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ON SOME PROPERTIES OF GENERALIZED δ-SUPPLEMENTED MODULES AND (GENERALIZED) f-δ-SUPPLEMENTED MODULES

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Abstract. In this paper, we give some properties of generalized δ-supplemented modules together with δ-radical and δ-reduced module concepts. Moreover we define (generalized) f-δ-supplemented modules and investigate some characterizations of these modules.

Keywords: generalized δ-supplemented modules; generalized f- δ-supplemented modules; associative commutative ring.

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1. Introduction

Throughout this paper, we use R to denote an associative commutative ring with identity and all modules are unitary left R-modules. Let M be an R-module. By $N \leq M$ we mean that N is a submodule of M. A submodule L of a module M is called small in M (denoted by $L << M$) if for every proper submodule K of M, $L + K \neq M$. The Jacobson radical of M is denoted by $Rad(M)$. Equivalently, $Rad(M)$ is the sum of all small submodules of M. Recall that a submodule $L \leq M$ is called essential, denoted by $L \trianglerighteq M$, if $L \cap K \neq 0$ for each

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nonzero submodule $K \leq M$. The singular submodule of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0 \text{ for some ideal } I \trianglerighteq M\}$. A module M is called singular if $Z(M) = M$. For further properties of singular modules we refer to [3].

An R-module M is called supplemented if every submodule N of M has a supplement that is a submodule K minimal with respect to $N + K = M$. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K << K$ [10]. Let M be an R-module and let N and K be any submodules of M with $M = N + K$. If $N \cap K \leq Rad(K)$ then K is called a generalized supplement of N in M. In [8] M is called generalized supplemented module (or briefly GS-module) if every submodule N of M has a generalized supplement K in M.

In [12], Zhou defined the concept of δ-small submodules as a generalization of small submodules. Let N be a submodule of M. N is said to be δ-small in M if $N + K \neq M$ for any proper submodule K of M with $\frac{M}{K}$ singular. $\delta(M) = \sum\{N \leq M \mid N <<_{\delta} M\} = \text{Rej}_M(\varphi) = \cap\left\{N \leq M \mid \frac{M}{N} \in \varphi\right\}$, where φ be the class of all singular simple modules. A submodule L of M is called a δ-supplement of N in M if $M = N + L$ and $N \cap L$ is δ-small in L and M is called δ-supplemented in case every submodule of M has a δ-supplement in M [4]. Let M be an R-module and, $N, K \leq M$ with $M = N + K$. If $N \cap K \leq \delta(N)$ then N is called a generalized δ-supplement of K in M. Following [6], M is called a generalized δ-supplemented module (or briefly δ-GS module) if every submodule N of M has a generalized δ-supplement K in M.

In this paper, we give some specialized properties of generalized δ-supplemented modules and (generalized) f-δ-supplemented modules.

2. Preliminaries

We begin by stating the following lemmas for the completeness.

Lemma 1. *Let N be a submodule of M. The following are equivalent:*

(1) $N <<_{\delta} M$;

(2) If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$;

(3) If $X + N = M$ with $\frac{M}{X}$ Goldie torsion, then $X = M$ "(see [12])".

Lemma 2. *Let M be a module.*

(1) For submodules N, K, L of M with $K \subseteq N$, we have

(a) $N <<_\delta M$ if and only if $K <<_\delta M$ and $\frac{N}{K} <<_\delta \frac{M}{K}$.

(b) $N + L <<_\delta M$ if and only if $N <<_\delta M$ and $L <<_\delta M$.

(2) If $K <<_\delta M$ and $f : M \longrightarrow N$ is a homomorphism, then $f(K) <<_\delta N$.

In particular, if $K <<_\delta M \subseteq N$, then $K <<_\delta N$.

(3) Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 <<_\delta M_1 \oplus M_2$ if and only if $K_1 <<_\delta M_1$ and $K_2 <<_\delta M_2$ "(see [12]*)".*

Proposition 3. *Let U and V be submodules of a module M. Assume that V is a δ-supplement of U in M. Then the following statements hold*

(1) If $W + V = M$ for some $W \subseteq U$, then V is a δ-supplement of W in M,

(2) If $K <<_\delta M$, then V is a δ-supplement of $U + K$ in M,

(3) For $K <<_\delta M$ we have $K \cap V <<_\delta V$ and so $\delta(V) = V \cap \delta(M)$,

(4) For $L \subseteq U$, $\frac{V+L}{L}$ is a δ-supplement of $\frac{U}{L}$ in $\frac{M}{L}$

(5) If $\delta(M) <<_\delta M$, or $\delta(M) \subseteq U$ and if $p : M \longrightarrow \frac{M}{\delta(M)}$ is the canonical projection, then $\frac{M}{\delta(M)} = p(U) \oplus p(V)$ "(see [5]*)".*

Proposition 4. *Let A,B be submodules of M such that B is a generalized δ-supplement submodule of A in M. Then:*

(1) If $W + B = M$ for some $W \subseteq A$ then B is a generalized δ-supplement of W.

(2) If $K <<_\delta M$ then B is generalized δ-supplement of $A + K$.

(3) For $K <<_\delta M$ then $K \cap B <<_\delta B$ and so $\delta(B) = B \cap \delta(M)$.

(4) For $L \subseteq A$, $\frac{B+L}{L}$ is a generalized δ-supplement of $\frac{A}{L}$ in $\frac{M}{L}$ "(see [11])".

3. $\mathbf{f} - \delta$ Supplemented Modules

Definition 1. *Let M be an R-module. If every finitely generated submodule of M has a δ-supplement in M, then M is called finitely δ-supplemented module or briefly f-δ-supplemented module.*

Proposition 5. *Let M be an f-δ-supplemented module and $L <<_\delta M$. Then, $\frac{M}{L}$ is also f-δ-supplemented.*

Proof. Let $\frac{K}{L}$ be a finitely generated submodule of $\frac{M}{L}$. It follows that $\frac{K}{L} = \langle k_1+L, k_2+L, ..., k_n+L\rangle$ for some $k_1, k_2, ..., k_n \in K$. If we say $S = \langle k_1, k_2, ..., k_n\rangle$ then it can be seen easily $K = S+L$. Because S is finitely generated in M, there exists a submodule V in M which is a δ-supplement for S. By hypothesis, V is also a δ-supplement for K. Therefore, $\frac{V+L}{L}$ is a δ-supplement of $\frac{K}{L}$ in $\frac{M}{L}$.

Proposition 6. *Let M be an f-δ-supplemented module and $\delta(M) <<_\delta M$. Then every finitely generated submodule of $\frac{M}{\delta(M)}$ is a direct summand.*

Proof. It is clear that $\frac{M}{\delta(M)}$ is f-δ-supplemented from the previous proposition. Let $\frac{K}{\delta(M)} \leq \frac{M}{\delta(M)}$ be a finitely generated submodule. By hypothesis, there is a δ-supplement $\frac{V}{\delta(M)}$ of $\frac{K}{\delta(M)}$ in $\frac{M}{\delta(M)}$. That means, $\frac{K}{\delta(M)} + \frac{V}{\delta(M)} = \frac{M}{\delta(M)}$ and $\frac{K}{\delta(M)} \cap \frac{V}{\delta(M)} = \frac{K\cap V}{\delta(M)} <<_\delta \frac{V}{\delta(M)}$. Since $K\cap V$ is δ-small in M, we can write $K\cap V \leq \delta(M)$. Hence $\frac{K}{\delta(M)} \cap \frac{V}{\delta(M)} = 0_{\frac{M}{\delta(M)}}$ and this completes the proof.

Proposition 7. *Let M be an f-δ-supplemented module and L be a generated submodule of M. Then $\frac{M}{L}$ is also f-δ-supplemented.*

Proof. Let $\frac{K}{L}$ be a finitely generated submodule of $\frac{M}{L}$.Since $\frac{K}{L}$ and L finitely generated so is K. From hypothesis there exists a δ-supplement V of K in M. Then $\frac{V+L}{L} \leq \frac{M}{L}$ is also a δ-supplement for $\frac{K}{L}$.

4. Generalized $\delta-$Supplemented Modules

Proposition 8. *Let M be an R-module and $U, V \leq M$. V is a generalized δ-supplement of U if and only if $U+V = M$ and $Rm <<_\delta V$ for every $m \in U\cap V$.*

Proof. Let V be a generalized δ-supplement of U. Then $U+V = M$ and $U\cap V \leq \delta(V)$. Since, $\delta(V)$ is the sum of all δ-small submodules of V we can write $m = m_1+m_2+...+m_n$ for every $m \in U\cap V$ such that $m_i \in V_i <<_\delta V, \forall i = 1,2,...,n$. Since $V_i <<_\delta V$, also $Rm_i <<_\delta V$. And so, $Rm <<_\delta V$ since $Rm \subseteq Rm_1+Rm_2+...+Rm_n$.

Conversely, let $U+V = M$ and $Rm <<_\delta V$ for every $m \in U\cap V$. Since $\delta(V) = \sum_{L<<_\delta V} L$ for every element m in $U\cap V, m \in Rm \leq \delta(V)$. Hence, $U\cap V \leq \delta(V)$.

A module M is called *radical* if $Rad\ M = M$, and M is called *reduced* if it has no nonzero radical submodule. See [13] for details for the notion of reduced and radical modules. By using these concepts we can give following definition.

Definition 2. *A module M is called δ-radical if $\delta(M) = M$, and M is called δ-reduced if it has no nonzero δ-radical submodule.*

Corollary 9. *Let V be a δ-radical submodule of M. Then, V is a generalized δ-supplement of every submodule in M including V.*

Proposition 10. *Let $U, V \leq M$ and V be a generalized δ-supplement of U that is not δ-radical. Then there is a maximal essential submodule of M including U.*

Proof. *By hypothesis, there is a maximal submodule K of M such that $\frac{V}{K}$ simple singular. Here, K is a maximal submodule of V so $V = K + Rv$ for $v \in V \backslash K$. Assume that $M = U + K$. Then $v = u + k$, $u \in U$, $k \in K$ and so $u \in U \cap V$. Because V is a generalized δ-supplement of U we can say $U \cap V \leq \delta(V) \leq K$. And $v \in K$ is obtained but this contradicts with $M \neq U + K$. Since $\frac{M}{U+K} \cong \frac{V}{K}$, $U + K$ is maximal and essential in M.*

Corollary 11. *Let M be a δ-reduced module. If U has a generalized δ-supplement in M, U is included by a maximal essential submoule in M.*

Proposition 12. *Let M be an R-module. If every proper submodule of M is contained in a maximal submodule then $\delta(M) <<_\delta M$.*

Proof. Let $L \leq M$ with $\delta(M) + L = M$, and let $\frac{M}{L}$ be singular. If $L \neq M$, then there exists a maximal submodule K in M containing L. Since $\frac{M}{K}$ is simple and singular, $\delta(M) \leq K$. So $M = K$ is obtained but this contradicts with the maximality of K. Hence $L = M$ is obtained.

Corollary 13. *If M is a finitely generated R-module then $\delta(M) <<_\delta M$*

Corollary 14. *Let M be an R-module and V be a generalized δ-supplement of U. If V is finitely generated then V is also δ-supplement of U in M.*

Proposition 15. *Let M be an R-module and V be a generalized δ-supplement of U in M. If U is a maximal and singular submodule of M then $U \cap V = \delta(V)$.*

Proof. *$U \cap V \leq \delta(V)$ sinve V is a generalized δ-supplement of U. Additively, $\delta(V) \leq \delta(M)$ and $\delta(M) \leq U$ since U is maximal and singular and so $\delta(V) \leq U \cap V$.*

Proposition 16. *Let M be an R-module and V be a generalized δ-supplement of U. If $K \leq \delta(M)$ then $K \cap V \leq \delta(V)$.*

Proof. Assume that $K \cap V \nsubseteq \delta(V) = \cap\{N \leq V \mid \frac{V}{N} \in \varphi\}$ where φ be the class of all singular simple modules. Then, there is an element N in $\delta(V)$ such that $K \cap V \nsubseteq V$. So, $N + Rm = V$. Following this $M = U + V = U + N + Rm$ is obtained. Note that $\frac{M}{U+N}$ is singular and $Rm <<_{\delta} M$ since $K \leq \delta(M)$. Hence, $U + N = M$. By using modular law, $V = (U \cap V) + N$ and $\delta(V) + N = V$, since $U \cap V \leq \delta(V)$. It is easy to see that $\delta(V) \leq N$. At the end we have $N = V$ is obtained. This is a contradiction. So, $K \cap V \leq \delta(V)$.

Corollary 17. *Let M be an R-module and V be a generalized δ-supplement of U. Then $\delta(V) = V \cap \delta(M)$.*

Proof. *Trivially, $\delta(V) = V \cap \delta(M)$. By the previous proposition, it is easy to see that $V \cap \delta(M) \leq \delta(V)$.*

Proposition 18. *Let M be an R-module and V be a generalized δ-supplement of U. If K is a maximal submodule of V with $\frac{V}{K}$ singular, then $U + K$ is maximal in M and $\frac{M}{U+K}$ is singular.*

Proof. *By hypothesis $U + V = M$ and $U \cap V \leq \delta(V)$. First we will show $U + K \neq M$. Assume that $U + K = M$. Since K is maximal in V, $K + Rx = V$ for $x \in V - K$. We have $M = U + V = U + K + Rx$ so $Rx \subseteq U + K$ hence $x \in U + K$. Therefore, $x = y + k$ such that $y \in U$, $k \in K$. It is easy to see that $y \in U \cap V \leq \delta(V) \leq K$. This contradicts with $x \notin K$. Then, there is an element $m \in M \backslash (U + K)$. Since $m \in M = U + V$ we can write $m = u + v$ such that $u \in U$, $v \in V$. Here $v \notin U + K$ and so $v \notin K$. Following this, $U + K + Rm = U + K + Rv$ can be shown simply. $K + Rv = V$ since K is maximal in V. This means that $U + K + Rm = M$. Namely $U + K$ is a maximal submodule of M. Additively, $\frac{M}{U+K}$ is singular since, $\frac{M}{U+K} \cong \frac{V}{K+(U\cap V)} \leq \frac{V}{K}$.*

Proposition 19. *Let M be an R-module, $\delta(M) \subseteq U$, V be a generalized δ-supplement of U and $p : M \longrightarrow \frac{M}{\delta(M)}$ be natural epimorphism. Then, $\frac{M}{\delta(M)} = p(U) \oplus p(V)$.*

Proof. *By hypothesis, $U+V=M$ and $U\cap V\leq\delta(V)$. It is clear that $p(U)=\frac{U}{\delta(M)}$ and $p(V)=\frac{V+\delta(M)}{\delta(M)}$. Following this, we have $p(U)+p(V)=\frac{U}{\delta(M)}+\frac{V+\delta(M)}{\delta(M)}=\frac{U+V+\delta(M)}{\delta(M)}=\frac{M}{\delta(M)}$ and $p(U)\cap p(V)=\frac{U}{\delta(M)}\cap\frac{V+\delta(M)}{\delta(M)}=\frac{(U\cap V)+\delta(M)}{\delta(M)}=0_{\frac{M}{\delta(M)}}=\{\delta(M)\}$.*

Proposition 20. *Let M be an R-module and V be a generalized δ-supplement of U. Then, $\frac{V+L}{L}$ is a generalized δ-supplement of $\frac{U}{L}$ in $\frac{M}{L}$ where $L\leq U$.*

Proof. *Clearly, $U+V=M$ and $U\cap V\leq\delta(V)\leq\delta(V+L)$. Following this $\frac{U}{L}+\frac{V+L}{L}=\frac{M}{L}$ and $\frac{U}{L}\cap\frac{V+L}{L}=\frac{(U\cap V)+L}{L}=p(U\cap V)$ such that $p:V+L\longrightarrow\frac{V+L}{L}$ is natural epimorphism where $p(\delta(V+L))\leq\delta(\frac{V+L}{L})$. So, $\frac{(U\cap V)+L}{L}\leq\delta(\frac{V+L}{L})$ since $p(U\cap V)\leq p(\delta(V+L))$.*

Definition 3. *A module M is said to be δ-local if $\delta(M)<<_{\delta}M$ and $\delta(M)$ is a maximal submodule of M* [2].

Proposition 21. *Any δ- local module is generalized δ-supplemented.*

Proof. *Let $N\leq M$ be a proper submodule of M. Since, $\delta(M)$ is a maximal submodule of M, we have either $N\leq\delta(M)$ or $\delta(M)+N=M$. If $N\leq\delta(M)$ then trivially M is a generalized δ-supplement of N in M. Now suppose that $\delta(M)+N=M$. Since $\delta(M)<<_{\delta}M$, we have by lemma 1, $N\oplus Y=M$ for some projective semisimple submoule $Y\leq\delta(M)$. Cleary Y is a generalized δ-supplement of N in M. Therefore, M is generalized δ-supplemented.*

Proposition 22. *Let M be an R-module. ThenM is a generalized δ-supplement of $\delta(M)$.*

Proposition 23. *Let M be an R-module and V be a generalized δ-supplement of U in M such that $\delta(V)<<_{\delta}V$. then V is also a δ-supplement of U.*

Proposition 24. *U, $V\leq M$ and V is a generalized δ-supplement of U in M. If $U\cap V$ is a δ-supplement of U then V is a δ-supplement in M.*

Proof. *By hypothesis,$U+V=M$ and $U\cap V\leq\delta(V)$. Let $U\cap V$ be a δ-supplement of $X\leq U$. Then $X+U\cap V=U$ and $X\cap(U\cap V)=X\cap V<<_{\delta}U\cap V$. So we have $M=U+V=(X+U\cap V)+V=X+V$. Additionally $X\cap V<<_{\delta}V$ since $X\cap V<<_{\delta}U\cap V\leq V$.*

Corollary 25. *Generalized δ-supplement of a semisimple submodule is also a δ-supplement.*

Proof. *Let M be an R-module, U be a semsimple submodule and V be a generalized δ-supplement of U. Since $U \cap V$ is a direct sum it is a δ-supplement. And so, V is a δ-supplement by the previous proposition.*

5. Generalized $\mathbf{f} - \delta$ Supplemented Modules

Definition 4. *Let M be an R-module. If every finitely generated submodule has a generalized δ-supplement then M is called finitely generated δ-supplement module (denoted briefly f-δ-GS).*

Proposition 26. *Let M be an f-δ-GS module and L be a finitely generated submodule. Then, $\frac{M}{L}$ is f-δ-GS.*

Proof. $\frac{K}{L} \leq \frac{M}{L}$ be any finitely generated submodule. It can be shown that K is finitely generated. From hypothesis, K has a generalized δ-supplement $N \leq M$. Then $\frac{N+L}{L}$ is a generalized δ-supplement of $\frac{K}{L}$ in $\frac{M}{L}$.

Proposition 27. *Let M be an f-δ-GS module. If $\delta(M)$ is finitely generated then every finitely generated submodule of $\frac{M}{\delta(M)}$ is a direct sum.*

Proof. $\frac{M}{\delta(M)}$ is an f-δ-GS module by the previous proposition. Let $\frac{K}{\delta(M)} \leq \frac{M}{\delta(M)}$ be a finitely generated submodule. Then $\frac{K}{\delta(M)}$ has a generalized δ-supplement $\frac{X}{\delta(M)}$ such that $\frac{K}{\delta(M)} + \frac{X}{\delta(M)} = \frac{M}{\delta(M)}$ and $\frac{K}{\delta(M)} \cap \frac{X}{\delta(M)} \leq \delta(\frac{X}{\delta(M)})$. And so, $\frac{K}{\delta(M)} \cap \frac{X}{\delta(M)} = \frac{K \cap X}{\delta(M)} \leq \delta(\frac{X}{\delta(M)}) \leq \delta(\frac{M}{\delta(M)}) = \{\delta(M)\}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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