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NUMBER OF SPANNING TREES OF NEW JOIN GRAPHS

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Abstract. In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we derive simple formulas of the complexity, number of spanning trees, of some new join graphs, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

Keywords: Number of spanning trees;Chebyshev polynomials; Join graphs.

Mathematics Subject Classification: 05C05, 05C50.

1. Introduction

In this workwe deal with simple and finite undirected graphs $G=(V,E)$, where V is the vertex setand E is the edgeset. For a graph G, a spanning tree in G is a tree which has the same vertex set as G. The number of spanning trees in G, also called, the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time). A classical result of Kirchhoff, [19] can be used to determine the number of spanning trees for $G=(V,E)$.Let $V=\{v_1,v_2,.....,v_n\}$, then the Kirchhoff matrix H defined as $n\times n$ characteristic matrix $H=D-A$,where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H=[a_{ij}]$ defined as follows: (i) $a_{ij}=-1 v_i$ and v_j are adjacent and $i\neq j$, (ii) a_{ij} equals the degree of vertex v_i if $i=j$, and (iii) $a_{ij}=0$ otherwise. All of co-factors of H are equal to $\tau(G)$. There are

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other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_1 \geq .... \geq \mu_p$ denote the eignvaluesof H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [18] shown that, $\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}\mu_k$ .The formula for the number of spanning trees in a d-regular graph G can be expressed as $\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}(d-\mu_k)$ where $\lambda_0 = d, \lambda_1, \lambda_2, ..., \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such result is due to Cayley[4] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}, n \geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1}q^{p-1}, p,q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g., [5,22]. Another result is due to Sedlacek [23] who derived a formula for the wheel on $n+1$ vertices, W_{n+1}, he showed that $\tau(W_{n+1}) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2$, for $n \geq 3$. Sedlacek [24] also later derived a formula for the number of spanning trees in a Mobius ladder, M_n, $\tau(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$ for $n \geq 2$.Another class of graphs for which an explicit formula has been derived is based on a prism.Boesch, et .al. [2,3].

Daoud,[6-16] later derived formulas for the number of spanning trees for many graphs.

The Lucas numbers are the sequence $\{L_n\}_{n=1}^{\infty}$ defined by the linear recurrence equation

$$L_n = L_{n-1} + L_{n-2} \qquad (1)$$

With $L_0 = 2, L_1 = 1$ and $L_2 = 3$. The values of L_n for $n = 1, 2, 3, ...$ are $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ...$,[1,17].

The Fibonacci numbers are the sequence $\{F_n\}_{n=1}^{\infty}$ defined by the linear recurrence equation:

$$F_n = F_{n-1} + F_{n-2} \qquad (2)$$

With $F_0 = 0$ and $F_1 = F_2 = 1$. The values of F_n for $n = 1, 2, 3, \ldots\ldots$ are $1, 1, 2, 3, 5, 8, 13, 21, \ldots\ldots.$,[21].

Now, we can introduce the following lemma:

Lemma 1.1

$\tau(G) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A})$ where $\bar{A}$, $\bar{D}$ are the adjacency and degree matrices of $\bar{G}$, the complement of G , respectively, and I is the $n \times n$ unit matrix.

The advantageof thisformula is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations.

We begin from their definitions,Yuanping, et. al. [25].

Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2x \end{pmatrix}$$, where all other elements are zeros.

Further we recall that the Chebyshev polynomials of the first kind are defined by:

$$T_n(x) = \cos(n \arccos x) \quad (3)$$

The Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)} \quad (4)$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \quad (5)$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)), n \geq 1 \quad (6)$$

Furthermore by using standard methods for solving the recursion (3), one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}], n \geq 1 \quad (7)$$

Where the identity is true for all complex x (except at $x=\pm 1$ where the function can be taken as the limit).

The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x)=2^{n-1}\prod_{j=1}^{n-1}(x-\cos\frac{j\pi}{n}) \tag{8}$$

One further notes that

$$U_{n-1}(-x)=(-1)^{n-1}U_{n-1}(x) \tag{9}$$

These two results yield another formula for $U_n(x)$,

$$U_{n-1}{}^2(x)=4^{n-1}\prod_{j=1}^{n-1}(x^2-\cos^2\frac{j\pi}{n}) \tag{10}$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter:

$$U_{n-1}{}^2(\sqrt{\frac{x+2}{4}})=\prod_{j=1}^{n-1}(x-2\cos\frac{2j\pi}{n}) \tag{11}$$

Furthermore one can show that

$$U_{n-1}{}^2(x)=\frac{1}{2(1-x^2)}[1-T_{2n}]=\frac{1}{2(1-x^2)}[1-T_n(2x^2-1)], \tag{12}$$

and

$$T_n(x)=\frac{1}{2}[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n]. \tag{13}$$

Another interesting fact follows by comparing (7) with the will known closed form formula for the Fibonacci numbers F_n .

$$F_n=\frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n-(\frac{1-\sqrt{5}}{2})^n] \tag{14}$$

namely ,

$$U_{n-1}(\frac{3}{2})=F_{2n}=\frac{1}{\sqrt{5}}[(\frac{3+\sqrt{5}}{2})^n-(\frac{3-\sqrt{5}}{2})^n] \tag{15}$$

Also we can prove that :

$$2T_n(\frac{3}{2})=L_{2n}=(\frac{3+\sqrt{5}}{2})^n+(\frac{3-\sqrt{5}}{2})^n \tag{16}$$

Lemma 2.1 Let $B_n(x)$ be $n \times n$ matrix such that:

$$B_n(x) = \begin{pmatrix} x & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 1+x & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1+x & -1 \\ 0 & \cdots & \cdots & 0 & -1 & x \end{pmatrix}$$

Then

$$\det(B_n(x)) = (x-1)U_{n-1}\left(\frac{1+x}{2}\right).$$

Proof:

Straightforward induction using properties of determinants and above mentioned definition.

Lemma 2.2 Let $C_n(x)$ be $n \times n$ matrix, $n \geq 3, x > 2$ such that:

$$C_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & \cdots & 1 \\ 0 & x+1 & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & x+1 & 0 \\ 1 & \cdots & \cdots & 1 & 0 & x \end{pmatrix}$$

Then

$$\det(C_n(x)) = (n+x-2)U_{n-1}\left(\frac{x}{2}\right)$$

Proof: Straightforward induction using the properties of determinants and lemma 2.1.

Lemma 2.3 Let $D_n(x)$ be $n \times n$ matrix, $n \geq 3, x \geq 4$ such that :

$$D_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & x \end{pmatrix}$$

Then: $$\det(D_n(x)) = \frac{2(x+n-3)}{x-3}\left[T_n\left(\frac{x-1}{2}\right)-1\right]$$

Proof: Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial of the first and second kind.

Lemma 2.4

Let $E_n(x)$ be $n \times n$ matrix, $x \ge 2$ such that:

$$E_n(x) = \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix}$$

Then

$$\det(E_n) = (x + n - 1)(x - 1)^{n-1}$$

Proof From the definition of the circulant determinants , we have:

$$\det(E_n(x)) = \det\begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix} = \prod_{j=1}^{n}(x + \omega_j + \omega_j^2 + \omega_j^3 + ...+ \omega_j^{n-1})$$

$$= (x + 1 + 1 + ...+ 1) \times \prod_{j=1, \omega_j \neq 1}^{n} (x + \underbrace{\omega_j + \omega_j^2 + \omega_j^3 + ...+ \omega_j^{n-1}}_{=-1})$$

$$= (x + n - 1) \times (x - 1)^{n-1}.$$

Lemma 2.5,[20]

Let $A \in F^{n\times n}$, $B \in F^{n\times m}$, $C \in F^{m\times n}$ and $D \in F^{m\times m}$ assume that A, D ,

are nonsingular matrices. Then:

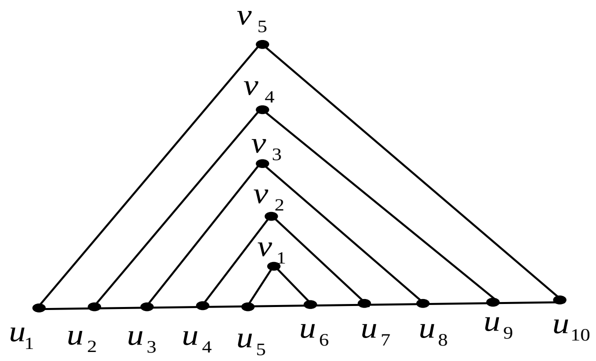
$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^{nm} \det(A - BD^{-1}C) \det D = (-1)^{nm} \det A \det(D - CA^{-1}B)$$

Formula in lemma 2.5 give some sort of symmetry in some matrices which facilitates our calculation of determinants.

3. Complexity of some new join Graphs

Definition 3.1

The join graph $P_{2n} \dot{\vee} N_n$ with $3n$ vertices, is formed from a simple path P_{2n} with $2n$ vertices $, u_1, u_2, ..., u_{2n}$ and null graph N_n with n vertices $, v_1, v_2, ..., v_n$ such that $v_n \in N_n$ adjacent with u_1 and u_{2n} in P_{2n} and $v_{n-1} \in N_n$ adjacent with u_2 and u_{2n-1} in P_{2n} and so on. See Fig. (3).



Fig(3) : $P_{10} \dot{\vee} N_5$

Theorem 3.2 For $n \ge 1$, $\tau(P_{2n} \dot{\vee} N_n) = 2^{n-1} L_{2n}$.

Proof: Applying lemma 1.1, we have:

$$\tau(P_{2n} \dot{\vee} N_n) = \frac{1}{(3n)^2} \det(3n\, I - \bar{D} + \bar{A})$$

$$= \frac{1}{(3n)^2} \det \begin{pmatrix}
3 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 0 \\
0 & 4 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & 0 & 1 \\
1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \iddots & \iddots & \iddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \iddots & \iddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 4 & 0 & \vdots & \ddots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 3 & 1 & \cdots & \cdots & 1 & 0 \\
1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & 1 \\
1 & \iddots & \iddots & \iddots & 1 & 1 & \ddots & 1 & \cdots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \ddots & 1 & 3
\end{pmatrix} = \frac{1}{(3n)^2} \det \begin{pmatrix} A_{(2n\times 2n)} & B_{(2n\times n)} \\ B^T_{(n\times 2n)} & C_{(n\times n)} \end{pmatrix}$$

$$= \frac{1}{(3n)^2} \det C \; \det(A - BC^{-1}B^T)$$

$$= \frac{1}{(3n)^2} \det \begin{pmatrix}
3 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 3
\end{pmatrix}_{n\times n} \times \left(\frac{1}{2n+4}\right)^{2n} \det \begin{pmatrix}
3n+15 & 5-2n & 9 & \cdots & 9 & 9 & \cdots & \cdots & 9 & 7-n \\
5-2n & 5n+19 & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \iddots & 9 \\
9 & \ddots & \ddots & \ddots & 9 & \vdots & \iddots & \iddots & \iddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 5-2n & 9 & 7-n & \iddots & \iddots & \vdots \\
9 & \cdots & 9 & 5-2n & 5n+19 & 3-3n & 9 & \cdots & \cdots & 9 \\
9 & \cdots & \cdots & 9 & 3-3n & 5n+19 & 5-2n & 9 & \cdots & 9 \\
\vdots & \iddots & \iddots & 7-n & 9 & 5-2n & \ddots & \ddots & \ddots & \vdots \\
\vdots & \iddots & \iddots & 9 & \vdots & 9 & \ddots & \ddots & \ddots & 9 \\
9 & \iddots & \iddots & \iddots & \vdots & \vdots & \ddots & \ddots & 5n+19 & 5-2n \\
7-n & 9 & \cdots & \cdots & 9 & 9 & \cdots & 9 & 5-2n & 5n+15
\end{pmatrix}$$

Straightforward induction using relations between Lucas numbers and properties of Chebyshev polynomials of first kind and second kind, yields:

$$\tau(P_{2n}\dot{\vee}N_n)=\frac{1}{(3n)^2}[2^{n-1}(n+2)][\frac{2(3n)^2}{n+2}T_n(\frac{3}{2})]=2^nT_n(\frac{3}{2})$$

$$=2^n\times\frac{1}{2}[(\frac{3+\sqrt{5}}{2})^n+(\frac{3-\sqrt{5}}{2})^n]=\frac{1}{2}[(3+\sqrt{5})^n+(3-\sqrt{5})^n]=2^{n-1}L_{2n}.$$

Definition 3.3

The join graph $P_{2n}\bigtriangledown P_n$ with $3n$ vertices, is formed from a simple path P_{2n} with $2n$ vertices $,u_1,u_2,....,u_{2n}$ and path P_n with n vertices $,v_1,v_2,....,v_n$ such that $v_n\in P_n$ adjacent with u_1 and u_{2n} in P_{2n} and $v_{n-1}\in P_n$ adjacent with u_2 and u_{2n-1} in P_{2n} and so on. See Fig. (2).

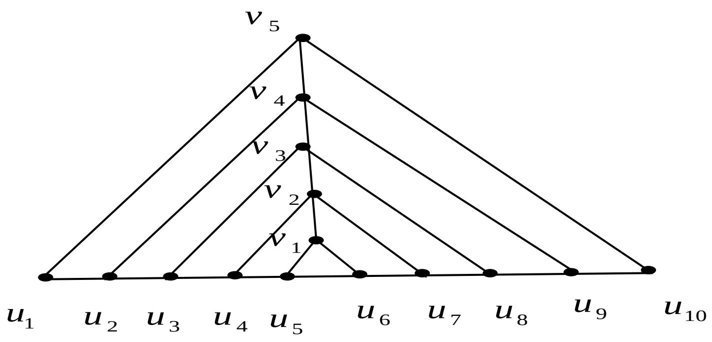


Fig.(2): $P_{10}\bigtriangledown N_5$

Theorem 3.4 Let For $n\geq 2$, $\tau(P_{2n}\bigtriangledown P_n)=2^{n-1}L_{2n}$.

Proof: Applying lemma 1.1, we have:

$$\tau(P_{2n}\bigtriangledown P_n)=\frac{1}{(3n)^2}\det(3nI-\bar{D}+\bar{A})$$

$$=\frac{1}{(3n)^2}\det\begin{pmatrix}
3 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & 0\\
0 & 4 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & 0 & 1\\
1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \iddots & \iddots & \iddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \iddots & \iddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \cdots & \cdots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 4 & 0 & \vdots & \ddots & \ddots & \ddots & 1\\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 3 & 1 & \cdots & \cdots & 1 & 0\\
1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 4 & 0 & 1 & \cdots & 1\\
1 & \iddots & \iddots & \iddots & 1 & 1 & \ddots & 1 & \cdots & \vdots & 0 & 5 & \ddots & \ddots & \vdots\\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & 1\\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 5 & 0\\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 4
\end{pmatrix}=\frac{1}{(3n)^2}\det\begin{pmatrix}A_{(2n\times 2n)} & B_{(2n\times n)}\\ B^T_{(n\times 2n)} & C_{(n\times n)}\end{pmatrix}$$

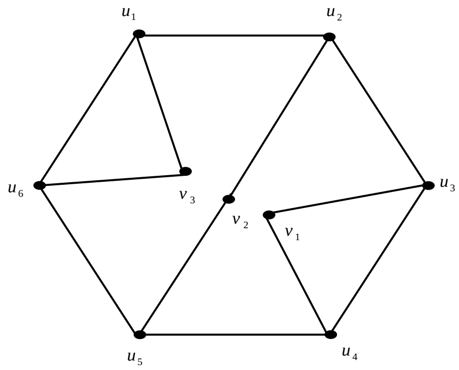
$$=\frac{1}{(3n)^2}\det C\ \det(A-BC^{-1}B^T)$$

Straightforward induction using properties of determinants and relations of Chebyshev polynomials of first kind and second kind, yields:

$$\tau(P_{2n} \triangledown P_n) = \frac{1}{(3n)^2}[(n+2)U_{n-1}(2)][\frac{2(3n)^2}{(n+2)}\frac{T_n(\frac{3}{2})U_{n-1}(\frac{5}{2})}{U_{n-1}(2)}] = 2T_n(\frac{3}{2})U_{n-1}(\frac{5}{2})$$
$$= \frac{1}{2^{2n}\sqrt{21}}[(3+\sqrt{5})^n + (3-\sqrt{5})^n][(5+\sqrt{21})^n - (5-\sqrt{21})^n].$$

Definition 3.5

The join graph $C_{2n} \dot{\vee} N_n$ on $3n$ vertices, is formed from a simple path C_{2n} with $2n$ vertices ,$u_1,u_2,....,u_{2n}$ and null N_n with n vertices ,$v_1,v_2,....,v_n$ such that $v_n \in N_n$ adjacent with u_1 and u_{2n} in C_{2n} and $v_{n-1} \in N_n$ adjacent with u_2 and u_{2n-1} in C_{2n} and so on. See Fig. (3).



Fig(3) : $C_6 \dot{\vee} N_3$

Theorem 3.6: for $n \geq 1$, $\tau(C_{2n} \dot{\vee} N_n) = 5 \times 2^{n-1} F_{2n}$.

Proof: Applying lemma 1.1, we have:

$$\tau(C_{2n} \dot{\vee} N_n) = \frac{1}{(3n)^2}\det(3n\, I - \bar{D} + \bar{A})$$

$$= \frac{1}{(3n)^2}\det\begin{pmatrix} 4 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 0 \\ 0 & 4 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \iddots & \iddots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \iddots & \iddots & \iddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \iddots & \iddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 4 & 0 & \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 4 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & 1 \\ 1 & \iddots & \iddots & \iddots & 1 & 1 & \ddots & 1 & \cdots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \ddots & 1 & 3 \end{pmatrix} = \frac{1}{(3n)^2}\det\begin{pmatrix} A_{(2n\times 2n)} & B_{(2n\times n)} \\ B^T_{(n\times 2n)} & C_{(n\times n)} \end{pmatrix}$$

$$= \frac{1}{(3n)^2}\det C\ \det(A - BC^{-1}B^T)$$

Straightforward induction using relations between Fibonacci numbers and properties of Chebyshev polynomials of first kind and second kind, yields:

$$\tau(C_{2n} \dot{\vee} N_n) = \frac{\sqrt{5}}{2}[(3+\sqrt{5})^n - (3-\sqrt{5})^n] = 5 \times 2^{n-1} F_{2n}.$$

Definition 3.7

The join graph $C_{2n} \triangledown P_n$ with $3n$ vertices, is formed from a simple path C_{2n} with $2n$ vertices $, u_1, u_2, ..., u_{2n}$ and path P_n with n vertices $, v_1, v_2, ..., v_n$ such that $v_n \in P_n$ adjacent with u_1 and u_{2n} in C_{2n} and $v_{n-1} \in P_n$ adjacent with u_2 and u_{2n-1} in C_{2n} and so on. See Fig. (2).

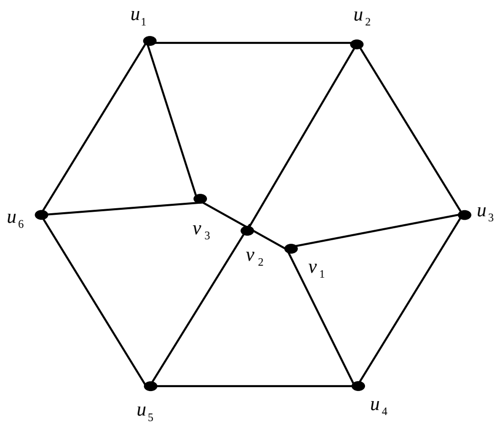


Fig.(4): $C_6 \triangledown P_3$

Theorem3.8 For $n \geq 2$,

$$\tau(C_{2n} \triangledown P_n) = \frac{5}{2^{2n}\sqrt{105}}[(3+\sqrt{5})^n - (3-\sqrt{5})^n][(5+\sqrt{21})^n - (5-\sqrt{21})^n].$$

Proof: Applying lemma 3.1, we have:

$$\tau(C_{2n} \triangledown P_n) = \frac{1}{(3n)^2}\det(3n\,I - \bar{D} + \bar{A})$$

$$= \frac{1}{(3n)^2}\det\begin{pmatrix}
4 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 0 \\
0 & 4 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \iddots & \iddots & 0 & 1 \\
1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \iddots & \iddots & \iddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \iddots & \iddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 4 & 0 & \vdots & \ddots & \ddots & \ddots & 1 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 4 & 1 & \cdots & \cdots & 1 & 0 \\
1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 4 & 0 & 1 & \cdots & 1 \\
1 & \iddots & \iddots & \iddots & 1 & 1 & \ddots & 1 & \cdots & \vdots & 0 & 5 & \ddots & \ddots & \vdots \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & 1 \\
\vdots & \iddots & \iddots & \iddots & \iddots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 5 & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 4
\end{pmatrix} = \frac{1}{(3n)^2}\det\begin{pmatrix} A_{(2n\times 2n)} & B_{(2n\times n)} \\ B^T_{(n\times 2n)} & C_{(n\times n)} \end{pmatrix}$$

$$= \frac{1}{(3n)^2} \det C \ \det(A - BC^{-1}B^T)$$

Straightforward induction using properties of determinants and relations of Chebyshev polynomials of first kind and second kind, yields:

$$\tau(C_{2n} \triangledown P_n) = \frac{5}{2^{2n}\sqrt{105}}[(3+\sqrt{5})^n - (3-\sqrt{5})^n][(5+\sqrt{21})^n - (5-\sqrt{21})^n]; n \geq 2.$$

4.Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and lemmas and their proofs.

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