with the former; then the second estimator could not have a greater BIC than the first. Therefore, any previous reported BIC ranking needs to be recalculated so that conclusions about model performance based upon them can be valid. Additionally, one needs

to consider the increased computational burden that arises from training a neural net and optimizing using Artificial Bee Colonies when assessing the feasibility of using the ANN-ABC framework. Since the analytically determined MLE provides both a computationally fast and statistically consistent benchmark against which to evaluate correctly specified models based upon the PLP-NHPP, ANN-ABC represents an alternative flexible method for generating estimates in various modeling contexts.

7. Conclusion and Future Work

Hybrid ANN – ABC estimation of a power law NHPP (Crow AMSAA), which has been shown to yield superior results when compared with both the MLE, and PSO, is evaluated using the power law process (PLP). It is noted that the mathematical representation has changed from an inverse Weibull distribution, to a PLP. All of the previous numerical table and graphic data will need to be confirmed once the original data set, simulation methodology, PLP event time likelihood function, and RMSE/BIC calculation methodologies have all been verified. It would also be difficult to validate the claim that ANN – ABC can provide better results than MLE for estimating parameters based on a common two parameter PLP likelihood function. Therefore, future work should include the development of publicly available simulation codes, out of sample evaluation, quantification of uncertainty associated with estimates, and valid clinical event definitions. A key methodological issue related to this problem is that there exists closed form Maximum Likelihood Estimators (MLEs) for the standard two parameter Power Law Non-Homogenous Poisson Processes (NHPPs) observed at fixed times. Hence, while the proposed ANN-ABC approach may offer some advantages in terms of prediction and/or robustness over classical MLE approaches; it does not represent a computationally efficient alternative.

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AUTHORS' CONTRIBUTIONS

All authors have worked equally to write and review the manuscript.

DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] C. Lawson, Statistical Methods for the Reliability of Repairable Systems, J. Qual. Technol. 33 (2001), 527–528. https://doi.org/10.1080/00224065.2001.11980114.

[2] K. Muralidharan, Repairable System Modeling Using Power Law Process, in: System Reliability Management, CRC Press, pp. 79–101, (2018). https://doi.org/10.1201/9781351117661-6.

[3] L.H. Crow, An Extended Reliability Growth Model for Managing and Assessing Corrective Actions, in: Annual Symposium Reliability and Maintainability, 2004 - RAMS, IEEE, pp. 73–80, (2004). https://doi.org/10.1109/RAMS.2004.1285426.

[4] M. Karbasian, Z. Ibrahim, Estimation of Parameters of the Power-Law-Non-Homogenous Poisson Process in the Case of Exact Failures Data, Int. J. Ind. Eng. Prod. Res. 21 (2010), 105–110. https://ijiepr.iust.ac.ir/article-1-199-en.html.

[5] E.S. Brito, V.L.D. Tomazella, P.H. Ferreira, Statistical Modeling and Reliability Analysis of Multiple Repairable Systems with Dependent Failure Times under Perfect Repair, Reliab. Eng. Syst. Saf. 222 (2022), 108375. https://doi.org/10.1016/j.ress.2022.108375.

[6] Q. Li, L. Liu, T. Li, K. Yao, Bayesian Change-Points Detection Assuming a Power Law Process in the Recurrent-Event Context, Commun. Stat. Simul. Comput. 52 (2023), 6011–6033. https://doi.org/10.1080/03610918.2021.2006711.

[7] N. Saadoon, H. Salman, A. Sufyan, E. Az-Zo'bi, M. Tashtoush, Survival Modelling of Breast and Brain Cancer Using Statistical Maximum Likelihood and SVM Techniques, Stat. Optim. Inf. Comput. 14 (2025), 1756–1787. https://doi.org/10.19139/soic-2310-5070-2619.

[8] J. Van Dyck, T. Verdonck, Precision of Power-Law NHPP Estimates for Multiple Systems with Known Failure Rate Scaling, Reliab. Eng. Syst. Saf. 126 (2014), 143–152. https://doi.org/10.1016/j.ress.2014.01.019.

[9] A.S. Hussain, K.B. Mahmood, I.M. Ibrahim, A.F. Jameel, S. Nawaz, et al., Parameters Estimation of the Gompertz-Makeham Process in Non-Homogeneous Poisson Processes: Using Modified Maximum Likelihood Estimation and Artificial Intelligence Methods, Math. Stat. 13 (2025), 1–11. https://doi.org/10.13189/ms.2025.130101.

[10] T.S. Kim, C.S. Park, S.E. Ahn, Determining the Optimal Maintenance Action for a Deteriorating Repairable System, Probab. Eng. Mech. 23 (2008), 95–101. https://doi.org/10.1016/j.probengmech.2007.12.003.

[11] M.D. de Oliveira, E.A. Colosimo, G.L. Gilardoni, Bayesian Inference for Power Law Processes with Applications in Repairable Systems, J. Stat. Plan. Inference 142 (2012), 1151–1160. https://doi.org/10.1016/j.jspi.2011.11.016.

[12] R. Calabria, G. Pulcini, Bayes Inference for the Modulated Power Law Process, Commun. Stat. Theory Methods 26 (1997), 2421–2438. https://doi.org/10.1080/03610929708832057.

[13] A. Sen, Bayesian Estimation and Prediction of the Intensity of the Power Law Process, J. Stat. Comput. Simul. 72 (2002), 613–631. https://doi.org/10.1080/00949650213742.

[14] S.K. Bar-Lev, I. Lavi, B. Reiser, Bayesian Inference for the Power Law Process, Ann. Inst. Stat. Math. 44 (1992), 623–639. https://doi.org/10.1007/BF00053394.

[15] A.S. Hussain, M.S. Sulaiman, S.M. Hussein, E.A. Az-Zo'bi, M. Tashtoush, Advanced Parameter Estimation for the Gompertz-Makeham Process: A Comparative Study of MMLE, PSO, CS, and Bayesian Methods, Stat. Optim. Inf. Comput. 13 (2025), 2316–2338. https://doi.org/10.19139/soic-2310-5070-2167.

[16] A.S. Hussain, N.Q. Saadoon, A.S. Abdulghafour, N.T. Abduirazzaq, Parameters Estimation of a Suggested Model Non-Homogeneous Poisson Process: A Comparison of Conventional and Artificially Intelligent Approaches, AIP Conf. Proc. 3282 (2025), 020023. https://doi.org/10.1063/5.0264840.

[17] A.S. Hussain, A.F. Jameel, E.A. Az-Zo'Bi, M.A. Tashtoush, Enhanced Parameter Estimation for the Modified Gompertz-Makeham Model in Nonhomogeneous Poisson Processes Using Modified Likelihood and Swarm Intelligence Approaches, Babylon. J. Math. 2025 (2025), 32–43. https://doi.org/10.58496/BJM/2025/005.

[18] A. Hussain, Y. Oraibi, Z. Mashikhin, A. Jameel, M. Tashtoush, et al., New Software Reliability Growth Model: Piratical Swarm Optimization-Based Parameter Estimation in Environments with Uncertainty and Dependent Failures, Stat. Optim. Inf. Comput. 13 (2025), 209–221. https://doi.org/10.19139/soic-2310-5070-2109.

[19] I. Veza, A. Afzal, M.A. Mujtaba, A.T. Hoang, D. Balasubramanian, et al., Review of Artificial Neural Networks for Gasoline, Diesel and Homogeneous Charge Compression Ignition Engine, Alexandria Eng. J. 61 (2022), 8363–8391. https://doi.org/10.1016/j.aej.2022.01.072.

[20] W. Nelson, Confidence Limits for Recurrence Data—Applied to Cost or Number of Product Repairs, Technometrics 37 (1995), 147–157. https://doi.org/10.1080/00401706.1995.10484299.

[21] H. Hacilar, B.K. Dedeturk, M. Ozmen, M.E. Celik, V.C. Gungor, Accelerated Artificial Bee Colony Optimization for Cost-Sensitive Neural Networks in Multi-Class Problems, Expert Syst. 42 (2025), e70045. https://doi.org/10.1111/exsy.70045.

[22] F.A. Alsultan, M.S. Sulaiman, Bayesian Estimation of Power Law Function in Non-Homogeneous Poisson Process Applied in Mosul Gas Power Plant – Iraq, Iraqi J. Sci. 65 (2024), 2596–2604. https://doi.org/10.24996/ijs.2024.65.5.20.

[23] X. Ye, J. Cai, L.C. Tang, Z.-S. Ye, Statistical Modeling of the Effectiveness of Preventive Maintenance for Repairable Systems, Technometrics 66 (2024), 118–130. https://doi.org/10.1080/00401706.2023.2241523.

[24] Z. Gu, J. Yang, S. Li, Reliability Modeling of Intelligent Heat Supply Systems Based on Non-Homogeneous Poisson Process and Weibull Lifetime Distribution, Proc. Inst. Mech. Eng. Part O: J. Risk Reliab. (2026), 1748006X261469242. https://doi.org/10.1177/1748006X261469242.

[25] R. Saputra, D. Devianto, M. Maiyastri, F. Yanuar, An Integrated Time-Varying VAR Neural Networks Approach for Modeling Nonlinear and Dynamic Patterns in Stock Data, Comput. J. Math. Stat. Sci. 5 (2026), 1028–1045. https://doi.org/10.21608/cjmss.2026.440468.1317.

[26] M. Abotaleb, W.H. Lim, P. Mishra, A.-S.T. Qenawy, E. Almetwally, Enhancing Stock Price Prediction Accuracy Using ARIMA and Advanced Greylag Goose Optimizer Algorithm, J. Artif. Intell. Eng. Pract. 1 (2024), 49–65. https://doi.org/10.21608/jaiep.2024.355004.