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THE GRAPH OF A COMMUTATIVE KU-ALGEBRA

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Abstract. In This paper, we introduce the associated graphs of commutative KU-algebra. Firstly, we define the KU-graph which is determined by all the elements of commutative KU-algebra as vertices. Secondly, the graph of equivalence classes of commutative KU-algebra is studied and several examples are presented. Also, by using the definition of graph folding, we prove that the graph of equivalence classes and the graph folding of commutative KU-algebra are the same, where the graph is complete bipartite graph.

Key words. KU-algebra; Annihilator ideal; graph of equivalence classes, graph folding.

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1. Introduction

The notion of BCK and BCI-algebras are first introduced by Imai and Iséki [11, 12 and 13]. Later on, in 1984, Komori [15] introduced a notion of BCC-algebras, and Dudek [7] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori. Accordingly, Dudek and Zhang [8] introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. Prabpayak and Leerawat [18, 19] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. Several authors

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studied the graph theory in connection with (commutative) semigroups and (commutative) rings, see [1, 2, 4, and 6]. Beck [3] introduced a coloring of commutative rings and he considered all the elements of a ring as vertices. The graph of equivalence classes of zero divisors of commutative rings is studied by Mulay [17]. In [14], Jun and Lee defined the notion of zero divisors and quasi-ideals in BCI-algebra and they have proved that if X is a BCK-algebra, then the associated graph of X is connected. Zahiri and Borzooei [21] defined a new graph to a BCI-algebra X, then this definition and last definition, which were introduced by Jun and Lee are the same. In this paper, we introduce the KU-graph of a commutative KU-algebra X , denoted by $G(X)$, as the (undirected) graph with all elements of X as vertices and for distinct $x, y \in X$, the vertices x and y are adjacent if and only if $x \dot{\wedge} y = 0$. Moreover, we study the graph $G_E(X)$ of equivalence classes of X and several examples are presented. Also, by using the definition of graph folding, we prove that the graph of equivalence classes and the graph folding of a commutative KU-algebra are the same, and every one, then both, is complete bipartite graph.

2. Preliminaries

In this section, we recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition2.1. [18] Let X be a nonempty set with a binary operation $*$ and a constant 0 . The triple $(X,*,0)$ is called a KU-algebra, if for all $x, y, z \in X$ the following axioms are satisfied:

(ku_1) $(x*y)*[(y*z))*(x*z)]=0$.

(ku_2) $x*0=0$.

(ku_3) $0*x=x$.

(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$.

(ku_5) $x*x=0$.

On a KU-algebra X we can define a binary relation $\leq$ on X by putting $x \leq y \Leftrightarrow y*x=0$. Then $(X,\leq)$ is a partially ordered set and **0** is its smallest element. Thus $(X,*,0)$ satisfies the following conditions: for all $x, y, z \in X$

$(ku_{1^\backslash})$ $(y*z)*(x*z) \leq (x*y)$.

$(ku_{2\backslash})$ $0 \le x$.

$(ku_{3\backslash})$ $x \le y, y \le x$ implies $x = y$.

$(ku_{4\backslash})$ $y * x \le x$.

Theorem 2.2. **[16]** In a KU-algebra X , the following axioms are satisfied: for all $x, y, z \in X$,

(1) $x \le y$ imply $y * z \le x * z$.

(2) $x * (y * z) = y * (x * z)$.

(3) $((y * x) * x) \le y$.

Definition2.3. [19, 20] Let I be a non empty subset of a KU-algebra X . Then I is said to be a KU-ideal of X , if

(I_1) $0 \in I$;

(I_2) $\forall x, y, z \in X$, if $x * (y * z) \in I$ and $y \in I$**,** imply $x * z \in I$.

Definition 2.4. We define $x \dot\wedge y = (y * x) * x$, then a KU-algebra $(X, *, 0)$ is said to be KU-commutative if it satisfies: for all x, y in X , $(y * x) * x = (x * y) * y$, i.e. $x \dot\wedge y = y \dot\wedge x$.

Theorem2.5. For a KU-algebra $(X, *, 0)$, the following are equivalent:

(a) X is commutative;

(b) $(y * x) * x \le (x * y) * y$;

(c) $((x * y) * y) * ((y * x) * x) = 0$.

Proof: clear.

Lemma 2.6. If X is commutative KU-algebra, then $x \dot\wedge (y * z) = (x \dot\wedge y) * (x \dot\wedge z)$.

Proof: If X is commutative KU-algebra,

then $(x \dot\wedge y) * (x \dot\wedge z) = (((y * x) * x) * ((z * x) * x))) \le \overbrace{(z * x) * (y * x)}^{by ku_{1\backslash}} \le y * z$. Also by $\overbrace{(x \dot\wedge y) * (x \dot\wedge z) \le (x \dot\wedge z) \le x}^{ku_{4\backslash}}$. It follows that $(x \dot\wedge y) * (x \dot\wedge z) \le x \dot\wedge (y * z)$.

Conversely, by ku_{1} and Theorem2.2 (3) we have

$((x \dot\wedge y) * (x \dot\wedge z)) * (x \dot\wedge (y * z)) = ((y * x) * x) * (z * x) * x)) * ((y * z) * x) * x)$

$\leq ((z * x) * (y * x)) * (y * z) \leq (y * z) * (y * z) = 0$, hence $x \dot\wedge (y * z) \leq (x \dot\wedge y) * (x \dot\wedge z)$. Therefore

$x \dot\wedge (y * z) = (x \dot\wedge y) * (x \dot\wedge z)$.□

We will refer to X is commutative KU-algebra unless otherwise indicated.

Definition 2.7. Let A be a subset of X. Then we define $ann(A) = \{x \in X : a \dot\wedge x = 0$ for all $a \in A\}$ and call it the KU-annihilator of A. If $A = \{a\}$, then we write $ann(a)$ instead of $ann(\{a\})$.

Lemma 2.8. Let A be a subset of X and $ann(A)$ be the KU-annihilator of A, then $ann(A)$ is an ideal of X.

Proof: Since $a \dot\wedge 0 = (0 * a) * a = a * a = 0$, then $0 \in ann(A)$.

Let $x * (y * z), y \in ann(A)$, then $a \dot\wedge (x * (y * z)) = 0$, which implies that $a \dot\wedge (y * (x * z)) = 0$, and by Lemma 2.6 $(a \dot\wedge y) * (a \dot\wedge (x * z)) = 0$. Since $y \in ann(A)$, then $0 * (a \dot\wedge (x * z)) = 0$, hence $a \dot\wedge (x * z) = 0$, i.e. $(x * z) \in ann(A)$. Therefore $ann(A)$ is an ideal of X.□

Lemma2.9. If $\phi \neq A, B \subseteq X$, then

(I) If $A \subseteq B, then\ ann(B) \subseteq ann(A)$;

(II) $ann(A \bigcup B) = ann(A) \bigcap ann(B)$;

(III) $ann(A) \bigcup ann(B) \subseteq ann(A \bigcap B)$.

Proof: (I) Suppose that

$x \in ann(B)$, $then (x * b) * b = 0, \forall b \in B$, $but\ A \subseteq B, therfore (x * b) * b = 0$, $\forall b \in A$

$i.e\ x \in ann(A)$, hence $ann(B) \subseteq ann(A)$.

(II) Since $A \subseteq A \bigcup B$ and $B \subseteq A \bigcup B$, we have by part (I) of Lemma 2.9 that,

$ann(A \bigcup B) \subseteq ann(A), ann(B)$, and hence $ann(A \bigcup B) \subseteq ann\,(A) \bigcap ann(B)$ $--(1)$

Conversely, if $x \in ann(A) \bigcap ann(B)$, then $x \in ann(A), ann(B)$, therefore $(x * a) * a = 0, \forall a \in A$ and $(x * b) * b = 0, \forall b \in B$. But if $c \in (A \bigcup B)$, then

$(x * c) * c = 0 \forall c \in (A \bigcap B)$ we have $x \in ann(A \bigcup B)$, hence

$ann\,(A) \bigcap ann(B) \subseteq ann(A \bigcup B)$ ------------- (2)

From (1) and (2), we have $ann(A \bigcup B) = ann(A) \bigcap ann(B)$.

(III): we have $A \supset A \bigcap B$, $B \supset A \bigcap B$ from (I) $ann(A) \subset ann(A \bigcap B)$ and $ann(B) \subset ann(A \bigcap B)$ which implies that $ann(A) \bigcup ann(B) \subseteq ann(A \bigcap B)$.

Lemma 2.10. If A is a nonempty subset of X, then $ann(A) = \bigcap_{a \in A} ann(a)$.

Proof: Since $A = \bigcup_{a \in A} \{a\}$, we have $ann(A) = ann\{\bigcup_{a \in A} \{a\}\} = \bigcap_{a \in A} ann(a)$.

Definition2.11. Define a relation ~ on X as follows:

$x \sim y \ if \ and \ only if \ ann(x) = ann(y), \forall x, y \in X$

Lemma2.12. the relation ~ (from Definition2.11) is an equivalence relation on X.

Proof: The reflexivity, symmetry, and transitivity follow very easily from Definition 2.11 showing ~ is an equivalence relation.

3. A graph of a commutative KU-algebra

In this section, we introduce the concepts of graph of X and the graph of equivalence classes of X. For a graph G, we denote the set of vertices of G as $V(G)$ and the set of edges as $E(G)$. A graph G is said to be complete if every two distinct vertices are joined by exactly one edge. A graph G is said to be bipartite graph if its vertex set $V(G)$ can be partitioned into disjoint subsets V_1 and V_2 such that, every edge of G joins a vertex of V_1 with a vertex of V_2. So, G is called a complete bipartite graph if every vertex in one of the bipartition subset is joined to every vertex in the other bipartition subset. Also, a graph G is said to be connected if there is a path between any given pairs of vertices, otherwise the graph is disconnected. For distinct vertices x and y of G, let d(x, y) be the length of the shortest path from x to y and if there is no such path we define d(x, y) = ∞. The diameter of G is $diam(G) = \sup\{d(x, y):$ x and y are distinct vertices of G}. The diameter is 0 if the graph consists of a single vertex. A connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. The neighborhood of a vertex $x \in G$ is the set of the vertices in G adjacent to x (i.e.) $N(x) = \{y \in V(G) : x - y\}$. In case $x \in X$, it is easy to see that $N(x) = ann(x)$ for all $x \neq 0$. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Two graphs G_1 and G_2 are said to be isomorphic if there exists a bijective mapping $f : V(G_1) \rightarrow V(G_2)$ such that

$x-y\in E(G_1)$ then $f(x)-f(y)\in E(G_2)$. A fan graph, denoted by F_n, is a path P_{n-1} plus an extra vertex x_0 connected to all vertices of the path P_{n-1}, where $P_{n-1}=\{x_1,x_2,...,x_{n-1}\}$. A graph G is called a star graph in case there is a vertex x in G such that every other vertex in G is an end, connected to x and no other vertex by an edge. For more details we refer to [5, 10].

Definition 3.1. A graph of X, denoted by $G(X)$ is an undirected simple graph whose vertices are the elements of X and two distinct elements $x,y\in X$ are adjacent if and only if $x \dot{\wedge} y=0$. This graph is called the KU-graph.

Theorem3.2. With notations as above. $G(X)$ is connected and $diam(G(X))\leq 3$.

Proof: Let $x,y\in X$ be two distinct vertices. We have the following two cases:

Case1: $x \dot{\wedge} y=0$. Then $d(x,y)=1$.

Case2: $x \dot{\wedge} y\neq 0$. Then there are $a,b\in X\setminus\{x,y\}$ with $a \dot{\wedge} x=b \dot{\wedge} y=0$. If $a=b$, then $x-a-y$ is a path of length 2; Thus $d(x,y)=2$. We may assume that $a\neq b$, if $a \dot{\wedge} b=0$, then $x-a-b-y$ is a path of length 3, and hence $d(x,y)\leq 3$. If $a \dot{\wedge} b\neq 0$, then $x \dot{\wedge}(a \dot{\wedge} b)=0$, $y \dot{\wedge}(a \dot{\wedge} b)=0$, thus $x-a \dot{\wedge} b-y$ is a path of length 2, so $d(x,y)=2$. In all of the cases, $diam(G(X))\leq 3$. From above, there exists a path between any two distinct elements in X and so $G(X)$ is connected.

Example 3.3. Let $X=\{0,a,b,c\}$ be a set, with the operation $*$ defined by the following table

*	0	a	b	c
0	0	a	b	c
a	0	0	a	c
b	0	0	0	c
c	0	a	b	0

Using the algorithms in Appendix A, we can prove that (X, *, 0) is a KU-algebra and it is easy to show that X is a commutative KU-algebra. By Definition 3.1, we determine the graph of X as follows: The set of vertices are $V(X)=\{0,a,b,c\}$ and the set of edges are $E(X)=\{\{0,a\},\{0,b\},\{0,c\},\{a,c\},\{b,c\}\}$, The Figure (1) shows the graph $G(X)$.

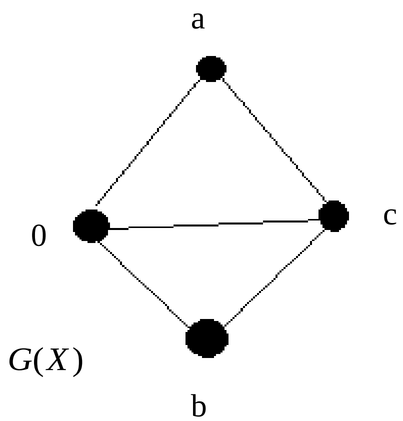


Fig. (1)

Now, we introduce the graph of equivalence classes of commutative KU-algebra X, which is constructed from classes of equivalence relation ~ in definition 2.11. For any $x,y\in X$, we say that x ~ y if and only if $ann(x)=ann(y)$. Note that ~ is an equivalence relation on X. Furthermore, if $x_1 \sim x_2$ and $x_1 \dot\wedge y=0$, then $y\in ann(x_1)=ann(x_2)$ and hence $x_2 \dot\wedge y=0$. We define $[x]$, the equivalence class of x, as follows: $[x]=\{z\in X: ann(z)=ann(x)\}$.

Lemma 3.4. Let $\{[x]: x\in X\}$ be the set of equivalence classes of X ,
where $[x]=\{z\in X: ann(z)=ann(x)\}$. Then $[x]\dot\wedge[y]=[x\dot\wedge y]$.

Proof: Since $ann(x)\subseteq ann(x\dot\wedge y)$, $ann(y)\subseteq ann(x\dot\wedge y)$, we have $[x\dot\wedge y]\subseteq[x],[y]$.
We claims that $[x]\dot\wedge[y]=[x\dot\wedge y]$. Let $[t]\subseteq[x],[y]$. Then $ann(x)\subseteq ann(t)$, $ann(y)\subseteq ann(t)$.
Now, we claim that $ann(x\dot\wedge y)\subseteq ann(t)$. Let $z\in ann(x\dot\wedge y)$, then $z\dot\wedge x\in ann(y)\subseteq ann(t)$. This gives $z\dot\wedge x\dot\wedge t=0$; that is $z\dot\wedge t\in ann(x)\subseteq ann(t)$. Hence $z\dot\wedge t=0$; that is $z\in ann(t)$. Then $ann(x\dot\wedge y)\subseteq ann(t)$. Thus $[t]\subseteq[x\dot\wedge y]$ and $[x],[y]\subseteq[x\dot\wedge y]$. Therefore, $[x]\dot\wedge[y]=[x\dot\wedge y]$.

Definition 3.5. Let X be as mention above. The graph of equivalence classes of X, denoted by $G_E(X)$ is a simple graph whose vertices are the set of equivalence classes $\{[x]; x \in X\}$ and two distinct classes $[x],[y]$ are adjacent in $G_E(X)$ if and only if $[x] \dot{\wedge} [y] = \{0\}$.

Example 3.6. Let $X = \{0, a, b, c, d, e\}$ be a set, with the operation $*$ defined by the following table:

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	c	b	c
b	0	a	0	b	a	d
c	0	a	0	0	a	a
d	0	0	0	b	0	b
e	0	0	0	0	0	0

Using the algorithms in Appendix A, we can prove that (X, *, 0) is a KU-algebra and it is easy to show that X is a commutative KU-algebra. Now, we determine the graph of X as follows: The set of vertices is $V(X) = \{0, a, b, c, d, e\}$, and the set of edges is $E(X) = \{\{0,a\},\{0,b\},\{0,c\},\{0,d\},\{0,e\},\{a,b\},\{a,c\}\}$, and the set of vertices of $G_E(X)$ is $\{[0],[a],[b],[d]\}$ since $ann(0) = X$, $ann(a) = \{0,b,c\}$, $ann(b) = ann(c) = \{0,a\}$, $ann(d) = ann(e) = \{0\}$, then $E(G_E(X) = \{\{[0],[a]\},\{[0],[b]\},\{[0],[d]\},\{[a],[b]\}\}$. The Figure (2) shows the graph $G(X)$ and the graph of equivalence classes $G_E(X)$.

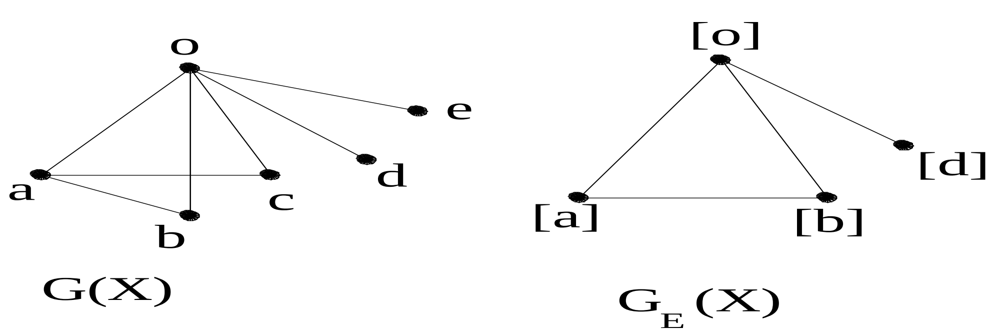


Fig .(2)

***Lemma* 3.7.** With notations as before.

1) $G_E(X)$ is a sub graph of $G(X)$;

2) $N(0) = X \setminus \{0\}$, *forall* $x \in X$. Then the KU-graph is a star graph.

Proof: straightforward.

***Theorem* 3.8.** Let $G_E(X)$ be the associated graph of equivalence classes of X. For any distinct vertices $[x],[y] \in G_E(X)$, if $[x] and [y]$ connected by an edge, then $ann(x)$ and $ann(y)$ are distinct KU-annihilator ideal of X.

Proof: suppose that $ann(x) = ann(y)$, then $x \sim y$. Hence $[x] = [y]$ this is a contradiction. Therefore, $ann(x)$ and $ann(y)$ are distinct KU-annihilator ideal of X.

The converse of this theorem is not true. In Example 3.6 it is easy to see that the vertices $[b]$ and $[d]$ are distinct KU-annihilators, but no edge joint between them.

***Theorem* 3.9.** Let X as mentioned above. If $G(X)$ is one of the following graphs:

(a) Complete graph;

(b) Fan graph,

then $G(X) \cong G_E(X)$.

Proof: (a) Suppose that $V(G(X)) = \{x_1, x_2, ..., x_n\}$. If $G(X)$ is the complete graph, then every pair of its vertices are adjacent. Thus $N(x_1) = \{x_2, x_3, ..., x_i\}, i = 2, ..., n$,

$N(x_2) = \{x_1, x_3, ..., x_i\}, i = 1, 3, ..., n$, …, $N(x_n) = \{x_1, x_2, ..., x_{n-1}\}$ .Then,

$ann(x_1) = N(x_1), ann(x_2) = N(x_2), ..., ann(x_n) = N(x_n)$ .Thus $ann(x_1) \neq ann(x_2) \neq ... \neq ann(x_n)$,

therefore every vertex of $G(X)$ is a equivalence class of $G_E(X)$, thus the vertices of $G_E(X)$ are distinct and the same number of vertices of $G(X)$, then there exist an isomorphic $f : G(X) \to G_E(X)$ satisfies $f(x_i) = [x_i]$ for each $i \in \{1, 2, ..., n\}$ and the mapping of edges $f : E(G(X)) \to E(G_E(X))$, which sends the edge $x_i - x_j$ in $E(G(X))$ to the edge $[x_i] - [x_j]$ in $E(G_E(X))$ is a well-defined bijection.

(b) If $G(X)$ is a fan graph, then there exist a path $P_{n-1}=\{x_1,x_2,...,x_{n-1}\}$ plus an extra vertex x_0 connected to all vertices of the path P_{n-1}. Thus

$N(x_0)=P_{n-1}, N(x_1)=\{x_0,x_2\}, N(x_2)=\{x_0,x_1,x_3\},..., N(x_n)=\{x_0,x_{n-1}\}$, so

$ann(x_1)\neq ann(x_2)\neq ..., ann(x_{r_1})\neq ann(x_{r_1+1})\neq ...\neq ann(x_r)$ then the vertices of $G_E(X)$ are distinct and the same number of vertices of $G(X)$, thus there exist an isomorphic $f:G(X)\to G_E(X)$ satisfies $f(x_i)=[x_i]$ for each $i\in\{1,2,...,n\}$, and the mapping of edges $f:E(G(X))\to E(G_E(X))$, which sends the edge x_i-x_j in $G(X)$ to the edge $[x_i]-[x_j]$ in $G_E(X)$ is a well-defined bijection. In all of the cases, $G(X)\cong G_E(X)$.

***Theorem* 3.10.** If $G(X)$ of X is the complete bipartite graph, then $G_E(X)$ is an edge.

Proof: Suppose that $G(X)$ is the complete bipartite graph with vertex set $V(G(X))=\{x_1,x_2,...,x_{r_1},x_{r_1+1},...,x_r\}$. This set can be split into two sets $V_1=\{x_1,x_2,...,x_{r_1}\}$ and $V_2=\{x_{r_1+1},...,x_r\}$ such that each vertex of V_1 is joined to each vertex of V_2 by exactly one edge. Thus, the set of edges are:

$E(G(X))=\{x_1-x_{r_1+1}, x_1-x_{r_2+1},..., x_1-x_r, x_2-x_{r_1+1},..., x_2-x_r,...., x_{r_1}-x_{r_1+1}, x_{r_1}-x_{r_2+1},..., x_{r_1}-x_r\}$,

so $N(x_1)=\{x_{r_1+1},x_{r_2+1},...,x_r\}=N(x_2)=...=N(x_{r_1})=V_2$ and

$N(x_{r_1+1})=\{x_1,x_2,...,x_{r_1}\}=N(x_{r_2+1})=...=N(x_r)=V_1$, then $ann(x_1)=ann(x_2)=..., ann(x_{r_1})=V_2$ and $ann(x_{r_1+1})=ann(x_{r_2+2})=..., ann(x_r)=V_1$.

Then there are two distinct equivalence classes $[x_1]$ and $[x_{r_1+1}]$ in $G_E(X)$, which are adjacent. Thus $G_E(X)$ is an edge.

Lemma 3.11. Let G and H be two graphs of commutative KU-algebras and $G\cong H$. For all $x\in V(G), y\in V(H)$, if $f(x)=y$, then $f(N(x))=N(y)$.

Proof: straightforward.

Theorem 3.12. If $G(X)\cong G(Y)$ for X and Y, then $G_E(X)\cong G_E(Y)$.

Proof: Suppose that $V(G(X)) = \{x_1, x_2, ..., x_n\}$ and $V(G(Y)) = \{y_1, y_2, ..., y_n\}$ such that the isomorphism $f : G(X) \to G(Y)$ satisfies $f(x_i) = y_i$ for each $i \in \{1,2,..., n\}$. By Lemma 3.11, $f(N(x_i)) = N(y_i)$ for each i, thus $f(ann(x_i)) = ann(y_i)$ and the mapping of edges $f : E(G_E(X)) \to E(G_E(Y))$, which sends the edge $[x_i]-[x_j]$ in $G_E(X)$ to the edge $[y_i]-[y_j]$ in $G_E(Y)$ is a well-defined bijection. Thus $G_E(X) \cong G_E(Y)$.

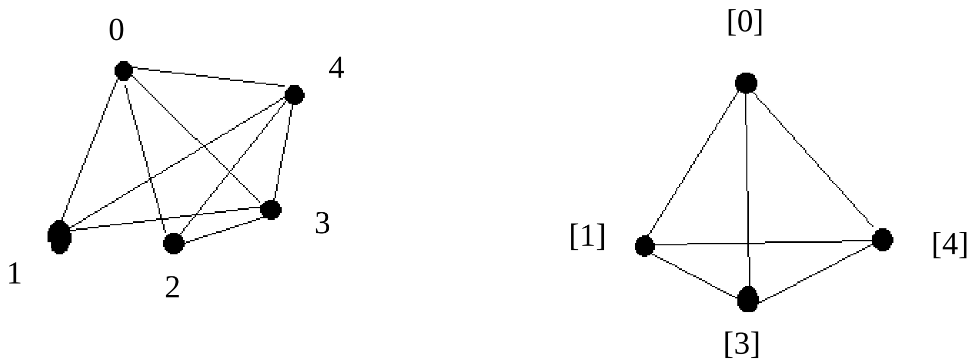
The converse of this theorem is false as illustrated in the following example.

***Example 3.13*. (a)** Let $X = \{0,1,2,3,4\}$ be a set, with the operation $*$ defined by the following table

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	1	2	0	4
4	0	1	2	3	0

Using the algorithms in Appendix A, we can prove that $(X,*,0)$ is a KU-algebra and it is easy to show that X is a commutative KU-algebra. We determine the graph of X as follows: The set of vertices is $V(X) = \{0,1,2,3,4\}$ and the set of edges is $E(X) = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$, and the set of vertices of $G_E(X)$ is $\{[0],[1],[3],[4]\}$ since $ann(0) = X$, $ann(1) = ann(2) = \{0,3,4\}$, $ann(3) = \{0,1,2,4\}$, $ann(4) = \{0,1,2,3\}$, then $E(G_E(X) = \{\{[0],[1]\},\{[0],[3]\},\{[0],[4]\},\{[1],[3]\},\{[1],[4]\},\{[3],[4]\}\}$.

Hence Figure(3) shows the graph $G(X)$ and the graph of equivalence classes $G_E(X)$.



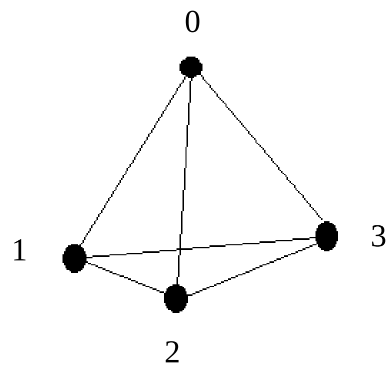
$G(X)$ Fig. (3) $G_E(X)$

(b) Let $Y = \{0,1,2,3\}$ be a set, with the operation $*$ defined by the following table

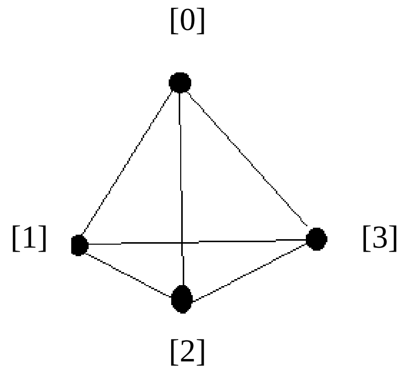
$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Using the algorithms in Appendix A, we can prove that (Y, *, 0) is a KU-algebra and it is easy to show that Y is a commutative KU-algebra. We determine the graph of Y as follows: The set of vertices is $V(Y) = \{0,1,2,3\}$ and the set of edges is $E(Y) = \{\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{1,3\},\{2,3\}\}$, then $G(Y)$ is complete graph.

Therefore, $G(Y) \cong G_E(Y)$ see Figure (4).



$G(Y)$



$G_E(Y)$

Fig. (4)

We have that $G_E(X) \cong G_E(Y)$ but $G(X) \ncong G(Y)$.

4. Graph folding

In this section, we describe the graph folding of a graph of commutative KU-algebra.

Definition 4.1. [9] ***(Graph folding)*** Let G_1 and G_2 be two graphs and $F: G_1 \to G_2$ be a continuous function. Then F is called a graph map, if

(i) for each vertex $x \in V(G_1), F(x)$ is a vertex in $V(G_2)$;

(ii) for each edge $e \in E(G_1)$, $\dim(F(e)) \leq \dim(e)$.

A graph map $F: G_1 \to G_2$ is called a graph folding if and only if F maps vertices to vertices and edges to edges, i.e., for each $x \in V(G_1)$, then $F(x) \in V(G_2)$ and for $e \in E(G_1)$, then $F(e) \in E(G_2)$. The graph folding is non trivial if and only if $No.V(F(G_1)) \leq No.V(G_1)$, also $No.E(F(G_1)) \leq No.E(G_1)$.

The set of graph foldings between graphs G_1 and G_2 is denoted by $\eta(G_1, G_2)$ and the set of graph folding of G_1 into itself by $\eta(G_1)$.

Example 4.2. Let $X = \{0, a, b, c\}$ be a set, with the operation $*$ defined by the following table

$*$	0	a	b	c
0	0	a	b	c
a	0	0	a	b
b	0	0	0	a
c	0	0	0	0

Using the algorithms in Appendix A, we can prove that (X, *, 0) is a KU-algebra and it is easy to show that X is a commutative KU-algebra. We determine the graph of X as follows:

The set of vertices is $V(X) = \{0, a, b, c\}$ and the set of edges is $E(X) = \{e_1 = \{0, a\}, e_2 = \{0, b\}, e_3 = \{0, c\}\}$, then it is clear that this graph is a complete bipartite graph (star graph), as shown in Fig (5).

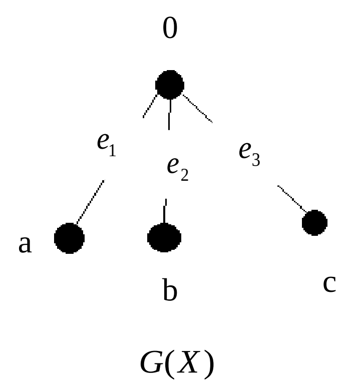
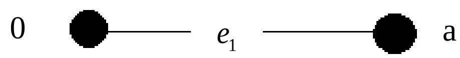


Fig. (5)

Now, we can define a graph map $F : G(X) \to G(X)$ by:

$F\{b, c\} = \{a, a\}$ and $F\{e_2, e_3\} = e_1$. It is clear that this graph map is a graph folding, such that $F(G(X)) = G'(X)$, this graph shown in Fig. (6)



$G'(X)$

Fig. (6)

Thus, the complete bipartite graph (star graph) $G(X)$ can be folded onto an edge.

Theorem 4.3. Any complete bipartite graph $G(X)$ of X can be folded to an edge.

Proof: Let $G(X)$ be a complete bipartite graph of a commutative KU-algebra X with vertex set $V(G)=\{x_1,...,x_{r_1},x_{r_1+1},...,x_r\}$. The vertex set can be split into two sets $V_1=\{x_1,...,x_{r_1}\}$ and $V_2=\{x_{r_1+1},...,x_r\}$ such that each vertex of V_1 is adjacent to each vertex of V_2 by one edge, hence $E(G(X))=\{x_1-x_{r_1+1},x_1-x_{r_2+1},...,x_1-x_r,x_2-x_{r_1+1},...,x_2-x_r,....,x_{r_1}-x_{r_1+1},x_{r_1}-x_{r_2+1},...,x_{r_1}-x_r\}$. Let

$$F(x_i)=\begin{cases} x_1 & if \quad i=1,...,r_1 \\ x_{r_1+1} & if \quad i=r_1+1,...,r \end{cases}.$$

Thus, the image of any edge of $E(G(X))$ will be the edge $x_1-x_{r_1+1}$. Moreover, this map is a graph folding.

Theorem 4.4. Let X be a commutative KU-algebra. If $G(X)$ is the complete bipartite graph then $G_E(X)$ and the graph folding of X are the same.

Proof: By using Theorem3.10 and Theorem 4.3, we obtain the result.

Conclusion

Graphs are a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, we have studied two types of graphs $G(X)$ and $G_E(X)$ of a commutative KU-algebras and discussed few results of these types, such as if $G(X)$ is the complete graph and the fan graph, then $G(X)\cong G_E(X)$. Also, if $G(X)$ is the complete bipartite graph, then $G_E(X)$ is an edge. Furthermore, we proved that if $G(X) \cong G(Y)$, then $G_E(X) \cong G_E(Y)$ but the converse is not true.

In the last section, the graph folding is defined and we proved that if the KU-graph is complete bipartite graph, then the graph $G_E(X)$ and the graph folding of a commutative KU-algebra are the same.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCH-algebra -Hilbert algebra -BF-algebra -J-algebra -WS-algebra -CI-algebra- SU-algebra -BCL-algebra -BP-algebra -Coxeter algebra -BO-algebra and so forth.

The main purpose of our future work is to investigate the folding and unfolding to other types of graphs on other algebraic systems.

Appendix A. Algorithms

<u>Algorithm for KU-algebras</u>

Input (X : set, $*$: binary operation)

Output (" X is a KU-algebra or not")

Begin

If $X = \phi$ then go to (1.);

EndIf

If $0 \notin X$ then go to (1.);

EndIf

Stop: =false;

$i := 1$;

While $i \leq |X|$ and not (Stop) do

If $x_i * x_i \neq 0$ then

Stop: = true;

EndIf

$j := 1$

While $j \leq |X|$ and not (Stop) do

If $((y_j * x_i) * x_i) \neq 0$ then

Stop: = true;

EndIf

EndIf

$k := 1$

While $k \leq |X|$ and not (Stop) do

If $(x_i * y_j) * ((y_j * z_k) * (x_i * z_k)) \neq 0$ then

Stop: = true;

EndIf

EndIf While

EndIf While

EndIf While

If Stop then

(1.) Output (" X is not a KU-algebra")

Else

Output (" X is a KU-algebra")

EndIf

End

Conflict of Interests

The author declares that there is no conflict of interests.

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